FOURTH EXERCISES FOR GMT

Exercise 1 (3 points)**.** Proposition 5.5 from the lectures says that rectifiable sets have many projections of positive length. So, you might think that a "very" rectifiable set has many projections of "large" length in some quantitative sense. This exercise[1] makes you reconsider! Fix $n \in \mathbb{N}$, and consider the set E_n whose construction is shown in Figure 1.

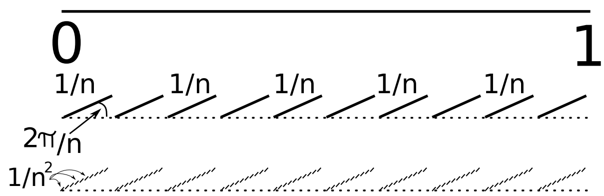

FIGURE 1. The first three steps in the construction of E_n.

More precisely, the construction of E_n happens in n steps. Start with $[0,1]$. Then, split $[0,1]$ into n pieces of length $1/n$, and rotate them counterclockwise by $2\pi/n$. Then, split the n pieces into n further pieces (each), and rotate all of them by $2\pi/n$. Now you have the situation of Figure 1. **Keep doing this for n steps**: the resulting set is called E_n.

(i) Clearly E_n is a finite union of line segments. How many, how long?
(ii) Show that E_n is "very" rectifiable: E_n be covered by a curve of length 10.
(iii) Show that E_n has small projections: $\mathcal{H}^1(\pi_e(E_n)) \lesssim 1/n$ for all $e \in S^1$.

Exercise 2 (2 points)**.** This exercise is a slightly stronger version of Lemma 5.10 in the lectures. First, a *half-line* is a set of the form $\ell_e^+(x) = \{x + te : t \geq 0\}$, where $e \in S^1$ and $x \in \mathbb{R}^2$. Then, a *half-cone* associated to a non-trivial arc $J \subset S^1$ is

$$C^+(x, J) = \bigcup_{e \in J} \ell_e^+(x).$$

Assume that $E \subset \mathbb{R}^2$ is a set with $\mathcal{H}^1(E) < \infty$ such that for every $x \in E$ there exists a half-cone $C_x = C^+(x, J_x)$ and a radius $r_x > 0$ such that

$$E \cap B(x, r_x) \cap C_x = \{x\}.$$

Prove that E is 1-rectifiable. *Hint*: countable decomposition + reduction to Lemma 5.10.

Exercise 3 (1 point)**.** Let $E \subset \mathbb{R}^2$ be compact, $\varepsilon > 0$, and $E(\varepsilon) = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, E) = \varepsilon\}$. Prove that $E(\varepsilon)$ is 1-rectifiable. *Hint*: previous exercise.

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[1]This is an unpublished example of T. Hrycak from the 90's.