Department of Mathematics and Statistics
Quasiconformal mappings
Exercise Set 5 / October 1, 2018
due by 15.10.2018 (before the exercise class)
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1. Calculate the Cauchy transform $\mathcal{C}h(z)$ and the Beurling transform $Sh(z)$ for
   (a) $h(z) = \chi_{B(z_0,r)}$, $z_0 \in \mathbb{C}$ and $r > 0$.
   (b) $h(z) = \frac{z}{\bar{z}}\chi_{\mathbb{D}}(z)$
2. Let $\|S\|_p = \sup\{\|S(g)\|_p \colon \|g\|_p = 1\}$ be the operator-norm of the Beurling transform S on $L^p(\mathbb{C})$. Show for $2 \le p < \infty$ that $\|S\|_p \ge p - 1$.
3. Let μ and ν be functions in $L^\infty(\mathbb{C})$ such that $|\mu(z)|+|\nu(z)| \le k < 1$ for a.e. $z \in \mathbb{C}$. Assume also that the coefficients are supported on $\mathbb{D}$, i.e. $\mu(z) = \nu(z) = 0$, when $|z| > 1$. Show that the equation
$$\bar\partial f(z) = \mu(z)\partial f(z) + \nu(z)\overline{\partial f(z)}, \quad \text{a.e. } z \in \mathbb{C}$$
   has a unique solution $f \in W^{1,2}_{loc}(\mathbb{C})$.
   Hint: Modify the arguments in Lemma 6.13 and Theorem 6.14.
4. Show that the λ-lemma from lecture notes does not generalize to real analytic motions. Construct a function $\Psi \colon (-1,1)\times\mathbb{C} \to \mathbb{C}$ with the following properties
   (a) $z \mapsto \Psi(t,z)$ is injective in $\mathbb{C}$ for each fixed $t \in (-1,1)$.
   (b) $t \mapsto \Psi(t,z)$ is real analytic on $(-1,1)$ for each fixed $z \in \mathbb{C}$
   (c) $\Psi(0,z) = z$ for all $z \in \mathbb{C}$
   (d) However, $z \mapsto \Psi(t,z)$ is not continuous in $\mathbb{C}$ (when $t \neq 0$)