Department of Mathematics and Statistics
Quasiconformal mappings
Exercise Set 4 / September 24, 2018
due by 8.10.2018 (before the exercise class)
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1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping. Show that f is η_K-quasisymmetric, where
$$\eta_K(t) = C \max\{t^K, t^{1/K}\}, \quad t \geq 0,$$
with some constant $C = C(K)$. *Hint:* Use the Hölder continuity estimate Theorem 4.4.

2. Let f be a quasiconformal map in a domain $\Omega \subset \mathbb{C}$ with complex dilatation $\mu = \mu_f$.

   (a) Show that the complex dilatation μ_g of the inverse map $g = f^{-1}$ is
$$\mu_g(f(z)) = -\mu_f(z)\partial f(z)/\overline{\partial f(z)}.$$

   (b) If h is another quasiconformal map in Ω, then calculate
$$\mu_{h\circ f^{-1}}(f(z)) = \frac{\mu_h(z) - \mu_f(z)}{1 - \mu_h(z)\overline{\mu_f(z)}}, \quad z \in \Omega.$$

3. Consider the mapping $f(z) = z|z|^{i\gamma} = z\exp(i\gamma \log|z|)$, $z \in \mathbb{C}$ and $\gamma \in \mathbb{R}$, with $f(1) = 1$. Sketch the arc $f([0,1])$. By calculating the complex dilatation, show that f is K-quasiconformal and find the optimal estimate for K in terms of γ. Next, by calculating the Jacobian observe that f is area-preserving map. Use the above facts to deduce that f is also L-bilipschitz. Finally, find the optimal relation between the bilipschitz constant L and γ.

4. Let $\sigma\colon \Omega \to \mathbb{R}^{2\times 2}$ be symmetric and $\det \sigma(x) \equiv 1$, with Ω simply connected. Verify directly, the content of Theorem 5.5 in this special case. That is, show that $u \in W^{1,2}_{loc}(\Omega)$ satisfies the equation $\operatorname{div}(\sigma \nabla u) = 0$ if and only if $\bar\partial f = \mu \partial f$, where $f = u + iv$, and v is the conjugate of u and
$$\mu = \frac{\sigma_{22} - \sigma_{11} - 2i\sigma_{12}}{2 + \sigma_{11} + \sigma_{22}}.$$