FIFTH EXERCISES FOR GMT

TUOMAS ORPONEN

Exercise 1 (2 points). Let $E \subset \mathbb{R}^d$, $d \geq 3$, be a Borel set with $\dim_{\mathrm{H}} E > 2$. Prove that the interior of $\pi_e(E)$ is non-empty for $\mathcal{H}^{d-1}$ almost every $e \in S^{d-1}$. *Hint*: Pick $\mu \in \mathcal{M}(E)$ with $I_s(\mu) < \infty$ for some $s > 2$, and argue that $\widehat{\pi_e \mu} \in L^1(\mathbb{R})$ for $\mathcal{H}^{d-1}$ almost every $e \in S^{d-1}$. How does this prove the claim?

Exercise 2 (2 points). Recall *Fourier dimension* from Definition 6.22 in the lectures. Show that $\dim_{\mathrm{F}} E \leq \dim_{\mathrm{H}} E$ for all $E \subset \mathbb{R}^d$. Then, show that

$$\dim_{\mathrm{F}}([0,1] \times \{0\}) = 0.$$

For the next exercise, recall that the convolution of two measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ is defined as the push-forward $\mu * \nu := \Sigma(\mu \times \nu)$, where $\Sigma \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is the map $\Sigma(x,y) = x + y$. It is then clear that $\operatorname{spt}(\mu * \nu) \in \mathcal{M}(\operatorname{spt}\mu + \operatorname{spt}\nu)$. In the sequel, you may use without extra justification that

$$\widehat{\mu * \nu}(\xi) = \hat{\mu}(\xi)\hat{\nu}(\xi), \qquad \xi \in \mathbb{R}^d.$$

Exercise 3 (2 points). Assume that $R \subset (\mathbb{R}^d, +)$ is an additive subgroup with $\dim_{\mathrm{F}} R > 0$. Prove that $R = \mathbb{R}^d$. *Hint*: Pick a measure $\mu \in \mathcal{M}(R)$ with some Fourier decay, and show that sufficiently high convolution powers are in L^2. Finally, use Steinhaus' theorem.

Exercise 4 (1 point). Assume that $R \subset (\mathbb{R}, +, \cdot)$ is a Borel subring with $\dim_{\mathrm{H}} R > 1/2$. Prove that $R = \mathbb{R}$. *Hint*: Use Theorem 6.26 from the lectures, combined with the previous exercise.

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