Complex Analysis I
MAST31006

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May 11, 2019

Contents

2019 Foreword

These lecture notes form the material to the elementary course on Complex Analysis which I have taught at the University of Helsinki in 2018-2019. The purpose has been to introduce the foundation of the analysis of complex valued functions defined on the complex plane: Differentiability, power series expansions, and integration of functions along suitable curves. As an application of Cauchy's global integral theorem we have studied integration of rational functions on the real line as well as integration of trigonometric-rational forms on the real line. For mapping properties we have considered especially Möbius transformations and their invariance properties.

These notes are based on my lecture notes in Finnish which were based on Kari Astala's Lecture Notes and Terry Tao's Lecture Notes. But I have benefitted also from H. A. Priestley's book *Introduction to Complex Analysis* as well as Elias Stein's and Rami Shakarchi's book on Complex Analysis and also some other sources which are mentioned in the reference list.

I wish to thank my students for attending the lectures and homework class. I wish to express my gratitude especially to those students who were very dedicated to the course the whole semester. So thank you- Alicia, Aleksis, Eetu, Elli, Fatima, Henri, Janne, Jose, Joonas, Khalid, Laura, Lauri, Matti, Miika, Mikael, Miko, Outi, Paul, Pekka, Riku, Risto, Robert, Sara, Sauli, Simo, Toivo, Tuomas, Ville and Ville. I wish to thank Outi Boman for being an excellent scribe and organizer and supervisor to the homework writing groups. Also, I am extremely grateful to Ilmari Lehmusoksa for drawing excellent pictures.

University of Helsinki, Finland

May 2019

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2018 Foreword

These lecture notes form the material to the elementary course on Complex Analysis which I taught at the University of Helsinki in winter and spring 2018. The purpose has been to introduce the foundation of the analysis of complex valued functions defined on the complex plane: Differentiability, power series expansions, and integration of functions along suitable curves. As an application of Cauchy's global integral theorem we have studied integration of rational functions on the real line. For mapping properties we have considered especially Möbius transformations and their invariance properties.

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University of Helsinki, Finland

May 2018

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1 Background

1.1 The vector space $\mathbb{R}^2$

We have studied the 2-dimensional vector space,

$$\mathbb{R}^2 = \left\{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\right\},$$

where $\mathbb{R}$ is the set of real numbers. The vector space $\mathbb{R}^2$ is equipped with two operations: addition between elements of $\mathbb{R}^2$ and scalar multiplication between an element of $\mathbb{R}^2$ and an element of $\mathbb{R}$. That is, for every $a = (a_1, a_2)$ and $b = (b_1, b_2)$,

$$a + b = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

and for $\lambda \in \mathbb{R}$,

$$\lambda a = \lambda(a_1, a_2) = (\lambda a_1, \lambda a_2).$$

Proposition 1.1.1. *Recall that* $(\mathbb{R}^2, +)$ *is an Abelian group.*

Proof.

(A1) For all $a, b \in \mathbb{R}^2$ we have $a + b \in \mathbb{R}^2$.

(A2) For all $a, b, c \in \mathbb{R}^2$, $(a + b) + c = a + (b + c)$.

(A3) There exists a unique $0 \in \mathbb{R}^2, 0 = (0, 0)$, such that $a + 0 = 0 + a = a$ for every $a \in \mathbb{R}^2$.

(A4) For every $a = (a_1, a_2) \in \mathbb{R}^2$ there exists exactly one $-a \in \mathbb{R}^2$,
$-a = -(a_1, a_2) = -1(a_1, a_2) = (-a_1, -a_2)$, such that $a + (-a) = -a + a = 0$.

(A5) For all $a, b \in \mathbb{R}^2$: $a + b = b + a$.

□

Remarks 1.1.2.

(A2) the operation + is associative.

(A3) the element 0 is called zero element.

(A4) the element $-a$ is called the additive inverse of a

(A5) since the operator is commutative, $(\mathbb{R}^2, +)$ is an Abelian group.

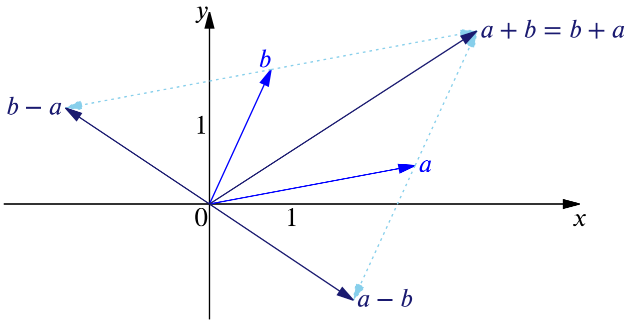


Figure 1.1: Addition and subtraction in the vector space $(\mathbb{R}^2, +)$ can be interpreted geometrically via the parallelogram law.

Remark 1.1.3. The vector space $(\mathbb{R}^2, +)$ is the Euclidean plane.

Remark 1.1.4. Recall that $(\mathbb{R}^2, +)$ is a real vector space, since $(\mathbb{R}^2, +)$ is an Abelian group and the following statements hold:

(V1) for all $\lambda \in \mathbb{R}, v \in \mathbb{R}^2 : \lambda v \in \mathbb{R}^2$.

(V2) for all $\lambda, \mu \in \mathbb{R}, v \in \mathbb{R}^2 :\ (\lambda\mu)v = \lambda(\mu v)$.

(V3) for the unit element $1 \in \mathbb{R}$ and every $v \in \mathbb{R}^2 :\ 1v = v$.

(V4) for all $\lambda \in \mathbb{R}^2$ and for all $v, u \in \mathbb{R}^2 :\ \lambda(v + u) = \lambda v + \lambda u$.

(V5) for all $\lambda, \mu \in \mathbb{R}$ and $v \in \mathbb{R}^2 :\ (\lambda + \mu)v = \lambda v + \mu v$.

Definition 1.1.5. Recall that we have introduced the inner product between $a = (a_1, a_2) \in \mathbb{R}^2$ and $b = (b_1, b_2) \in \mathbb{R}^2$ such that

$$\big(a|b\big) = a_1 b_1 + a_2 b_2.$$

This is a mapping $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. Another notation is $\langle a|b\rangle$.

Remark 1.1.6. The notation $a \cdot b$ **is not used** in this Complex Analysis course, since it is reserved for the multiplication which makes $(\mathbb{R}^2, +, \cdot)$ a field.

1.2 The complex plane $\mathbb{C}$

We introduce multiplication between any two elements a and b from $\mathbb{R}^2$ such that the multiplication takes as input $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^2$ and gives as output $c \in \mathbb{R}^2$. We define $\cdot :\ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)$$

for every pair $a = (a_1, a_2) \in \mathbb{R}^2$ and $b = (b_1, b_2) \in \mathbb{R}^2$. In doing so we obtain that $(\mathbb{R}^2, +, \cdot)$ is a field.

We write $\mathbb{C} = (\mathbb{R}^2, +, \cdot)$ and call $\mathbb{C}$ the complex plane.

Remark 1.2.1. Here lies the difference between our vector calculus (Vektorianalyysi II) course and this first complex analysis course.

Remark 1.2.2. Multiplication is a little bit messy in Cartesian co-ordinates, but it becomes nicer in polar form, which we introduce later. This polar form gives a geometric meaning for multiplication of two vectors.

In the vector calculus course we studied functions $f :\ \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbb{R}^2$ is the Euclidean plane. We looked at their differentiability and the properties of their integrals.

Now we study functions $f :\ \mathbb{C} \to \mathbb{C}$, where $\mathbb{C} = (\mathbb{R}^2, +, \cdot)$ is the complex plane.

It turns out that for a function $f :\ \mathbb{R}^2 \to \mathbb{R}^2$ to have a derivative at every point in an open ball in $\mathbb{R}^2$, that is, to be real differentiable in an open ball, is a weaker property than to have a complex derivative in every point of an open ball, that is, to be complex differentiable in an open ball.

In summary: to be real differentiable in an open ball is easier than to be complex differentiable in the corresponding ball.

Example 1.2.3. Let $f :\ \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x, -y)$. The mapping f is real differentiable on the whole plane $\mathbb{R}^2$, since its partial derivatives exist and are continuous.

However, it turns out that

$$f :\ \mathbb{C} \to \mathbb{C}, \qquad (x, y) \mapsto (x, -y),$$

is not complex differentiable.

Before studying the complex differentiability and analyticity of functions, it will be useful to go through the elementary properties of the elements of $\mathbb{C}$.

Remark 1.2.4. We give an outline of why $(\mathbb{R}^2, +, \cdot) =: \mathbb{C}$ is a field.

(F1) $(\mathbb{R}^2, +)$ is an Abelian group.

(F2) $(\mathbb{R}^2 \setminus \{0\}, \cdot)$ is an Abelian group:

  (B1) For all $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^2$: $a \cdot b \in \mathbb{R}^2$.

(B2) For all $a, b, c \in \mathbb{R}^2$: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(B3) There exists $(1, 0) \in \mathbb{R}^2$ such that for every $a \in \mathbb{R}^2$:

$$a \cdot (1,0) = a = (1,0) \cdot a.$$

(B4) For each $a \in \mathbb{R}^2 \setminus \{0\}$, $a = (a_1, a_2)$, there exists a unique inverse element of a,

$$\left(\frac{a_1}{a_1^2 + a_2^2}, \frac{-a_2}{a_1^2 + a_2^2}\right) =: a^{-1} \in \mathbb{R}^2$$

such that $a \cdot a^{-1} = a^{-1} \cdot a = \big(1, 0\big) =: 1$.

(B5) For all $a, b \in \mathbb{R}^2$: $a \cdot b = b \cdot a$.

(F3) For all $a, b, c \in \mathbb{R}^2$:

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

This is called the 1st distributional law.

Remarks 1.2.5.

(B2) The operation $\cdot$ is associative.

(B3) The element $(1, 0)$ is called the unit element.

Note: $(a_1, a_2) \cdot (1, 0) = (a_1 \cdot 1 - a_2 \cdot 0, a_1 \cdot 0 + a_2 \cdot 1) = (a_1, a_2)$ and (B5) yields

$$(1, 0) \cdot (a_1, a_2) = (a_1, a_2).$$

(B4) The point a^{-1} is the inverse of a and we also write $\frac{1}{a}$.

Let $a = (a_1, a_2) \in \mathbb{R}^2 \setminus \{0\}$. We have to find $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$ such that

$$x \cdot a = a \cdot x = (1, 0),$$

that is,

$$x \cdot a = (x_1, x_2) \cdot (a_1, a_2) = (x_1 a_1 - x_2 a_2, x_1 a_2 + x_2 a_1) = (1, 0).$$

This holds if and only if

$$\begin{cases} a_1 x_1 - a_2 x_2 & = 1 \quad \text{and} \\ a_1 x_2 + a_2 x_1 & = 0; \end{cases}$$

here $a_1^2 + a_2^2 > 0$. There exists exactly one such (x_1, x_2); since the determinant

$$\begin{vmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{vmatrix} = a_1^2 + a_2^2 \neq 0;$$

the corresponding linear system has a unique solution:

$$(x_1, x_2) = \left(\frac{a_1}{a_1^2 + a_2^2}, \frac{-a_2}{a_1^2 + a_2^2}\right).$$

Remarks 1.2.6.

1. We sometimes write $z_1 \cdot z_2 = z_1 z_2$, where $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}$.
2. Let $z \in \mathbb{C}$. We define $z^0 = (1, 0)$ and for each $n \in \mathbb{N}$: $z^n = z \cdot z^{n-1}$.

Remark 1.2.7. The set

$$\big\{(x,0) \mid x \in \mathbb{R}\big\} \subset \mathbb{C}$$

is a subfield of the field $\mathbb{C}$. There is a mapping $f : \mathbb{R} \to \big\{(x,0) \mid x \in \mathbb{R}\big\}$, which is a field isomorphism. This means that we can identify $x \in \mathbb{R}$ with $(x,0) \in \mathbb{C}$. Hence $\mathbb{R} \subset \mathbb{C}$, that is, the set of real numbers is a proper subset of the set of complex numbers.

Hence especially we can write $\big(1,0\big) = 1$.

Definition 1.2.8. The elements of $\mathbb{C}$ are called complex numbers.

Definition 1.2.9. The complex number $(0,1) \in \mathbb{C}$ is called the imaginary unit and we write

$$(0,1) =: i.$$

Remark 1.2.10. Note that

$$i^2 = i \cdot i = (0,1) \cdot (0,1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1,0) = -1,$$

but we cannot take any roots, yet. We have not defined what the square root of $z \in \mathbb{C}$ is.

2 The elementary properties of complex numbers

2.1 Complex numbers can be written in Cartesian form

Proposition 2.1.1. *Every complex number $z \in \mathbb{C}$ can be expressed in a unique way $z = x + iy$, where $x, y \in \mathbb{R}$.*

Proof. Let $z \in \mathbb{C}$ be fixed. Then $z = (x, y)$ for some $x, y \in \mathbb{R}$. In particular,

$$z = (x, y) = (x, 0) + (0, y) = (1, 0)(x, 0) + (0, 1)(y, 0) = (x, 0) + i(y, 0) = x + iy.$$

It is unique:

Let $z = x + iy = a + ib$, where $x, y, a, b \in \mathbb{R}$. Hence,

$$(x, 0) + (0, 1)(y, 0) = (a, 0) + (0, 1)(b, 0).$$

By calculating the products we get

$$(x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y) = (a, 0) + (0 \cdot b - 1 \cdot 0, 0 \cdot 0 + 1 \cdot b),$$

which implies that $(x, y) = (a, b)$. Thus $x = a$ and $y = b$. □

Definition 2.1.2. Let $z = x + iy$, where $x, y \in \mathbb{R}$ are fixed. The real part of z is

$$x =: \mathrm{Re}(z) \in \mathbb{R}$$

and the imaginary part of z is

$$y =: \mathrm{Im}(z) \in \mathbb{R}.$$

The set $\{z \in \mathbb{C} \mid \mathrm{Im}(z) = 0\}$ is the real axis and the set $\{z \in \mathbb{C} \mid \mathrm{Re}(z) = 0\}$ is the imaginary axis.
The complex conjugate of z is defined by

$$\overline{z} = x - iy,$$

and the magnitude of z is

$$|z| = \sqrt{x^2 + y^2}.$$

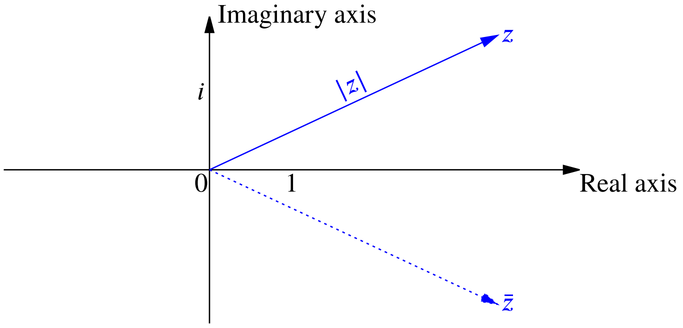


Figure 2.1: Complex number z, its magnitude $|z|$ and its complex conjugate $\overline{z}$.

Remark 2.1.3. Two complex numbers are equal if and only if they have the same real part and the same imaginary part.

Example 2.1.4.

$$\begin{aligned}
z &= \sqrt{3} + i,\\
\overline{z} &= \sqrt{3} - i,\\
|z| &= \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2,\\
\mathrm{Re}(z) &= \sqrt{3},\\
\mathrm{Im}(z) &= 1
\end{aligned}$$

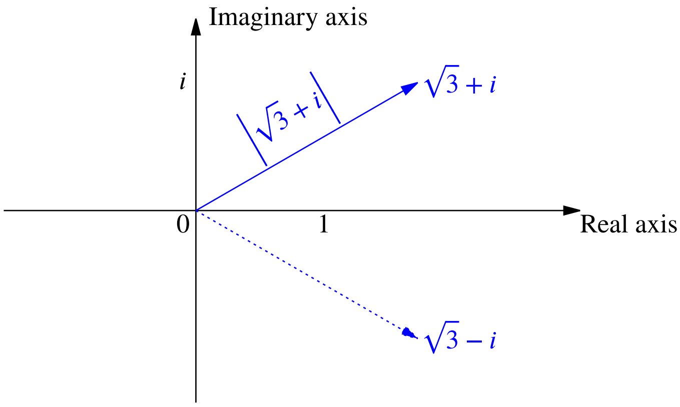


Figure 2.2: Complex number $\sqrt{3}+i$ from example 2.1.4, its magnitude $\left|\sqrt{3}+i\right| = 2$ and its complex conjugate $\sqrt{3}-i$.

Basic complex operations:

Proposition 2.1.5. *For all $z_1, z_2, z \in \mathbb{C}$:*

$$\begin{aligned} &(i) \quad \overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2}, \\ &(ii) \quad \overline{z}_1 \cdot \overline{z}_2 = \overline{z_1 \cdot z_2}, \\ &(iii) \quad \overline{\overline{z}} = z. \end{aligned}$$

Proof. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z = x + iy$, where $x_1, x_2, y_1, y_2, x, y \in \mathbb{R}$.

$$\begin{aligned} &(i) \quad \overline{z}_1 + \overline{z}_2 = x_1 - iy_1 + x_2 - iy_2 = (x_1 + x_2) - i(y_1 + y_2) = \overline{z_1 + z_2}, \\ &(ii) \quad \overline{z}_1 \cdot \overline{z}_2 = (x_1, -y_1)(x_2, -y_2) = \big(x_1x_2 - y_1y_2, -(x_1y_2 + x_2y_1)\big) = \overline{z_1 \cdot z_2}, \\ &(iii) \quad \overline{\overline{z}} = \overline{(x, -y)} = \big(x, -(-y)\big) = (x, y) = z. \end{aligned}$$

□

Note: given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.

The proofs to the following propositions are similar direct calculations as in the previous one, and are left as an exercise to the reader.

Proposition 2.1.6. *For all $z \in \mathbb{C}$:*

$$\begin{aligned} |z| &= |-z|, \\ |\overline{z}| &= |z|, \\ z \cdot \overline{z} &= |z|^2. \end{aligned}$$

Remark 2.1.7. $\overline{z} \cdot z = |z|^2$, thus if $z \in \mathbb{C} \setminus \{0\}$, then

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{\overline{z}}{|z|^2}.$$

Proposition 2.1.8. *For every $z \in \mathbb{C}$,*

$$\begin{aligned} \mathrm{Re}(z) &= \frac{1}{2}(z + \overline{z}) \quad \textit{and} \\ \mathrm{Im}(z) &= \frac{1}{2i}(z - \overline{z}). \end{aligned}$$

The magnitude $|z|$ behaves well with respect to the multiplication and division operations:

Proposition 2.1.9.

$$\begin{aligned} |z_1 \cdot z_2| &= |z_1||z_2|, && \textit{for all } z_1, z_2 \in \mathbb{C}, \\ \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|}, && \textit{for all } z_1 \in \mathbb{C} \textit{ and } z_2 \in \mathbb{C} \setminus \{0\}, \\ |z|^n &= |z^n|, && \textit{for all } z \in \mathbb{C}. \end{aligned}$$

With respect to addition and subtraction, we have the triangle inequality:

Theorem 2.1.10: The triangle inequality.

$$|z_1 + z_2| \le |z_1| + |z_2|, \qquad \textit{for all } z_1, z_2 \in \mathbb{C}.$$

More generally,

$$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|, \qquad \textit{for all } z_1, z_2 \in \mathbb{C}.$$

Proof.

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2) \cdot \overline{(z_1 + z_2)} = (z_1 + z_2) \cdot (\overline{z}_1 + \overline{z}_2) = z_1 z_1 + z_1 \overline{z}_2 + \overline{z}_1 z_2 + z_2 \overline{z}_2 \\ &= |z_1|^2 + |z_2|^2 + z_1 \overline{z}_2 + \overline{z_1 \overline{z}_2} = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1 \overline{z}_2) \\ &\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2, \end{aligned}$$

which implies $|z_1 + z_2| \le |z_1| + |z_2|$.

For the lower inequality, we have

$$|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|,$$

which implies

$$|z_1| - |z_2| \le |z_1 - z_2|, \qquad \text{for all } z_1, z_2 \in \mathbb{C}.$$

Similarly,

$$|z_2| = |z_2 - z_1 + z_1| \le |z_2 - z_1| + |z_1|.$$

Note that $|z_2 - z_1| = |z_1 - z_2|$, which gives us

$$|z_2| - |z_1| \le |z_1 - z_2|, \qquad \text{for all } z_1, z_2 \in \mathbb{C}.$$

Hence we have shown that $\big||z_1| - |z_2|\big| \le |z_1 + z_2|$. □

2.2 Complex numbers can be written in polar form

Definition 2.2.1: Polar coordinates. Let us recall the connection between Cartesian coordinates and polar coordinates from the vector calculus course.

If the point $(x, y) \in \mathbb{R}^2 \setminus \{0\}$ is given, we can find $r = \sqrt{x^2 + y^2} > 0$ and $\alpha \in [0, 2\pi]$, counting counter-clockwise, such that

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ x = r\cos\alpha, \\ y = r\sin\alpha. \end{cases}$$

Conversely, if $r > 0$ and $\alpha \in [0, 2\pi]$ are given, we can find x and y.

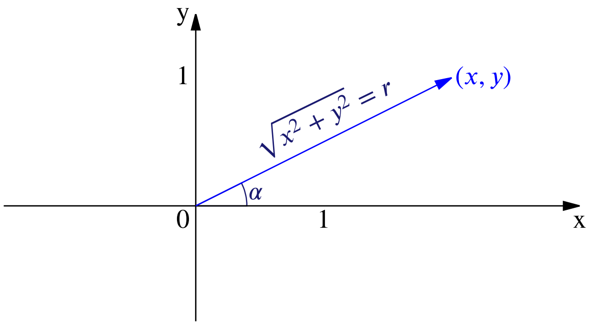


Figure 2.3: Polar coordinates r and α of vector $(x, y) \in \mathbb{R}^2 \setminus \{0\}$.

Let us remove the restriction on α. Let $\alpha \in \mathbb{R}$ and note that

$$\begin{cases} \cos\alpha = \frac{x}{r}, \\ \sin\alpha = \frac{y}{r}. \end{cases}$$

These equations determine α only up to an integer multiple of 2π.

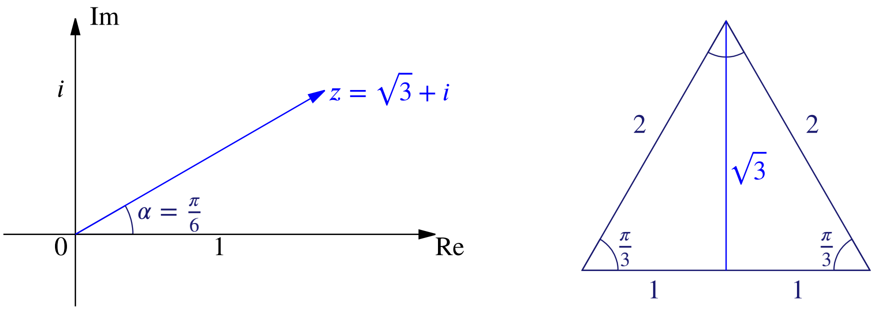
A complex number $z \in \mathbb{C} \setminus \{0\}$ can be written in polar form as

$$z = r\cos\alpha + ir\sin\alpha, \tag{2.2.2}$$

where $r = |z| > 0$ is called the magnitude, modulus or absolute value of z and $\alpha \in \mathbb{R}$ is called the argument or phase of z. The argument α is, however, only determined up to an integer multiple of 2π.

Example 2.2.3.

$$z = \sqrt{3} + i$$
$$\arg\left(\sqrt{3} + i\right) = \frac{\pi}{6} + n2\pi, \qquad n \in \mathbb{Z}$$



(a) The complex number z and its argument. (b) An equilateral triangle.

Figure 2.4: The complex number z from example 2.2.3 and an equilateral triangle, which can be used to find the argument of z.

Note: The "set" of all possible arguments of z is denoted by $\mathrm{Arg}(z)$.

Theorem 2.2.4: Euler's formula. *The formula*

$$\cos\alpha + i\sin\alpha =: e^{i\alpha}, \qquad \alpha \in \mathbb{R},$$

is called Euler's formula. We take it now as a notation, but we will prove it later.

2.2.5. The relationship between the Cartesian and the polar form is given by Euler's formula:

$$z = x + iy = r\cos\alpha + ri\sin\alpha = r(\cos\alpha + i\sin\alpha) = re^{i\alpha}.$$

Remark 2.2.6. If $r_1 e^{i\alpha_1} = r_2 e^{i\alpha_2} \neq 0$, then $r_1 = r_2$ and $\alpha_1 = \alpha_2 + n2\pi$ for some $n \in \mathbb{Z}$.

2.2.7. There are infinitely many choices for the argument of any given complex number. To get a unique argument we restrict the argument to lie in an interval $(\alpha, \beta]$, where $\beta - \alpha = 2\pi$. The standard argument $\mathrm{Arg}_{(-\pi,\pi]}$ is defined as the unique argument, of a complex number z, that lies in the interval $(-\pi, \pi]$. For the general interval $(\alpha, \beta]$, where $\beta - \alpha = 2\pi$, the argument $\mathrm{Arg}_{(\alpha,\beta]}$ is defined as the unique argument of z that lies in the interval $(\alpha, \beta]$.

2.2.8. Recall the formulas for the functions sine and cosine when $\alpha, \beta \in \mathbb{R}$:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

2.2.9 The geometric meaning of the multiplication of complex numbers

For two complex numbers $z_1 = r_1 e^{i\alpha_1}$ and $z_2 = r_2 e^{i\alpha_2}$, by Euler's formula, the definition of multiplication and 2.2.8 we get

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i\alpha_1} e^{i\alpha_2} = r_1 r_2 (\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2) \\ &= r_1 r_2 \big((\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2) + i(\cos\alpha_1\sin\alpha_2 + \sin\alpha_1\cos\alpha_2)\big) \\ &= r_1 r_2 \big(\cos(\alpha_1 + \alpha_2) + i\sin(\alpha_1 + \alpha_2)\big) = r_1 r_2 e^{i(\alpha_1+\alpha_2)}. \end{aligned}$$

Notice the geometric meaning: when we multiply two complex numbers, the output is a complex number whose magnitude we obtain by multiplying the magnitudes of the input complex numbers. Likewise, the argument of the output complex number is obtained by summing the arguments of the input complex numbers.

If $z_0 \in \mathbb{C} \setminus \{0\}$ is a fixed complex number, then in the mapping $z \mapsto z_0 z$, the point z is the input and the output is

$$z_0 z = |z_0 z|\, e^{i(\mathrm{Arg}\, z_0 + \mathrm{Arg}\, z)}.$$

This means stretching and rotating in the complex plane $\mathbb{C}$.

Example 2.2.10. Let $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto iz$. The mapping f corresponds to a 90 degree rotation counter-clockwise.

2.2.11. We have proved

$$z_1 z_2 = r_1 r_2 e_1^{i(\varphi_1+\varphi_2)},$$
$$|z_1 z_2| = |z_1||z_2|, \qquad \text{for all } z_1, z_2 \in \mathbb{C},$$

where

$$z_1 = r_1 e^{i\varphi_1}, \quad z_2 = r_2 e^{i\varphi_2},$$
$$r_j > 0, \quad \varphi_j \in \mathbb{R}, \qquad j = 1, 2.$$

As well as

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \arg(z_2), \qquad \text{for all } z_1, z_2 \in \mathbb{C} \setminus \{0\}.$$

Remark 2.2.12. Let $\alpha \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. It might be the case that

$$\mathrm{Arg}_{(\alpha,\alpha+2\pi]}(z_1 z_2) \neq \mathrm{Arg}_{(\alpha,\alpha+2\pi]}(z_1) + \mathrm{Arg}_{(\alpha,\alpha+2\pi]}(z_2).$$

Example. Let $z_1 = -i = z_2$. Then

$$\mathrm{Arg}_{(-\pi,\pi]}(z_1) = \mathrm{Arg}_{(-\pi,\pi]}(z_2) = -\frac{\pi}{2}$$

and

$$\mathrm{Arg}_{(-\pi,\pi]}(z_1 z_2) = \mathrm{Arg}_{(-\pi,\pi]}(-1) = \pi.$$

Hence,

$$\mathrm{Arg}_{(-\pi,\pi]}(z_1 z_2) = \pi \neq -\frac{\pi}{2} + \left(-\frac{\pi}{2}\right) = -\pi = \mathrm{Arg}_{(-\pi,\pi]}(z_1) + \mathrm{Arg}_{(-\pi,\pi]}(z_2).$$

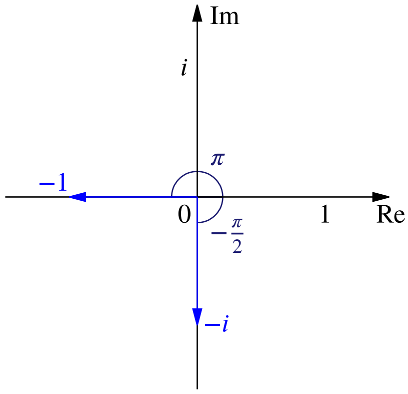


Figure 2.5: About Remark 2.2.12: $z_1 = -i = z_2$ and $\mathrm{Arg}_{(-\pi,\pi]}(z_1) = \mathrm{Arg}_{(-\pi,\pi]}(z_2) = -\frac{\pi}{2}$. $z_1 z_2 = (-i)(-i) = -1$ but $\mathrm{Arg}_{(-\pi,\pi]}(z_1 z_2) = \pi \neq \mathrm{Arg}_{(-\pi,\pi]}(z_1) + \mathrm{Arg}_{(-\pi,\pi]}(z_2) = -\frac{\pi}{2} + \left(-\frac{\pi}{2}\right) = -\pi$.

Proposition 2.2.13. *For all $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C} \setminus \{0\}$*

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Furthermore, if $z_1 \in \mathbb{C} \setminus \{0\}$, then

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

Proof. Since

$$|z_1| = \left|\frac{z_1}{z_2} z_2\right| = \left|\frac{z_1}{z_2}\right| |z_2|,$$

we obtain

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Also,

$$\arg(z_1) = \arg\left(\frac{z_1}{z_2} z_2\right) = \arg\left(\frac{z_1}{z_2}\right) + \arg(z_2).$$

□

<u>*Remark 2.2.14*</u>. Let $\alpha \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. It might be the case that

$$\mathrm{Arg}_{(\alpha,\alpha+2\pi]}\left(\frac{z_1}{z_2}\right) \neq \mathrm{Arg}_{(\alpha,\alpha+2\pi]}(z_1) - \mathrm{Arg}_{(\alpha,\alpha+2\pi]}(z_2).$$

Example. Let $z_1 = 1$ and $z_2 = -2$. Then,

$$\mathrm{Arg}_{(-\pi,\pi]}(z_1) = 0,$$
$$\mathrm{Arg}_{(-\pi,\pi]}(z_2) = \pi \qquad \text{and}$$
$$\mathrm{Arg}_{(-\pi,\pi]}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}_{(-\pi,\pi]}\left(-\frac{1}{2}\right) = \pi.$$

Hence,

$$\mathrm{Arg}_{(-\pi,\pi]}\left(-\frac{1}{2}\right) = \pi \neq 0 - \pi = \mathrm{Arg}_{(-\pi,\pi]}(z_1) - \mathrm{Arg}_{(-\pi,\pi]}(z_2).$$

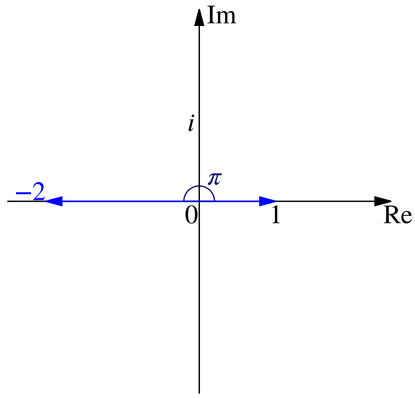


Figure 2.6: About Remark 2.2.14: $z_1 = 1$ and $z_2 = -2$, so $\operatorname{Arg}_{(-\pi,\pi]}(z_1) = 0$ and $\operatorname{Arg}_{(-\pi,\pi]}(z_2) = \pi$. So $\frac{z_1}{z_2} = -\frac{1}{2}$, but $\operatorname{Arg}_{(-\pi,\pi]}\left(-\frac{1}{2}\right) = \pi \neq 0 - \pi = \operatorname{Arg}_{(-\pi,\pi]}(z_1) - \operatorname{Arg}_{(-\pi,\pi]}(z_2)$.

Theorem 2.2.15: De Moivre's theorem. *Let $\alpha \in \mathbb{R}$. Then for any $n \in \mathbb{Z}$*

$$\left(\cos\alpha + i\sin\alpha\right)^n = \cos n\alpha + i\sin n\alpha.$$

Proof. Homework 1.4. □

2.2.16 Applications of De Moivre's theorem

1. Homework 1.5.(2)
2. Trigonometric formulas:

   Example.
   $$\cos 2\beta = 2\cos^2\beta - 1, \qquad \beta \in \mathbb{R}.$$

   Proof.
   $$\begin{aligned}\cos 2\beta &= \operatorname{Re}\left(\cos 2\beta + i\sin 2\beta\right)\\ &= \operatorname{Re}\left(\left(\cos\beta + i\sin\beta\right)^2\right)\\ &= \operatorname{Re}\left(\cos^2\beta - \sin^2\beta + i2\cos\beta\sin\beta\right)\\ &= \cos^2\beta - \sin^2\beta\\ &= \cos^2\beta - \left(1 - \cos^2\beta\right)\\ &= 2\cos^2\beta - 1.\end{aligned}$$
   □

3. The study of the n^{th} roots of complex numbers

2.3 The n^{th} roots of complex numbers

Let $n \geq 2$, $n \in \mathbb{N}$. A complex number $z \in \mathbb{C} \setminus \{0\}$ is called an n^{th} root of a complex number $a \in \mathbb{C} \setminus \{0\}$, if

$$z^n = a.$$

Proposition 2.3.1. *Let*

$$a = |a|\left(\cos\theta + i\sin\theta\right) \neq 0.$$

If

$$n \in \mathbb{N},\ n \geq 2,$$

then a has the n distinct n^{th} roots

$$z_k = |a|^{\frac{1}{n}} \left(\cos\frac{\theta + k2\pi}{n} + i\sin\frac{\theta + k2\pi}{n} \right), \qquad k = 0, 1, \ldots, n-1.$$

Remarks.

1. Each of the n^{th} roots of a has modulus $|a|^{\frac{1}{n}}$. Hence, all the n^{th} roots of a lie on a circle of radius $|a|^{\frac{1}{n}}$ in the complex plane.
2. Since the argument of each successive n^{th} root exceeds the argument of the previous root by $\frac{2\pi}{n}$, the n^{th} roots are equally spaced on this circle.

Proof. Let

$$a = |a| \left(\cos\theta + i\sin\theta \right) \neq 0.$$

Our goal is to find all complex numbers

$$z = |z| \left(\cos\varphi + i\sin\varphi \right)$$

such that

$$z^n = a.$$

De Moivre's theorem implies that

$$|a| \left(\cos\theta + i\sin\theta \right) = |z|^n \left(\cos\varphi + i\sin\varphi \right)^n = |z|^n \left(\cos n\varphi + i\sin n\varphi \right).$$

Hence,

$$\begin{cases} |z|^n &= a \\ \cos n\varphi &= \cos\theta \\ \sin n\varphi &= \sin\theta \end{cases}$$

and we obtain

$$\begin{cases} |z| &= |a|^{\frac{1}{n}} \\ n\varphi &= \theta + k2\pi, \qquad k \in \mathbb{Z}. \end{cases}$$

that is

$$\begin{cases} |z| = |a|^{\frac{1}{n}} \\ \varphi = \frac{\theta + k2\pi}{n}, \qquad k \in \mathbb{Z}. \end{cases}$$

Since by choosing $k = 0, 1, \ldots, n-1$, we obtain different values for z, we have

$$z_k = |a|^{\frac{1}{n}} \left(\cos\frac{\theta + k2\pi}{n} + i\sin\frac{\theta + k2\pi}{n} \right), \qquad \text{where } k = 0, 1, \ldots, n-1.$$

On the other hand, the numbers z_k satisfy the equation $z_k^n = a$. □

Example 2.3.2. Find the fourth roots of $a = -1$. That is, solve the equation

$$z^4 = -1,$$
$$z^4 + 1 = 0.$$

Solution. Since

$$a = -1 = \cos\pi + i\sin\pi,$$
$$z_k = \cos\frac{\pi + k2\pi}{4} + i\sin\frac{\pi + k2\pi}{4}, \qquad \text{where } k = 0, 1, 2, 3.$$

That is, as shown in Figure 2.7,

$$z_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}, \qquad z_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{-1+i}{\sqrt{2}},$$
$$z_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{-1-i}{\sqrt{2}} \quad \text{and} \quad z_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1-i}{\sqrt{2}}.$$

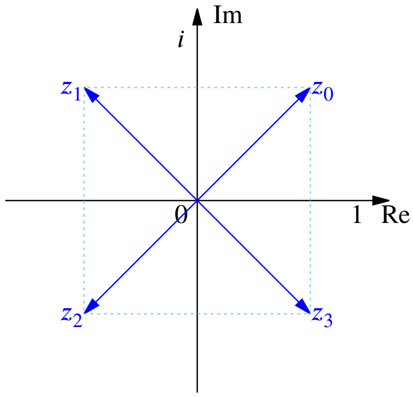


Figure 2.7: The four fourth roots of $a = -1$.

2.3.3 Special case: the n^{th} roots of unity

The solutions of the equation

$$z^n = 1, \qquad n \in \mathbb{N} \setminus \{0\},$$

are called the n^{th} roots of unity and are given by

$$z_k = \cos\frac{k2\pi}{n} + i\sin\frac{k2\pi}{n} = e^{i\frac{k2\pi}{n}}, \qquad k = 0, 1, \dots, n-1.$$

The roots represent the n vertices of a regular polygon of n sides inscribed in a circle of radius 1 with center at the origin.

Example 2.3.4. Find fourth roots of unity.

Solution.

$$z^4 = 1 \quad \Leftrightarrow \quad z^4 - 1 = 0.$$

We notice that

$$z^4 - 1 = \left(z^2 - 1\right)\left(z^2 + 1\right) = \left(z - 1\right)\left(z + 1\right)\left(z - i\right)\left(z + i\right).$$

So the equation becomes

$$\left(z - 1\right)\left(z + 1\right)\left(z - i\right)\left(z + i\right) = 0.$$

And the four fourth roots of unity, shown in Figure 2.8, are 1, i, -1 and $-i$.

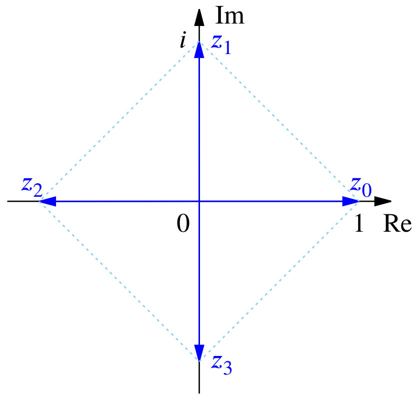


Figure 2.8: The four fourth roots of unity.

2.4 About history

We mention only some names from the history of complex numbers and complex analysis. As a further reference, we refer to MacTutor, a web page by the University of St Andrews:

`http://www-history.mcs.st-and.ac.uk/`

- Girolamo Gardano (1501-1576) Mentioned the square roots of negative numbers in his book 'Ars Magna', 1545, but it seems that he thought they were useless.
- Raphael Bombelli (1526-1572) Introduced the algebraic operations with complex numbers and used them to solve the 3rd degree equations, 'L'Algebra', 1572.
- Gottfried Wilhelm Leibnitz (1646-1716) Seemed to wonder if complex numbers are useful.
- Leonhard Euler (1707-1783) Started to use complex numbers actively. For example, Euler studied analytic functions of one complex variable.
- Caspar Wessel (1745-1818) A Norwegian-Danish engineer. The geometrical interpretation of complex numbers as vectors in the plane appeared in his writing from 1799.
- Jean Robert Argand (1768-1822) A French accountant, bookkeeper and an amateur mathematician. The geometric presentation of complex numbers as vectors in the plane is called Argand's diagram, although his publication is from 1806.
- Carl Friedrich Gauss (1777-1855) Started to study complex numbers systematically and to use their geometric properties.
- William Rowan Hamilton (1805-1865) Introduced the concept of $(\mathbb{R}^2, +, \cdot)$.

The complex function theory was developed in about 60 years by the following mathematicians. The properties of complex functions based on

- differentiation were developed by Bernhard Riemann (1826-1866),
- integration were developed by Augustin-Louis Cauchy (1789-1859),
- the power series were developed by Karl Theodor Wilhelm Weierstrass (1815-1897).

To summarise, complex numbers were discovered in the 1500's. However, the complex analysis was developed in about 60 years during the 1800's.

3 Topological notions of sets in the complex plane

We give some simple topological properties which are necessary in our study of functions. The norm

$$|\cdot| : \mathbb{C} \times \mathbb{C} \to [0, \infty),$$

defines a metric (or a distance function)

$$d : \mathbb{C} \times \mathbb{C} \to [0, \infty), \qquad d(z_1, z_2) = |z_1 - z_2|.$$

Namely, for all $z_1, z_2, z_3 \in \mathbb{C}$ the following statements are valid:

1. $|z_1 - z_2| \leq |z_1 - z_3| + |z_3 - z_2|$.
2. $|z_1 - z_2| = |z_2 - z_1|$.
3. $|z_1 - z_2| = 0$, if and only if $z_1 = z_2$.

The non-negative real number $|z_1 - z_2|$ is the distance between $z_1, z_2 \in \mathbb{C}$.

The space $(\mathbb{C}, |\cdot|)$ is a normed space and $(\mathbb{C}, d)$, where $d(z_1, z_2) = |z_1 - z_2|$ is defined by the norm $|\cdot|$, is a complete metric space.

3.1 Sets in the complex plane

Let $a \in \mathbb{C}$ and $r > 0$. The open disc, $\mathbb{D}(a, r)$, of radius r that is centered at a is defined by

$$\mathbb{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}.$$

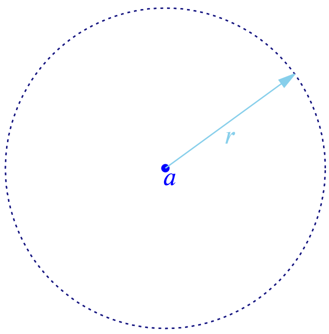
The closed disc, $\overline{\mathbb{D}}(a, r)$, of radius r that is centered at a is defined by

$$\overline{\mathbb{D}}(a, r) = \{z \in \mathbb{C} : |z - a| \leq r\}.$$

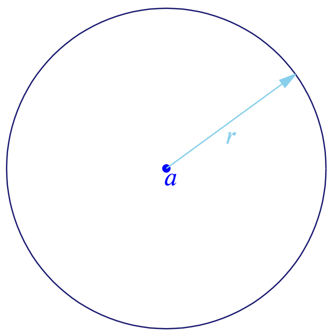
The boundary of both $\mathbb{D}(a, r)$ and $\overline{\mathbb{D}}(a, r)$ is the circle

$$C_r(a) = \{z \in \mathbb{C} : |z - a| = r\}.$$

The unit disc is the disc $\mathbb{D}(0, 1)$.



(a) The open disc $\mathbb{D}(a, r)$.



(b) The closed disc $\overline{\mathbb{D}}(a, r)$.

Figure 3.1: Open and closed disc.

Definition 3.1.1. Let S be a set in $\mathbb{C}$. A point z_0 is an interior point of S if there exists a real number $r > 0$, such that $\mathbb{D}(z_0, r) \subset S$. The interior of S, denoted by int S, consists of all of its interior points. That is,

$$\text{int } S = \{z \in \mathbb{C} : z \text{ is an interior point of } S\}.$$

Definition 3.1.2. A set S is open, if every point contained in S is an interior point of S. A set S is closed, if its complement, $\mathbb{C} \setminus S$, is open.

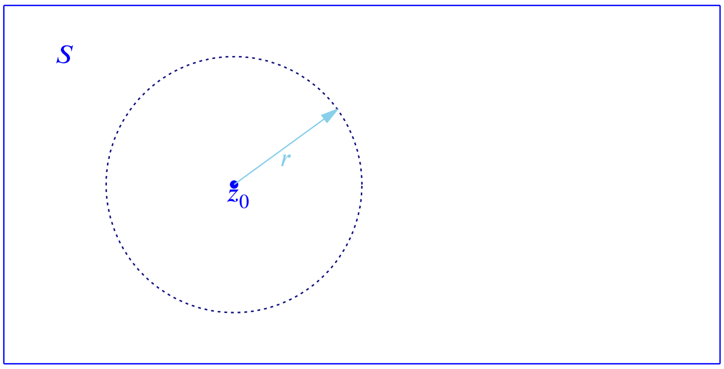


Figure 3.2: Let S be a set in $\mathbb{C}$. A point z_0 is an interior point of S if there exists $r > 0$ such that $\mathbb{D}(z_0, r) \subset S$.

Example 3.1.3 (Examples of open sets)**.**

1. $\mathbb{D}(z_0, r)$.
2. The punctured open disc, $\mathbb{D}(z_0, r) \setminus \{z_0\}$.
3. The punctured complex plane, $\mathbb{C} \setminus \{0\}$.
4. The upper half-plane, $\mathbb{H}^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

Example 3.1.4 (Examples of closed sets)**.**

1. $\overline{\mathbb{D}}(z_0, r)$.
2. The circle, $C_r(a)$.

Definitions 3.1.5.

- A point $z_0 \in \mathbb{C}$ is a <u>boundary point</u> of S if every disc $\mathbb{D}(z_0, r)$ contains points belonging to both S and its complement, $\mathbb{C} \setminus S$.
- The boundary of S is defined by

$$\partial S = \{z \in \mathbb{C} : z \text{ is a boundary point of } S\}.$$

- A point $z_0 \in \mathbb{C}$ is an <u>accumulation point</u> of S, if each punctured disc $\mathbb{D}(z_0, r) \setminus \{z_0\}$ contains at least one point of S.
- A point which is not an accumulation point is called an <u>isolated point</u> of S.
- The <u>closure</u> of S, denoted by $\overline{S}$, is the union of S and its accumulation points.
- The open set $U \subset \mathbb{C}$ is a <u>neighbourhood</u> of z_0, if $z_0 \in U$. For example, $\mathbb{D}(z_0, r)$ is a neighbourhood of z_0.

<u>*Remark*</u>. Open sets are useful for differentiation!

Definition 3.1.6. A set S is called <u>bounded</u> if there exist a real constant $M > 0$, such that $|z| < M$ whenever $z \in S$. In other words, the set S is contained in some large disc. If S is bounded, we define its diameter by

$$\operatorname{diam}(S) = \sup |z_1 - z_2|, \quad z_1, z_2 \in S.$$

Conversely, a set is called <u>unbounded</u>, if it's not bounded.

A set S is <u>compact</u>, if S is both closed and bounded.

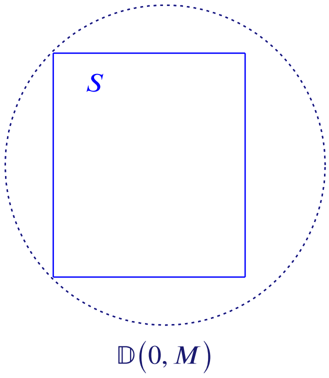


Figure 3.3: A compact set S in $\mathbb{C}$.

3.1.7 Connectedness

Recall that a continuous map,

$$\gamma : [a, b] \to S, \quad a < b,$$

is a path in S. We write

$$\begin{aligned} \gamma(t) &= (\gamma_1(t), \gamma_2(t)) \\ &= \gamma_1(t) + i\gamma_2(t) \\ &= \mathrm{Re}\big(\gamma(t)\big) + i\,\mathrm{Im}\big(\gamma(t)\big), \quad t \in [a, b]. \end{aligned}$$

The path γ joins the points $\gamma(a)$ and $\gamma(b)$.

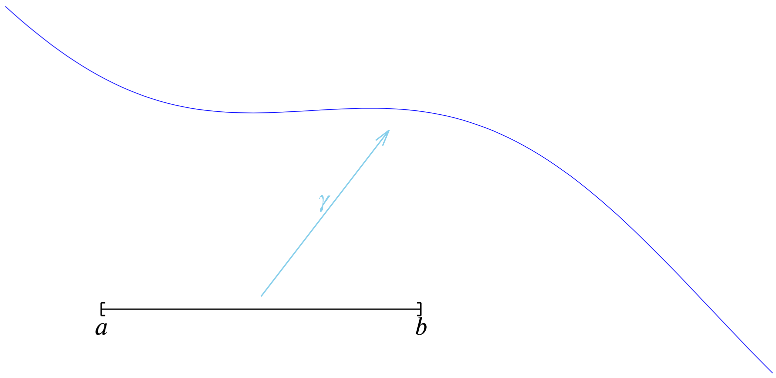
Definition. Let z_1, z_2 be given. The image of the path $\gamma(t) = z_1 + t(z_2 - z_1)$, where $t \in [0, 1]$, is a straight line segment:

$$\gamma([0, 1]) = [\gamma(0), \gamma(1)] = [z_1, z_2].$$

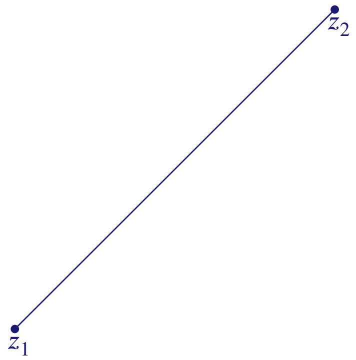
Definition. A finite number of line segments joined end-to-end forms a polygonal line or a polygonal route. So if we have points $z_0, \dots, z_n \in \mathbb{C}$, we call the union

$$[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n]$$

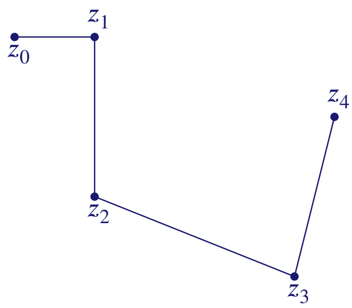
a polygonal route from z_0 to z_n.



(a) A continuous mapping $\gamma : [a, b] \to S$, where $a < b$, is a path in S.



(b) A line segment in $\mathbb{C}$.



(c) A polygonal route in $\mathbb{C}$.

Figure 3.4: A continuous mapping, a line segment and a polygonal route.

Definition. A non-empty subset S of $\mathbb{C}$ is polygonally connected if, given two points $z_1, z_2 \in S$, there is a polygonal route from z_1 to z_2 that lies entirely in S.

Definition. An **open** set S is connected if each pair of points $z_1, z_2 \in S$ can be joined by a polygonal line or route that lies entirely in S.

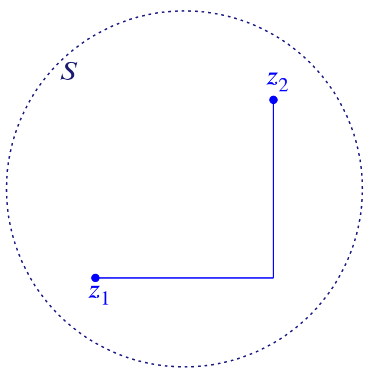


Figure 3.5: A polygonally connected open set S.

Remark. Connected sets are good for integration!

Definition (Domains)**.** Open sets which are connected are called domains.

3.2 Limits and continuity

3.2.1 Sequences

A complex sequence (z_n) is an assignment of a complex number z_n to each number $n \in \mathbb{N}$. We write $(z_n)_{n=1}^{\infty}$ or just (z_n).

Definitions.

- A sequence (z_n) is bounded if there exists a constant $M \in \mathbb{R}$, such that $|z_n| \leq M$ for all n.
- The sequence (z_n) converges to a limit $a \in \mathbb{C}$, written as $z_n \to a$ or $\lim_{n\to\infty} z_n = a$, if, given $\epsilon > 0$, there exists a number $N_\varepsilon \in \mathbb{N}$, such that $|z_n - a| < \varepsilon$ whenever $n \geq N_\varepsilon$.
- The sequence (ω_k) is a subsequence of the sequence (z_n), if there exist natural numbers $n_1 < n_2 < \dots$, such that
$$\omega_k = z_{n_k}, \quad \text{for } k = 1, 2, \dots .$$

Example. Let $a \in \mathbb{C}$ be given and $|a| \neq 1$. Let (z_n) be a sequence where $z_n = a^n$.

Now, if $|a| < 1$, then $|z_n| = |a|^n$, which converges to 0.

If $|a| > 1$, then $|a|^n \to \infty$ and hence the sequence (z_n) has no limit in $\mathbb{C}$.

3.2.2 Limits of functions

Let $f : S \to \mathbb{C}$ be a function on a set $S \subset \mathbb{C}$. Let a be an accumulation point of S. Then

$$\lim_{z\to a} f(z) = \omega$$

or

$$\lim_{\substack{z\to a \\ z\in S}} f(z) = \omega,$$

that is, $f(z) \to \omega$ as $z \to a$, if, given $\varepsilon > 0$, there exists $\delta = \delta_{a,\varepsilon} > 0$ such that

$$|f(z) - \omega| < \varepsilon,$$

whenever $z \in S$ and $0 < |z - a| < \delta$.

Remark. The limit, if it exists, is determined by the behaviour of $f(z)$ as z approaches a. The value of $f(a)$ is irrelevant and it may not even be defined if $a \notin S$.

Example. Let $f(z) = \dfrac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$, $z \neq 0$.

Now we have

$$\begin{aligned} f(z) &= 0, \text{ when } z \in \mathbb{R} \setminus \{0\} \\ f(z) &= 1, \text{ when } z \text{ is on the line } y = x. \end{aligned}$$

Hence, $\lim_{z\to 0} f(z)$ fails to exist.

3.2.3 Continuity

Let $f : S \to \mathbb{C}$ be a function. Then f is continuous at $a \in S$ if, given $\varepsilon > 0$, there exists $\delta = \delta_{a,\varepsilon} > 0$, such that

$$|f(z) - f(a)| < \varepsilon$$

whenever $a \in S$ and $|z - a| < \delta$. The function f is continuous on S if it is continuous at each $a \in S$.

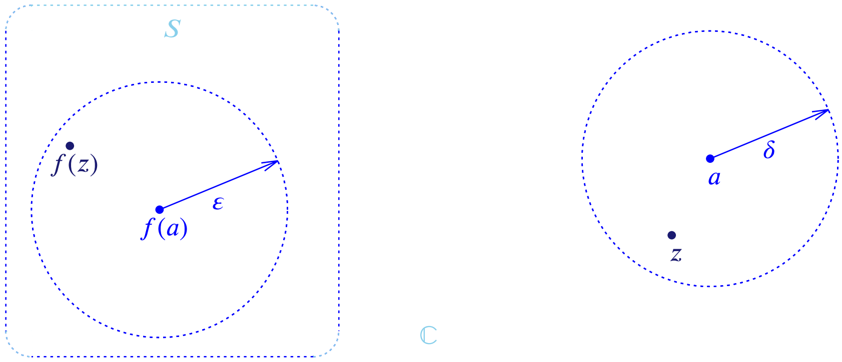


Figure 3.6: A function $f : S \to \mathbb{C}$ is continuous at $a \in S$ if, given $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(z) - f(a)| < \varepsilon$ whenever $a \in S$ and $|z - a| < \delta$.

Remarks.

- Sums and products of continuous functions are also continuous.
- Since the notions of convergence for complex numbers and points in $\mathbb{R}^2$ are the same, the function f of a complex variable $z = x + iy$ is continuous if and only if it is also continuous when viewed as a function of two real variables, x and y.
- If f is continuous, then the real-valued function

$$|f| : \mathbb{C} \to \mathbb{R}, \quad |f|(z) = |f(z)|$$

is continuous.

The following lemma provides a link between convergence in $\mathbb{C}$ and $\mathbb{R}$.

Lemma 3.2.4. *Let (z_n) be a complex sequence. Then (z_n) converges, if and only if the real-valued sequences $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$ both converge.*

If $z_n \to a$, then $|z_n| \to |a|$ and $\overline{z_n} \to \overline{a}$.

Example. Let $z_n = \left(1 + \frac{1}{m}\right) + i\left(1 - \frac{1}{m}\right)$, $m \in \mathbb{N}$.

Then $z_n \to 1 + i$.

Definition 3.2.5. Let $f : S \to \mathbb{C}$, $f = u + iv$, where $u, v : \mathbb{C} \to \mathbb{R}$. So $u = \mathrm{Re}\, f$ and $v = \mathrm{Im}\, f$.

Now we define the functions $u = \mathrm{Re}\, f : \mathbb{C} \to \mathbb{R}$ and $v = \mathrm{Im}\, f : \mathbb{C} \to \mathbb{R}$ by setting

$$u(z) = (\mathrm{Re}\, f)(z) = \mathrm{Re}(f(z)),$$
$$v(z) = (\mathrm{Im}\, f)(z) = \mathrm{Im}(f(z)).$$

Then, $f(z) \to \omega$ implies

$$\mathrm{Re}\, f(z) \to \mathrm{Re}\, \omega$$
$$\mathrm{Im}\, f(z) \to \mathrm{Im}\, \omega.$$

We also have $|f(z)| \to |\omega|$ and $\overline{f(z)} \to \overline{\omega}$.

Proposition 3.2.6. *Let $f : S \to \mathbb{C}$. Then f is continuous at $a \in S$ (or on S), if and only if both $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are continuous at $a \in S$ (or on S).*

3.3 The Cauchy convergence principle for $\mathbb{C}$

Definition. A sequence (z_n) is called a Cauchy sequence, or simply Cauchy, if

$$|z_n - z_m| \to 0, \qquad \text{as } n, m \to \infty.$$

Recall that $\mathbb{R}$ is complete: every Cauchy sequence of real numbers converges to a real number.

Since the complex sequence (z_n) is Cauchy if and only if $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$ are, then we can conclude that every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$.

Theorem. $\mathbb{C}$ *is complete.*

4 Analytic functions

4.1 Definitions

Let Ω be an open set in $\mathbb{C}$ and f a complex-valued function on Ω (we assume that f is a complex-valued function of one complex variable and $\Omega \neq \varnothing$).

Definition. The function f is complex differentiable at the point $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0+h)-f(z_0)}{h} \tag{4.1.1}$$

converges to a limit when $h \to 0$, where $h \in \mathbb{C} \setminus \{0\}$ and $z_0 + h \in \Omega$. Recall $\mathbb{C} = (\mathbb{R}^2, +, \cdot)$ is a field, and hence the quotient is well defined.

The limit of 4.1.1, when it exists, is denoted by $f'(z_0)$ and is called the complex derivative of f at z_0:

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}. \tag{4.1.2}$$

Note that $h \in \mathbb{C} \setminus \{0\}$ must approach 0 from any direction.

Remark. Complex differentiability which is defined only at one point or two points is not enough to build an interesting, meaningful theory, so we need complex differentiability to be defined also on a small disc $\mathbb{D}(z_0, \varepsilon)$, $\varepsilon > 0$.

Definition 4.1.3. The function f is said to be analytic at z_0 if there exists a disc $\mathbb{D}(z_0, r)$ such that f is complex differentiable at every point in the disc.

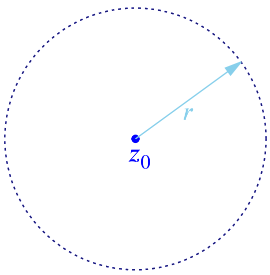


Figure 4.1: Open disc $\mathbb{D}(z_0, r)$.

Remark. In order for f to be analytic at z_0, f should have a complex derivative at every point in a small neighbourhood of z_0.

Definition 4.1.4. Let Ω be an open set in $\mathbb{C}$ and f a complex-valued function on Ω. The function f is analytic on Ω if it is analytic at every point of Ω.

Remark. Thus, analyticity is a stronger condition than differentiability. Now, we give some examples.

Examples 4.1.5.

1. Any constant function is analytic on the whole complex plane. Let $f(z) = c \in \mathbb{C}$ for all $z \in \mathbb{C}$. Then $f'(z) = 0$:
$$\frac{f(z+h)-f(z)}{h} = \frac{c-c}{h} = 0.$$

2. The function $f(z) = z$ is analytic on any open set Ω in $\mathbb{C}$ and $f'(z) = 1$:
$$\frac{f(z+h)-f(z)}{h} = \frac{z+h-z}{h} = 1.$$

3. The function $f(z) = z^2$ is analytic on any open set Ω in $\mathbb{C}$ and $f'(z) = 2z$:

$$\frac{f(z+h) - f(z)}{h} = \frac{(z+h)^2 - z^2}{h} = \frac{z^2 + 2zh + h^2 - z^2}{h} = 2z + h \overset{h\to 0}{\to} 2z.$$

Example 4.1.6. The function $f(z) = \overline{z}$ is not analytic. We have:

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h} = \begin{cases} 1 & \text{if } h \in \mathbb{R} \setminus \{0\} \\ -1 & \text{if } \mathrm{Re}(h) = 0, h \neq 0. \end{cases}$$

Hence there is no limit as $h \to 0$.

Note that, in terms of real variables, the function $f(z) = \overline{z}$ corresponds to the map $g : (x, y) \mapsto (x, -y)$, which is differentiable in the real sense. So, the existence of the real derivative need not guarantee that f is analytic.

Remark. The adjectives "regular" and "holomorphic" are sometimes used instead of "analytic".

Definition 4.1.7. If a function is analytic on the entire complex plane, it is said to be *entire*.

> Entire functions are the very best class of complex functions. (T. Tao)

Remark. Note that, if D is a domain, then f is analytic if and only if f is complex differentiable at every point in D.

4.2 Elementary properties

Lemma 4.2.1. *A function f is complex differentiable at $z_0 \in \Omega$ if and only if there exists a complex number a such that*

$$f(z_0 + h) - f(z_0) - ah = h\varphi(h), \tag{4.2.2}$$

where φ is a function defined for all h, when $|h|$ is very small and $\lim_{h\to 0} \varphi(h) = 0$.

Here, $a = f'(z_0)$.

Remark. The function φ depends on f and z_0, that is, $\varphi = \varphi_{f,z_0}$.

Proof.

"⇒": Let us define $\varphi(0) = 0$, and $\varphi(h) = \frac{f(z_0+h)-f(z_0)}{h} - f'(z_0)$, $h \neq 0$.

"⇐": From (4.2.2), $\frac{f(z_0+h)-f(z_0)}{h} = a + \varphi(h) \to a$ as $h \to 0$.

□

Corollary 4.2.3. *Let Ω be an open set in $\mathbb{C}$. Let f be a complex-valued function on Ω. If f is differentiable at $z_0 \in \Omega$, then f is continuous at the point $z_0 \in \Omega$.*

Proof. From (4.2.2), $\lim_{h\to 0} f(z_0 + h) - f(z_0) = 0$, and this is enough since $z_0 \in \Omega$ is an accumulation point of the open set Ω. □

Corollary 4.2.4. *An analytic function is continuous.*

Remark. There are continuous functions which are not analytic.

Example. $f(z) = \overline{z}$ is continuous on the whole plane but it is not analytic.

4.2.5 (Differentiation formulas)**.** Let Ω be an open set in $\mathbb{C}$. Let f and g be complex-valued functions on Ω. If the complex derivatives of f and g exist at a point $z_0 \in \Omega$, then

1. the function $f + g : \Omega \to \mathbb{C}$ is complex differentiable at z_0 and
$$(f+g)'(z_0) = f'(z_0) + g'(z_0),$$
2. the function $f \cdot g : \Omega \to \mathbb{C}$ is complex differentiable at z_0 and
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$
3. when $g(z_0) \neq 0$, then the function $\frac{f}{g}$ is well defined in some disc $\mathbb{D}(z_0, r), r > 0$ and
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof. The proof is left as homework to the reader. Use lemma 4.2.1 or the definition. □

Corollary 4.2.6. *Any polynomial*
$$p(z) = a_0 + a_1 z + \ldots + a_n z^n,$$
where $n \in \mathbb{N} = \{1, 2, \ldots\}$, $a_k \in \mathbb{C}$, $k = 0, 1, \ldots, n$ *and* $a_n \neq 0$, *is analytic on the entire plane, and*
$$p'(z) = a_1 + \ldots + n a_n z^{n-1}.$$

Corollary 4.2.7. *A rational function* $\frac{p(z)}{q(z)}$, *where* $p(z)$ *and* $q(z)$ *are polynomials, is analytic on any open set in which* $q(z)$ *is never zero.*

Examples 4.2.8.

1. The function $f(z) = \frac{1}{z}$ is analytic on any open set in $\mathbb{C}$ that does not contain the origin and $f'(z) = -\frac{1}{z^2}$.
2. The rational function $f(z) = \frac{1}{z^2+1}$ is analytic on any open set in $\mathbb{C}$ that does not contain $\pm i$.

Example 4.2.9. When is the function $g(z) = |z|^2$ analytic? Since $f(z) = \overline{z}$ is not analytic in any open set and $\overline{z} = \frac{|z^2|}{z}$ when $z \neq 0$, the function g is not analytic anywhere.

Remark. The function g is complex differentiable at the origin.
$$\frac{g(0+h) - g(0)}{h} = \frac{|h|^2}{h} = \overline{h} \overset{h\to 0}{\to} 0.$$

Remark. Note that the corresponding map $k : (x, y) \mapsto x^2 + y^2$ is differentiable in the real sense on the whole $\mathbb{R}^2$.

Theorem 4.2.10: Chain rule. *Let* Ω *and* U *be open sets in* $\mathbb{C}$. *Let* f *be analytic on* Ω *and let* g *be analytic on* U *and* $f(\Omega) \subseteq U$. *The composite function* $g \circ f$, *given by* $(g \circ f)(z) = g(f(z))$, *is analytic on* Ω *and for all* $z \in \Omega$,
$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

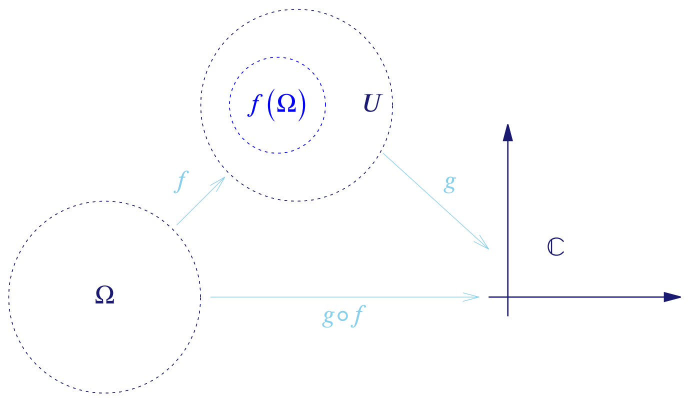


Figure 4.2: Chain rule (Theorem 4.2.10): the composite function $(g\circ f)(z) = g(f(z))$ is analytic on Ω if functions g and f are analytic on their respective domains.

Theorem 4.2.11: Derivative of inverse function. *Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be differentiable at $z \in \Omega$, with $f'(z) \neq 0$. If there is a neighbourhood U of the point $w := f(z)$ such that f has a continuous inverse function f^{-1} on U, then f^{-1} is differentiable at w and*

$$\left(f^{-1}\right)'(w) = \frac{1}{f'(z)} = \frac{1}{f'\left(f^{-1}(w)\right)}.$$

Proof. Since f has a derivative at z, we have

$$f(z+h) - f(z) = f'(z)h + h\varepsilon(h),$$

where $\varepsilon(h) \to 0$ when $h \to 0$.

Let $|k|$ be so small that $w + k \in U$. Then for every such $k \in \mathbb{C}$ there exists $h \in \mathbb{C}$ with $f^{-1}(w+k) = z + h$. Since f^{-1} is continuous in U,

$$\lim_{k\to 0}(z+h) = \lim_{k\to 0} f^{-1}(w+k) = f^{-1}(w) = z,$$

so $h \to 0$ when $k \to 0$. Hence

$$\begin{aligned}\frac{f^{-1}(w+k) - f^{-1}(w)}{k} &= \frac{z+h-z}{w+k-w} = \frac{h}{f(z+h) - f(z)} \\ &= \frac{h}{f'(z)h + h\varepsilon(h)} = \frac{1}{f'(z) + \varepsilon(h)} \\ &\to \frac{1}{f'(z)},\end{aligned}$$

when $k \to 0$. □

4.3 The relationship between the complex and real derivatives

The notion of complex differentiability differs from the notion of real differentiability of a function of two real variables. Let us associate to a complex valued function $f = u + iv$, of one complex variable, the mapping

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \quad g(x, y) = \left(u(x, y), v(x, y)\right).$$

Recall from the vector calculus course that the function g is differentiable at a point (x_0, y_0), if there exists an $\mathbb{R}$-linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$g\left((x_0, y_0) + h\right) - g(x_0, y_0) = Lh + \|h\|\,\varepsilon(h),$$

with $\|\varepsilon(h)\| \to 0$ as $\|h\| \to 0$.

Equivalently, we can write

$$\frac{\left\|g\left((x_0, y_0)+h\right)-g(x_0, y_0)-Lh\right\|}{\|h\|} \to 0,$$

as $\|h\| \to 0$, where $h = (h_1, h_2) \in \mathbb{R}^2 \setminus \{0\}$.

<u>*Remark*</u>. For a complex number $z = x+iy$, we write $f(z) = u(x, y)+iv(x, y)$, where $u\colon \mathbb{R}^2 \to \mathbb{R}$ and $v\colon \mathbb{R}^2 \to \mathbb{R}$.

The $\mathbb{R}$-linear transformation L is unique and is called the derivative of g at (x_0, y_0). If g is differentiable in the real sense, the partial derivatives of u and v exist, and the $\mathbb{R}$-linear transformation is described in the standard basis $\{e_1, e_2\} = \{(1, 0),\ (0, 1)\}$ of $\mathbb{R}^2$ by the Jacobian matrix of g:

$$\mathcal{J}_g(x, y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \backsim L.$$

The derivative in the real case is a matrix, while the complex derivative is a complex number $f'(z_0)$.

Our goal is to find the relationship between these two notions.

4.3.1. Let us assume that f has a complex derivative at the point z_0. We consider the limit in (4.1.2), where $h \in \mathbb{R}$, i.e. $h = h_1 + ih_2$ with $h_2 = 0$. Let us write $z = x + iy,\ z_0 = x_0 + iy_0$, and $f(z) = f(x, y)$. Then,

$$\begin{aligned} f'\left(z_0\right) &\overset{4.1.2}{=} \lim_{h\to 0} \frac{f\Big(\left(x_0, y_0\right)+h\Big)-f\left(x_0, y_0\right)}{h} \\ &= \lim_{h_1\to 0} \frac{f\left(x_0+h_1, y_0\right)-f\left(x_0, y_0\right)}{h_1} \\ &= \tfrac{\partial f}{\partial x}\left(x_0, y_0\right), \end{aligned}$$

where $\frac{\partial}{\partial x}$ denotes the usual partial derivative in the first variable by the definition of the partial derivatives.

Next we consider the limit (4.1.2) when $h = ih_2$, so $h_1 = 0$. We obtain

$$\begin{aligned} f'\left(z_0\right) &= \lim_{h\to 0} \frac{f\Big(\left(x_0, y_0\right)+h\Big)-f\left(x_0, y_0\right)}{h} \\ &= \lim_{h_2\to 0} \frac{f\left(x_0, y_0+h_2\right)-f\left(x_0, y_0\right)}{ih_2} \\ &= \frac{1}{i}\tfrac{\partial f}{\partial y}\left(x_0, y_0\right), \end{aligned}$$

where $\frac{\partial}{\partial y}$ is the partial derivative in the 2^{nd} variable. Recall that $i = (0, 1)$ and so $ih_2 = (0, h_2)$.

4.3.2. Hence, if f is analytic, we have shown that

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Writing $f = u + iv$, we find, after separating the real and imaginary parts and writing $\frac{1}{i} = -i$, that the partial derivatives of u and v exist. Moreover, they satisfy the equations

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(u+iv) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \\ \frac{1}{i}\frac{\partial f}{\partial y} = -i\frac{\partial}{\partial y}(u+iv) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, \end{cases}$$

which gives us the following identities:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{4.3.3}$$

These are called the Cauchy–Riemann (C–R) equations. The Cauchy–Riemann equations link together real and complex analysis.

4.3.4. Let us define two differential operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Theorem 4.3.5. *If f is analytic at z_0, then*

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0, \tag{4.3.6}$$

and

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0). \tag{4.3.7}$$

If we write $g(x, y) = f(z)$, then g is differentiable in the sense of real variables and

$$\det \mathcal{J}_g(x_0, y_0) = \left|f'(z_0)\right|^2. \tag{4.3.8}$$

Proof. For (4.3.6): Taking the real and imaginary parts, we see that the Cauchy–Riemann equations are equivalent to

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0.$$

The C–R equations imply:

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} + i^2\frac{\partial v}{\partial y}\right)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) \overset{\text{(C–R)}}{=} 0.\end{aligned}$$

Conversely, if

$$0 = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right),$$

then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

For (4.3.7):

$$\begin{aligned}f'(z_0) &= \frac{1}{2}f'(z_0) + \frac{1}{2}f'(z_0)\\ &\overset{(4.3.1)}{=} \frac{1}{2}\frac{\partial f}{\partial x}(z_0) + \frac{1}{2}\left(\frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right)\\ &\overset{(4.3.4)}{=} \frac{\partial f}{\partial z}(z_0).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\frac{\partial f}{\partial z} &\overset{(4.3.4)}{=} \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) \\ &\overset{\text{(C-R)}}{=} \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} - i\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right) \\ &= 2\frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) = 2\frac{\partial u}{\partial z}.\end{aligned}$$

□

We defined two differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let's recall theorem 4.3.5:

If f is analytic at z_0, then

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0 \text{ and } f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Also, if we write

$$g(x, y) = f(z),$$

then g is differentiable in the sense of two real variables, and

$$\det \mathcal{J}_g(\underbrace{x_0, y_0}_{=z_0}) = \left|f'(z_0)\right|^2.$$

Let us write $z_0 = (x_0, y_0)$. Now, our goal is to show that g is differentiable in the real sense. We have to show that there exists an $\mathbb{R}$-linear transformation

$$L_g : \mathbb{R} \to \mathbb{R}$$

such that

$$g((x_0, y_0) + H) - g(x_0, y_0) = L_g H + \|H\|\,\varepsilon(H), \tag{4.3.9}$$

where $\|\varepsilon(H)\| \to 0$ as $\|H\| \to 0$, $H = (h_1, h_2) \in \mathbb{R}^2$ and $h_1 + ih_2 \in \mathbb{C}$.

Recall from the vector calculus course that the $\mathbb{R}$-linear transformation L_g is unique and is called the derivative of g at (x_0, y_0).

Since f is complex differentiable at z_0 and the C–R equations are satisfied, we know that

$$f'(z_0) = \frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

By the definition of complex differentiability,

$$\begin{aligned}f(z_0 + h) - f(z_0) &= f'(z_0)\cdot h + h\varepsilon(h) \\ &= \left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)h + h\varepsilon(h) \\ &= \left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + h\varepsilon(h),\end{aligned}$$

where $\lim_{h\to 0}\varepsilon(h) = 0$.

Now we identify a complex number with the pair of its real and imaginary parts. Hence, by equation 4.3.9 and calculations from the start of this section,

$$g((x_0, y_0) + H) - g(x_0, y_0) = g((x_0, y_0) + (h_1, h_2)) - g(x_0, y_0)$$

$$\begin{aligned}
&= f(z_0+h) - f(z_0) \\
&= \left(\tfrac{\partial u}{\partial x} - i\tfrac{\partial u}{\partial y}\right)(h_1 + ih_2) + h\varepsilon(h) \\
&= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + h\varepsilon(h) \\
&= L_g h + h\varepsilon(h) = L_g H + \|H\|\,\varepsilon(H),
\end{aligned}$$

where $\varepsilon(h) = (\varepsilon_1(h), \varepsilon(h_2))$ and $H \neq (0,0)$.

Note that

$$h \cdot \varepsilon(h) = \|H\| \underbrace{\frac{(h_1\varepsilon_1(h) - h_2\varepsilon_2(h), h_2\varepsilon_2(h) + h_2\varepsilon_1(h))}{\|H\|}}_{=:\varepsilon(H)},$$

where $\left\|\varepsilon(H)\right\| \to 0$ as $\|H\| \to 0$.

So g is differentiable in the sense of two real variables.

We obtain

$$\begin{aligned}
\det \mathcal{J}_g(x_0, y_0) &= \tfrac{\partial u}{\partial x}\tfrac{\partial v}{\partial y} - \tfrac{\partial v}{\partial x}\tfrac{\partial u}{\partial y} \\
&= \left(\tfrac{\partial u}{\partial x}\right)^2 + \left(\tfrac{\partial u}{\partial y}\right)^2 \\
&= \left|\tfrac{\partial u}{\partial x} - i\tfrac{\partial u}{\partial y}\right|^2 = \left|2\tfrac{\partial u}{\partial z}\right|^2 = \left|f'(z_0)\right|^2.
\end{aligned}$$

The next theorem gives an important converse:

Theorem 4.3.10. *Suppose that $f = u + iv$ is a complex-valued function defined on an open set $\Omega \subset \mathbb{C}$. If u and v are differentiable in the real sense at the point $z_0 \in \Omega$, and their partial derivatives satisfy the C–R equations at point z_0, then f has a complex derivative at z_0 and*

$$f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0).$$

Proof. Let $z_0 = (x_0, y_0) \in \Omega$ and $h = h_1 + ih_2 = (h_1, h_2)$. Since u and v are differentiable at z_0, we have

$$\begin{aligned}
u\big(z_0+h\big) - u\big(z_0\big) &= \mathrm{D}u\big(z_0\big)h + \|h\|\,\varepsilon_1(h) \\
&= (\nabla u\big(z_0\big)|h) + \|h\|\,\varepsilon_1(h) \\
&= \tfrac{\partial u}{\partial x}(z_0)h_1 + \tfrac{\partial u}{\partial y}(z_0)h_2 + \|h\|\,\varepsilon_1(h)
\end{aligned}$$

and

$$\begin{aligned}
v(z_0+h) - v(z_0) &= \mathrm{D}v\big(z_0\big)h + \|h\|\,\varepsilon_2(h) \\
&= (\nabla v\big(z_0\big)|h) + \|h\|\,\varepsilon_2(h) \\
&= \tfrac{\partial v}{\partial x}(z_0)h_1 + \tfrac{\partial v}{\partial y}(z_0)h_2 + \|h\|\,\varepsilon_2(h),
\end{aligned}$$

where $\varepsilon_k(h) \to 0$ as $h \to 0$, $k = 1, 2$.

Let us write $\varepsilon(h) = \varepsilon_1(h) + i\varepsilon_2(h)$. We obtain using the complex functions

$$\begin{aligned}
f(z_0+h) - f(z_0) &= (u(z_0+h) - u(z_0)) + i(v(z_0+h) - v(z_0)) \\
&= \left(\tfrac{\partial u}{\partial x}(z_0)h_1 + \tfrac{\partial u}{\partial y}(z_0)h_2\right) + i\left(\tfrac{\partial v}{\partial x}(z_0)h_1 + \tfrac{\partial v}{\partial y}(z_0)h_2\right) + \|h\|\,\varepsilon(h).
\end{aligned}$$

By the C–R equations

$$f(z_0+h) - f(z_0) = \left(\tfrac{\partial u}{\partial x}(z_0)h_1 - \tfrac{\partial v}{\partial x}(z_0)h_2\right) + i\left(\tfrac{\partial v}{\partial x}(z_0)h_1 + \tfrac{\partial u}{\partial x}(z_0)h_2\right) + \|h\|\,\varepsilon(h)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h_1 + ih_2) + \|h\|\,\varepsilon(h)$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \cdot h + h\frac{\|h\|\,\varepsilon(h)}{h},$$

where $\varepsilon_0 = \dfrac{\|h\|\,\varepsilon(h)}{h}$, if $h \neq 0, \varepsilon_0(0) = 0$ and $\varepsilon_0(h) \to 0$ as $h \to 0$.

Hence f is complex differentiable at z_0 and

$$f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0).$$

□

Corollary 4.3.11. *The function $f = u + iv$ is complex differentiable at z_0 if and only if u and v are differentiable in the real sense at z_0 and their partial derivatives satisfy the Cauchy–Riemann equations at z_0.*

Proof. One implication is Theorem 4.3.10. The other implication follows from the proof of Theorem 4.3.5. □

Remark 4.3.12. Recall the following sufficient condition for $u\colon \mathbb{R}^2 \to \mathbb{R}$ to be differentiable in the real sense:

Let $\Omega \subset \mathbb{R}^2$ be an open set and $u\colon \Omega \to \mathbb{R}$. If the partial derivatives of u exist at every point in Ω and are continuous at every point in Ω, then u is differentiable on Ω.

There is a weaker sufficient condition: If the partial derivatives of u exist in a small ball $\mathbb{B}(z, r)$ with some $r > 0$ and are continuous at z, then u is differentiable at z.

In real life the following corollary is very important:

Corollary 4.3.13. *Let Ω be an open set in $\mathbb{C}$ and let $f = u + iv$ be a complex-valued function on Ω. If u and v have continuous partial derivatives on Ω and satisfy the C–R equations on Ω, then f is analytic on Ω.*

Examples 4.3.14.

1. Let $f(z) = 2xy + i(x^2 + y^2)$. Now, let $u\colon \mathbb{R}^2 \to \mathbb{R}$ and $v\colon \mathbb{R}^2 \to \mathbb{R}$ be such that $u(x, y) = 2xy$ and $v(x, y) = x^2 + y^2$. The partial derivatives of u and v are

$$u_x = \frac{\partial u}{\partial x} = 2y, \quad u_y = \frac{\partial u}{\partial y} = 2x \quad \text{and}$$
$$v_x = \frac{\partial v}{\partial x} = 2x, \quad v_y = \frac{\partial v}{\partial y} = 2y.$$

   Now u_x, u_y, v_x, v_y exists and are continuous, thus we know that f is differentiable in the real sense of two variables. C–R equations are

$$\begin{cases} u_x = & v_y \quad \text{and} \\ u_y = & -v_x. \end{cases}$$

   The only points where the C–R equations are satisfied are the points where $x = 0$, i.e., the points on the imaginary axis. Hence, f is complex differentiable at these points and

$$f'(iy) = u_x(iy) + iv_x(iy) = 2y.$$

   Therefore f fails to be analytic (there does not exist an open disk where the function has complex derivatives, see Figure 4.3).

2. Let $f(z) = (\overline{z})^2$. Is f analytic?

   Now, let $u\colon \mathbb{R}^2 \to \mathbb{R}, u(x, y) = \operatorname{Re} f(x, y) = x^3 - 3xy^2$ and $v\colon \mathbb{R}^2 \to \mathbb{R}, v(x, y) = \operatorname{Im} f(x, y) = y^3 - 3x^2y$. Homework.

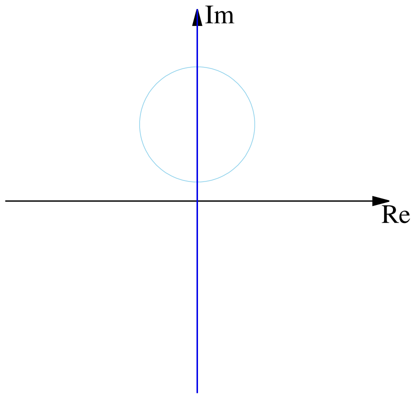


Figure 4.3: Function f in example 4.3.14.1 is complex differentiable at the imaginary axis, but it is not analytic anywhere as there does not exist an open disk where the function has complex derivatives.

Remark 4.3.15. Let $f(x+iy) = \sqrt{|x|\,|y|}$, where $x, y \in \mathbb{R}$. Show that f satisfies the C–R equations at the origin, but f is not complex differentiable at the origin.

Remark 4.3.16. Let $f(x+iy) = (x^2+y^2) + i2yx$. Now the partial derivatives

$$\begin{cases} \frac{\partial f}{\partial x} & = 2x + i2y \quad \text{and} \\ \frac{\partial f}{\partial y} & = -2y + i2x \end{cases}$$

satisfy the C–R equations. Do as in example 4.3.14.

Remark 4.3.17. We defined the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The notation for this is clear/reasonable if we write

$$x = \frac{1}{2}(z+\overline{z}) \text{ and } y = \frac{1}{2i}(z-\overline{z}),$$

working formally using the chain rule, we have the following equations:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right).$$

4.3.18 On harmonic functions

Let $f = u + iv$. The (C–R) equations

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

have an interesting consequence:
If we only know the real part of an analytic function, we can deduce the imaginary part v (up to a constant), by integrating the C–R equations.

Theorem 4.3.19. *If $f_1 = u + iv_1$ and $f_2 = u + iv_2$ are two analytic functions which are defined on the same domain and which have the same real part u, then we know that $v_1 = v_2 + C$ for some constant C.*

We need following Lemma 4.3.20 from the vector calculus course:

Lemma 4.3.20. *If k is a function on a domain D such that*

$$\frac{\partial k}{\partial x} = \frac{\partial k}{\partial y} = 0 \quad \text{on } D,$$

then k is a constant.

Proof. Since $\frac{\partial k}{\partial x} = 0$, k is constant on every horizontal line segment in D by the fundamental theorem of calculus. As also $\frac{\partial k}{\partial y} = 0$ similarly k is constant. Since every two points in D can be connected be vertical and horizontal line segments, k is constant. □

Proof. (of the theorem) From the C–R equations we have that

$$\frac{\partial u}{\partial x} = \frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v_1}{\partial x} = -\frac{\partial v_2}{\partial x},$$

thus

$$\frac{\partial(v_1 - v_2)}{\partial y} = 0 = \frac{\partial(v_1 - v_2)}{\partial x}.$$

Then by the above Lemma 4.3.20 $v_1 - v_2 = C$. □

Definition 4.3.21. If $u + iv$ is an analytic function, then v is called a harmonic conjugate of u, and vice versa.

The previous Theorem 4.3.19 tells us that up to a constant, every function u has only one harmonic conjugate. Note that not every function has a harmonic conjugate.

Let us take the C–R equations:

$$(1) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad (2) \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

and differentiate (1) with respect to x and (2) with respect to y.

$$(3) \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad (4) \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Now we assume that

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}.$$

It is enough to assume that $v : \mathbb{R}^2 \to \mathbb{R}$ is C^2 in the real sense. Adding (3) and (4) together we obtain

$$(5) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which is known the Laplace equation $\Delta^2 u = 0$. Functions which satisfy this rule are called harmonic.

Remark 4.3.22. We have just shown that in order for u to be the real part of an analytic function, then u must be harmonic.

Recall the Laplace operator:

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{4.3.23}$$

Let Ω be an open set and $f : \Omega \to \mathbb{R}$ be a C^2-function (a function which is twice continuously differentiable in the real sense, we write $f \in C^2(\Omega)$).

If $\Delta f(x, y) = 0$ for all $(x, y) \in \Omega$, then f is called harmonic. Using the C–R equations we obtain the following property for analytic functions:

Proposition 4.3.24. *The real part and the imaginary part of an analytic function are harmonic.*

<u>Warning:</u> We will use the fact that an analytic function is twice continuously differentiable (in fact, it is $f \in C^\infty(\Omega)$), but we will prove this property later.

Let Ω be an open set. Let $f : \Omega \to \mathbb{C}$ be analytic, and the real part of f, $\operatorname{Re} f : \Omega \to \mathbb{C}$, $(\operatorname{Re} f)(z) = \operatorname{Re} f(z)$. By the C–R equations, we have:

$$\begin{aligned}\Delta \operatorname{Re} f &= \tfrac{\partial^2 \operatorname{Re} f}{\partial x^2} + \tfrac{\partial^2 \operatorname{Re} f}{\partial y^2} \\ &= \tfrac{\partial}{\partial x}\tfrac{\partial}{\partial x} \operatorname{Re} f + \tfrac{\partial}{\partial y}\tfrac{\partial}{\partial y} \operatorname{Re} f \\ &= \tfrac{\partial}{\partial x}\tfrac{\partial}{\partial y} \operatorname{Im} f - \tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x} \operatorname{Im} f = 0.\end{aligned}$$

Hence, $\Delta \operatorname{Re} f = 0$ and $\operatorname{Re} f$ is harmonic. In the same way, $\Delta \operatorname{Im} f = 0$, and $\operatorname{Im} f$ is harmonic.

There exists the following theorem, which we do not prove:

Theorem. *If u is harmonic on $\mathbb{D}(z_0, r)$ with $r > 0$, then there exists an analytic map f on $\mathbb{D}(z_0, r)$ such that $u = \operatorname{Re} f$.*

Example. $u(x, y) = ax + by + c$, where $a, b \in \mathbb{R}$, is harmonic since $\Delta u(x, y) = 0$.

Theorem 4.3.25. *Let f be analytic on a disc $\mathbb{D}$. If $\operatorname{Re} f$ is constant in $\mathbb{D}$, then f is constant in $\mathbb{D}$.*

Proof. Let $f = u + iv = \operatorname{Re} f + i \operatorname{Im} f$. If $u = \operatorname{Re} f$ is constant, then $u_x = u_y = 0$. Since f is analytic, the C–R equations imply that $v_x = v_y = 0$.

From the vector calculus course we know that this means that f is constant. □

Theorem 4.3.26. *If f is analytic on a disc $\mathbb{D}$ and $\operatorname{Im} f$ or $|f|$ or $\arg f$ is constant, then f is constant.*

<u>*Remark 4.3.27*</u>. Different ways to show that f is not complex differentiable at z_0:

1. If f is discontinuous at z_0, then f is not complex differentiable at z_0.
2. If the limit for $\frac{f(z)-f(z_0)}{z-z_0}$ fails to exist when $z \to z_0$, then f is not complex differentiable at z_0. For an example, see Homework 3.1.
3. if the C–R equations are not satisfied at $(x_0, y_0) = z_0$, then f is not complex differentiable at z_0. Here $f = u + iv$.
4. If f is not differentiable in the sense of two real variables at (x_0, y_0), then f is not complex differentiable at $z_0 = (x_0, y_0)$.
5. Use the differentiation formulas for sum, multiplication and quotient. For an example, see Homework 3.3.

<u>Important!</u> Let $f = u + iv : \Omega \to \Omega, \Omega \subset \mathbb{C}$ open. In this chapter we proved:

Theorem 4.3.28. *If f is complex differentiable at $z_0 \in \Omega$, then the partial derivatives of u and v exist and they obey the R–C equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad f'(x) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0).$$

Example 4.3.29. Function $z \mapsto z^2$ is analytic in the whole complex plane, as

$$(x + iy) \mapsto (x + iy)^2 = \underbrace{x^2 - y^2}_{u} + \underbrace{i2xy}_{v}.$$

Now

$$\begin{cases} \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial y} = -2Y = -\frac{\partial u}{\partial x} \end{cases}$$

so z^2 satisfies C–R equations. By the product formula $f'(z_0) = 2z_0$ and

$$f'\big(x_0 + iy_0\big) = 2\big(x_0 + iy_0\big) = \frac{\partial u}{\partial x}\left(z_0\right) + i\frac{\partial v}{\partial x}\left(z_0\right).$$

Theorem 4.3.30. *Suppose that $f = u + iv \colon \Omega \mapsto \mathbb{C}, \Omega \subset \mathbb{C}$ open. If u and v are differentiable in the real sense at $z_0 \in \Omega$ and their partial derivates obey the C–R equations at z_0, then f has a complex derivative at z_0 and*

$$f'\big(x_0 + iy_0\big) = \frac{\partial u}{\partial x}\left(z_0\right) + i\frac{\partial v}{\partial x}\left(z_0\right).$$

Remark 4.3.31. (Important!) If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous at z_0, then the following is true:
$f(z)$ is complex differentiable at z_0 if and only if the C–R equations

$$\frac{\partial f}{\partial x}\big(z_0\big) = \tfrac{1}{i}\frac{\partial f}{\partial y}\big(z_0\big)$$

hold.

4.3.32

Let $\Omega \subset \mathbb{R}^2$ be an open set. Consider $f : \Omega \to \mathbb{R}^2$, a function of two variables. Recall that f is differentiable at the point z_0 if there exists an $\mathbb{R}$-linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ – the derivative, or differential $\mathcal{D}f(z_0)$ – such that

$$f(z_0 + h) - f(z_0) = \mathcal{D}f(z_0)h + \|h\|\,\varepsilon(h),$$

where $\|\varepsilon\| \to 0$ as $\|h\| \to 0$ and $z_0, h \in \mathbb{R}^2$.

On the other hand, the fact that the corresponding function $f : \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$, is complex differentiable at z_0 means that there exists a complex number α such that

$$f(z_0 + h) - f(z_0) = \alpha \cdot h + h\varepsilon(h),$$

where $\lim_{h\to 0} \varepsilon(h) = 0$.

Hence, f is complex differentiable at z_0 if and only if f is differentiable in the real sense, and the derivative $\mathcal{D}f(z_0)$ is of the form $\mathcal{D}f(z_0)h = \alpha \cdot h$, with $\alpha \in \mathbb{C}$.

But the $\mathbb{R}$-linear mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ of this type are exactly the $\mathbb{C}$-linear mappings. So we have:

Remark. The function f, defined in $\mathbb{D}\big(z_0, r\big) \subset \mathbb{C}$, is complex differentiable at z_0 if and only if f is differentiable in the real sense and the derivative $\mathcal{D}f(z_0)$ is $\mathbb{C}$-linear.

The plane $\mathbb{R}^2$ ($\mathbb{C}$) is an $\mathbb{R}$-vector space. Hence, we can consider the mappings $L \colon \mathbb{C} \to \mathbb{C}$, which are $\mathbb{R}$-linear:

$$\begin{aligned} L(z + w) &= L(z) + L(w), & z, w &\in \mathbb{C} \\ L(\lambda z) &= \lambda L(z), & \lambda &\in \mathbb{R}. \end{aligned}$$

There is a corresponding matrix of L with respect to the basis $\{(1,0) = 1, (0,1) = i\}$:

$$L = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{kj} \in \mathbb{R}.$$

So that

$$\begin{aligned} L\colon\ (x,y) &\mapsto (a_{11}x + a_{12}y, a_{21}x + a_{22}y) \\ L(1,0) &= (a_{11}, a_{21}) = a_{11} + ia_{21} \\ L(0,1) &= (a_{12}, a_{22}) = a_{12} + ia_{22}. \end{aligned}$$

If $z = x + iy$, then by the $\mathbb{R}$-linearity of L:

$$\begin{aligned} L(z) &= L(x+iy) \\ &= xL(1,0) + yL(0,1) \\ &= a_{11}x + a_{12}y + i\left(a_{21}x + a_{22}y\right) \\ &= \alpha(x+iy) + \beta(x-iy), \end{aligned}$$

with $\alpha = \frac{1}{2}\left(a_{11} + a_{22} - ia_{12} + ia_{21}\right)$ and $\beta = \frac{1}{2}\left(a_{11} - a_{22} + ia_{12} + ia_{21}\right)$.

Hence, $L(z) = \alpha z + \beta\overline{z}$, where $\alpha, \beta \in \mathbb{C}$, is the general expression of an $\mathbb{R}$-linear mapping.

On the other hand, $\mathbb{C}$ is also a $\mathbb{C}$-vector space. So $L\colon\ \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear, if

$$\begin{aligned} L(z+\omega) &= L(z) + L(\omega), \quad && z, \omega \in \mathbb{C} \\ L(\lambda z) &= \lambda L(z), && \lambda \in \mathbb{C}. \end{aligned}$$

Now $L(z) = \alpha z$, because

$$\begin{aligned} L(x+iy) = L(x) + L(iy) &= L(x) + iL(y) \\ &= xL(1) + iyL(1) \\ &= \underbrace{(x+iy)}_{z}\underbrace{L(1)}_{\alpha}. \end{aligned}$$

We note from the previous calculations that $\beta = 0$ if and only if $a_{11} = a_{22}$ and $a_{12} = -a_{21}$.

Hence, in the Jabocian matrix

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix},$$

$u_x = v_y$ and $u_y = -v_x$, which means that the C–R equations are satisfied.

Example. As a first approximation the surface of soap films (or any elastic surface) is the graph of harmonic function.

5 Complex Series

5.1 Series of complex terms

Let (z_n) be a complex sequence. An expression

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \ldots \tag{5.1.1}$$

is an infinite series. The numbers

$$\sum_{n=1}^{N} z_n =: S_N, \qquad N = 1, 2, \ldots \tag{5.1.2}$$

are called the partial sums of the series.

Definition 5.1.3. If the sequence of partial sums converges to some number $S \in \mathbb{C}$, that is, $\lim_{N\to\infty} S_N = S$, then we define

$$\sum_{n=1}^{\infty} z_n = S,$$

and we say that the series $\sum_{n=1}^{\infty} z_n$ converges to (the sum) S. We call S the sum of the series $\sum_{n=1}^{\infty} z_n$. A series which converges to some number $S \in \mathbb{C}$ is called a convergent series. If a series does not converge, we say that it diverges or that it is divergent.

Remark. It follows from the definition that questions of the convergence of series are questions of the convergence of sequences.

Proposition 5.1.4: Linear combinations of series. *If*

$$\sum_{n=1}^{\infty} z_n = s \quad \textit{and} \quad \sum_{n=1}^{\infty} w_n = t,$$

then

$$\sum_{n=1}^{\infty} \left(z_n + w_n\right) = s + t \quad \textit{and} \quad \sum_{n=1}^{\infty} \lambda z_n = \lambda s, \qquad \lambda \in \mathbb{C}.$$

Proposition 5.1.5. *The complex series $\sum_{n=1}^{\infty} z_n$ converges if and only if the real series*

$$\sum_{n=1}^{\infty} \mathrm{Re}\left(z_n\right) \quad \textit{and} \quad \sum_{n=1}^{\infty} \mathrm{Im}\left(z_n\right)$$

both converge.

The propositions above tell us that we can write

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \mathrm{Re}\left(z_n\right) + i \sum_{n=1}^{\infty} \mathrm{Im}\left(z_n\right),$$

whenever the two series on the right-hand side converge, or the series on the left-hand side converges.

Remarks 5.1.6.

1. If a series of complex numbers converges, then the n^{th} term converges to zero as n tends to infinity. Note that the converse is generally not true: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\lim_{n\to\infty} \frac{1}{n} = 0$.
2. The terms of a convergent series are bounded. That is, when the series $\sum_{n=1}^{\infty} z_n$ converges, then there exists a positive constant M such that $\left|z_n\right| \le M$ for all positive integers n.

Definition 5.1.7. The series $\sum_{n=1}^{\infty} z_n$ is said to be absolutely convergent if the series

$$\sum_{n=1}^{\infty} |z_n|$$

of real numbers $|z_n| = \sqrt{\mathrm{Re}(z_n)^2 + \mathrm{Im}(z_n)^2}$ converges.

Proposition 5.1.8: Absolute convergence vs. convergence. *The absolute convergence of a series of complex numbers implies the convergence of that series.*

In establishing the fact that the sum of a given series is a given number S, it is often convenient to define the remainder ϱ_N after N terms:

Definition 5.1.9. Let $S \in \mathbb{C}$ be given. Let $S_N = \sum_{j=1}^{N} z_j$ be the partial sum of the series $\sum_{j=1}^{\infty} z_j$. The remainder ϱ_N after N terms is $\varrho_N = S - S_N$. Thus $S = S_N + \varrho_N$, and since $|S - S_N| = |\varrho_N|$, we see that a series $\sum_{n=1}^{\infty} z_n$ converges to a number $S \in \mathbb{C}$ if and only if the sequence of remainders tends to zero.

Testing for convergence

Let $\sum_{n=1}^{\infty} z_n$ be a complex series. The associated series $\sum_{n=1}^{\infty} |z_n|$ has real, non-negative terms. Hence well-known tests (from real analysis) for the convergence of a series with non-negative terms can be applied. Combining this with Proposition 5.1.8 we obtain a sufficient condition for the convergence of $\sum_{n=1}^{\infty} z_n$.

Remark. The geometric series $\sum z^n$

- converges and $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, if $|z| < 1$
- fails to converge, if $|z| \geq 1$.

5.1.10 The comparison test

Suppose that $\sum_{n=1}^{\infty} w_n$ is a convergent series with $w_n \geq 0$ for all n and suppose that for some constant $k > 0, |z_n| \leq k w_n$ for all n. Then $\sum_{n=1}^{\infty} z_n$ converges absolutely and hence it converges.

5.1.11 d'Alembert's ratio test

Suppose that $\sum_{n=1}^{\infty} z_n$ is a series of complex terms such that the limit

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| =: q$$

exists.

If $0 \leq q < 1$, then $\sum_{n=1}^{\infty} z_n$ converges absolutely.

If $q > 1$, then $\sum_{n=1}^{\infty} z_n$ diverges.

If $q = 1$, then the test gives no information.

5.1.12 The n^{th} root test

Suppose that (z_n) is a complex sequence such that the limit

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} =: \ell$$

exists.

If $\ell < 1$, then $\sum_{n=1}^{\infty} z_n$ converges absolutely.

If $\ell > 1$, then $\sum_{n=1}^{\infty} z_n$ diverges.

If $\ell = 1$, then the test gives no information.

Remark (for further reading). There exists a useful test called Raabe's test.

Examples 5.1.13.

1. The series
$$\sum_{n=1}^{\infty} \frac{(-i)^n}{n^2}$$
converges, since it is absolutely convergent:
$$\sum_{n=1}^{\infty} \left| \frac{(-i)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$
Recall from real analysis that a series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

2. The series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} i$$
is convergent. Let $x_n = 0$ and $y_n = \frac{(-1)^n}{n!}$. Evidently, the series $\sum_{n=1}^{\infty} x_n$ is convergent. Likewise, $\sum_{n=1}^{\infty} y_n$ is convergent:
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = e^{-1}.$$
And so
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} i = \frac{i}{e}.$$

3. The series
$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + i\frac{1}{n} \right)$$
diverges, since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

4. The series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2i)^{2n-1}}{(2n-1)!}$$
converges.

   Let
$$z_n = \frac{(-1)^{n-1} (2i)^{2n-1}}{(2n-1)!} = \frac{1}{2i} \frac{(-1)^{n-1} (2i)^{2n}}{(2n-1)!},$$
then
$$\left| \frac{z_{n+1}}{z_n} \right| = \left| \frac{(2n-1)! \, (-1)^n (2i)^{2n+2}}{(-1)^{n-1} (2i)^{2n} \left(2(n+1)-1\right)!} \right| = \frac{4}{2n(2n+1)} \underset{n\to\infty}{\longrightarrow} 0.$$
Hence, the series converges by d'Alembert's ratio test.

5. Let us prove that
$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad \text{whenever } |z| < 1.$$
We achieve this with the help of remainders. Recall that $(1-z)(1+z+\cdots+z^n) = 1 - z^{n+1}$, so
$$1 + z + \cdots + z^n = \frac{1-z^{n+1}}{1-z}, \qquad z \neq 1.$$

Now, we may write the partial sum

$$S_N(z) = \sum_{n=0}^{N} z^n = 1 + z + \cdots + z^N, \qquad z \neq 1,$$

as

$$S_N(z) = \frac{1 - z^{N+1}}{1 - z}, \qquad z \neq 1.$$

If $S(z) = \frac{1}{1-z}$, then

$$\varrho_N(z) = S(z) - S_N(z) = \frac{z^{N+1}}{1 - z}, \qquad z \neq 1.$$

Thus

$$\left|\varrho_N(z)\right| = \frac{|z|^{N+1}}{|1 - z|}$$

and the remainders $\varrho_N(z)$ tend to zero when $|z| < 1$, but not when $|z| \geq 1$. Hence

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}, \qquad \text{whenever } |z| \leq 1.$$

5.2 Power series

A power series is an expansion of the form

$$\sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n = a_0 + a_1 \left(z - z_0\right) + \cdots + a_n \left(z - z_0\right)^n + \ldots,$$

where z_0 and the coefficients a_n are complex constants and z may be any point in a stated domain containing z_0. In such a series involving a variable z, we shall denote sums, partial sums and remainders by $S(z)$, $S_N(z)$ and $\varrho_N(z)$, respectively. Our last example 5.1.13(5) gives an example of a power series.

In the power series expansion we shall often assume, without loss of generality, that $z_0 = 0$.

Theorem 5.2.1. (Theorem of Abel) *If $\sum_{n=0}^{\infty} a_n z^n$ converges at some $z_1 \in \mathbb{C}$, then the series converges absolutely at each $z \in \mathbb{D}(0,|z_1|)$.*

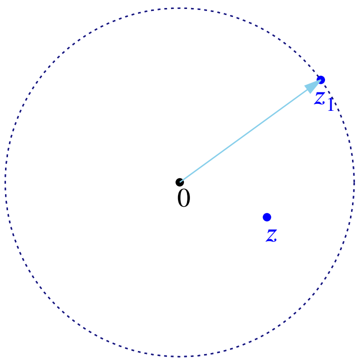


Figure 5.1: Theorem 5.2.1: If a power series converges at some $z_1 \in \mathbb{C}$, then it converges absolutely at each $z \in \mathbb{D}(0,|z_1|)$.

Proof. Since $\sum a_n z^n$ converges at $z_1 \neq 0$, it follows that $\lim_{k\to\infty}\left|a_k z_1^k\right| = 0$. So there exists a real constant M such that $\left|a_k z_1^k\right| \leq M$ for all $k \in \mathbb{N}$. Hence for each k

$$\left|a_k z^k\right| = \left|a_k z_1^k\right| \left|\frac{z}{z_1}\right|^k \leq M q^k,$$

where $q = \frac{|z|}{|z_1|} < 1$, when $|z| < |z_1|$, that is, $z \in \mathbb{D}(0,|z_1|)$. The series $\sum q^k$ converges as a geometric series in $\mathbb{D}(0,|z_1|)$. Hence $\sum M q^k$ converges and by the comparison test $\sum |a_k z^k|$ converges, when $|z| < |z_1|$ and the original series $\sum a_k z^k$ converges absolutely when $|z| < |z_1|$. □

Corollary 5.2.2. *If $\sum a_k z^k$ fails to converge at some $z_2 \in \mathbb{C}$, then $\sum a_k z^k$ diverges for all $|z| > |z_2|$.*

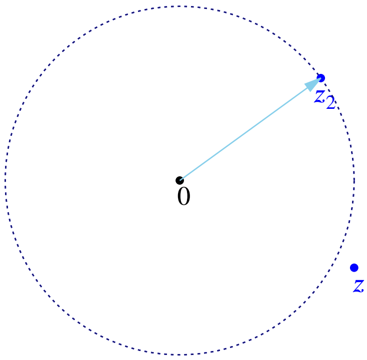


Figure 5.2: Corollary 5.2.2: If a power series fails to converge at some $z_2 \in \mathbb{C}$, then it diverges for all $|z| > |z_2|$.

Definition 5.2.3. The radius of convergence of the power series $\sum a_n(z - z_0)^n$ is defined to be

$$R := \sup \left\{ |z - z_0| \ : \ \sum |a_n(z - z_0)^n| \text{ converges at } z \right\}.$$

Here we write $R = \infty$, if $\sum |a_n(z - z_0)|^n$ converges for arbitrarily large $z - z_0$.

Lemma 5.2.4. *(Radius of convergence) Let $\sum a_k(z - z_0)^k$ be a power series with radius of convergence R. Then there are three possibilities:*

1. $R = 0:$ *the series $\sum a_k(z - z_0)^k$ converges only at the point z_0.*
2. $0 < R < \infty:$ *the series $\sum a_k(z - z_0)^k$ converges absolutely for all z with $|z - z_0| < R$ and the series $\sum a_k(z - z_0)^k$ fails to converge for any z with $|z - z_0| > R$*
3. $R = \infty:$ *the series $\sum a_k(z - z_0)^k$ converges absolutely for all $z \in \mathbb{C}$.*

Definition 5.2.5. If $0 < R < \infty$ then the disc $\mathbb{D}(z_0, R)$ is called the disc of convergence. If $R = 0$ the disc of convergence is $\emptyset$. If $R = \infty$ then the disc of convergence is $\mathbb{C}$.

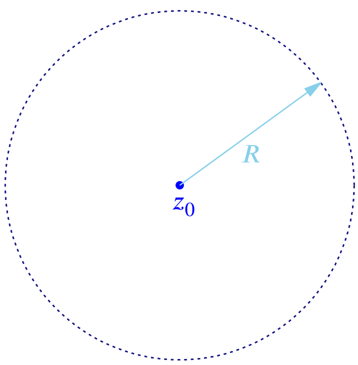


Figure 5.3: Definition 5.2.5: The disc of convergence $\mathbb{D}(z_0, R)$.

Remark. On the boundary of the disc of convergence, $|z - z_0| = R$, the situation is delicate, as one can have either convergence or divergence.

Examples 5.2.6. For each of the following power series, calculate the radius of convergence:

$$(1) \sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad (2) \sum_{n=1}^{\infty} \frac{z^n}{n!}, \qquad (3) \sum_{n=1}^{\infty} n! z^n.$$

(1) We apply the Ratio test to $\sum \left|\frac{1}{n} z^n\right|$. For $z \neq 0$,

$$\left|\frac{z^{n+1}}{n+1} \frac{n}{z^n}\right| = \frac{n}{n+1}|z| \to |z|,$$

as $n \to \infty$. Hence $\sum \left|\frac{1}{n} z^n\right|$ converges if $|z| < 1$ and fails to converge if $|z| > 1$. We conclude that $R = 1$.

(2) We apply the Ratio test to $\sum \left|\frac{z^n}{n!}\right|$. For $z \neq 0$,

$$\left|\frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n}\right| = \frac{|z|}{n+1} \to 0,$$

as $n \to \infty$. Hence $\sum \left|\frac{z^n}{n!}\right|$ converges for all $z \in \mathbb{C}$. We deduce that $R = \infty$.

(3) We apply the Ratio test to $\sum |n! z^n|$. For $z \neq 0$,

$$\left|\frac{(n+1)! z^{n+1}}{n! z^n}\right| = \big(n+1\big)|z| \to \infty,$$

as $n \to \infty$. Hence $\sum |n! z^n|$ converges only if $z = 0$. So $R = 0$.

For calculating the radius of convergence we may apply the Ratio test and the nth root test. But we need the limits and sometimes they do not exist. The formula of Hadamard solves the problem!

Note. We use the following convention that

$$\frac{1}{0} = \infty \quad \text{and} \quad \frac{1}{\infty} = 0.$$

Remark. Recall the definition of limes superior. Let (x_k) be a sequence of real numbers. Then

$$\overline{\lim_{k\to\infty}} x_k = \limsup_{k\to\infty} x_k = \inf_{k\geq 1} \sup \left\{x_k, x_{k+1}, \dots\right\}.$$

Example. Let (x_k) be a sequence where $x_k = (-1)^k$, $k \geq 1$. Note that the sequence

$$(x_k) = (-1, 1, -1, \dots)$$

fails to converge, but

$$\limsup_{k\to\infty} x_k = 1.$$

Theorem 5.2.7: Hadamard's theorem. *The radius of convergence of the power series* $\sum a_k (z - z_0)^k$ *is given by the formula*

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{\frac{1}{k}}. \tag{5.2.8}$$

Examples 5.2.9. Find the radius of convergence for the following series:

1.
$$\sum_{n=1}^{\infty} \frac{z^{n^2}}{n^2} = z + \frac{z^4}{4} + \frac{z^9}{9} + \frac{z^{16}}{16} + \cdots = \sum_{j=1}^{\infty} a_j z^j,$$

where
$$a_j = \begin{cases} \frac{1}{j}, & \text{when } j = n^2 \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

The Hadamard formula 5.2.8 gives

$$\frac{1}{R} = \limsup_{j\to\infty} \left|a_j\right|^{\frac{1}{j}} = \limsup_{n\to\infty} \left|\frac{1}{n^2}\right|^{\frac{1}{n^2}} = \sup_{k\to\infty} \frac{1}{k^{\frac{1}{k}}} = 1,$$

since
$$k^{\frac{1}{k}} = e^{\log k^{\frac{1}{k}}} = e^{\frac{1}{k}\log k} \to 1.$$

2.
$$\sum_{n=1}^{\infty} a^{n^2} z^n, \ a \in \mathbb{C}.$$

The Hadamard formula 5.2.8 gives

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \left|a_n\right|^{\frac{1}{n}} = \overline{\lim_{n\to\infty}} |a|^{\frac{n^2}{n}} = \overline{\lim_{n\to\infty}} |a|^n .$$

Hence,
$$R = \begin{cases} 0, \text{ if } |a| > 1 \\ 1, \text{ if } |a| = 1 \\ \infty, \text{ if } |a| < 1. \end{cases}$$

Proof of Hadamard's theorem. Let $L = \frac{1}{R}$ where R is defined by the Hadamard's formula 5.2.8. Suppose that $L \neq 0, \infty$. If $|z| < R$, choose $\varepsilon > 0$ such that

$$(L + \varepsilon)|z| = r < 1.$$

By the definition of L, we have $\left|a_n\right|^{\frac{1}{n}} \leq L + \varepsilon$ for all large n. Hence,

$$\left|a_n\right||z|^n \leq (L+\varepsilon)^n |z|^n = r^n.$$

By the Comparison test with the geometric series $\sum r^n$ we can show that $\sum a_n z^n$ converges.

If $|z| > R$, then a similar argument proves that there exists a sequence of terms in the series whose absolute values go to infinity, and hence the sequence, and also the series diverges. ☐

Power series provide a very important class of analytic functions that are easy to manipulate.

The following theorem is very important.

Theorem 5.2.10. *The power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

defines an analytic function in its disc of convergence. The derivative of f is also a power series obtained by differentiating the series for f term by term, that is,

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Moreover, f' has the same radius of convergence as f.

Remark. There is the obvious candidate for the derivative:

$$\sum na_n z^{n-1}.$$

But this assumes that we differentiate the series 'term-by-term'. Note that

$$\sum na_n z^{n-1} = \sum \frac{\mathrm{d}}{\mathrm{d}z} a_n z^n$$

whereas

$$f'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\sum a_n z^n\right),$$

the differentiation and summation are performed in different orders here. Both operations are performed by taking a limit. But recall that limiting processes need not commute with one another. So, the validity of term-by-term differentiation of a power series needs a proof.

Example 5.2.11. The geometric series $\sum z^n$ has radius of convergence 1, and provides a power series expansion of $\frac{1}{1-z}$ for $|z| < 1$. By Theorem 5.2.10,

$$\frac{1}{(1-z)^2} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{1-z}\right) = 1 + 2z + 3z^2 + \dots, \qquad \text{when } |z| < 1.$$

Lemma 5.2.12. *The power series $\sum a_n z^n$ and $\sum na_n z^{n-1}$ have the same radius of convergence.*

Proof. Let R be the radius of convergence of $\sum a_n z^n$. We will show that $\sum |a_n z^n|$ converges for $|z| < R$. We choose ρ so that $|z| < \rho < R$ and assume $z \neq 0$.

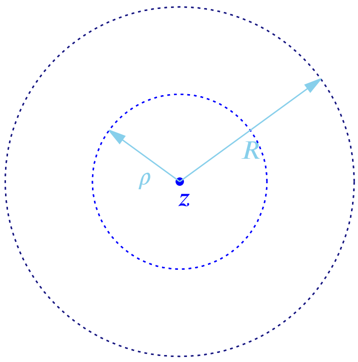


Figure 5.4: In the proof of Lemma 5.2.12 the disc of convergence is $\mathbb{D}(z, R)$.

Then

$$|na_n z^{n-1}| = \frac{n}{|z|}\left(\frac{|z|}{\rho}\right)^n |a_n \rho|^n.$$

As $\left(\frac{|z|}{\rho}\right)^n < 1$, the series $\sum n\left(\frac{|z|}{\rho}\right)^n$ converges by the ratio test.

$$\left(\frac{(n+1)\left(\frac{|z|}{\rho}\right)^{n+1}}{n\left(\frac{|z|}{\rho}\right)^n} = \frac{n+1}{n}\frac{|z|}{\rho} \to \frac{|z|}{\rho} < 1, \qquad \text{as } n \to \infty\right)$$

Hence, there is a constant M such that

$$n\left(\frac{|z|}{\rho}\right)^n \leq M, \qquad \text{for all } n.$$

Thus,

$$\left|na_n z^{n-1}\right| \le \frac{M}{|z|}\left|a_n \rho^n\right|.$$

The result follows from the Comparison test.

Conversely, suppose that

$$\sum \left|na_n z^{n-1}\right|$$

converges. Then,

$$\left|a_n z^n\right| \le |z|\left|na_n z^{n-1}\right|, \qquad n \ge 1,$$

so

$$\sum \left|a_n z^n\right|$$

converges by the Comparison test. □

Proof for Theorem 5.2.10. To prove that the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

defines an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

in its disc of convergence, we must show that the series

$$g(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$$

gives the derivative of f.

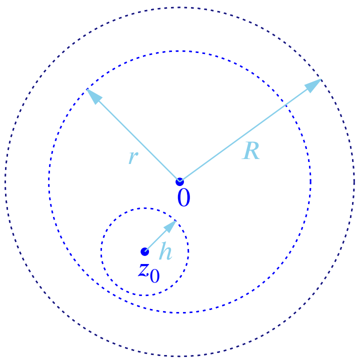


Figure 5.5: The disc of convergence is $\mathbb{D}(0, R)$, the disc $\mathbb{D}(0, r) \subset \mathbb{D}(0, R)$ and the chosen point $z_0 \in \mathbb{D}(0, r)$ with disc $\mathbb{D}(z_0, h) \subset \mathbb{D}(0, r)$ in the proof of Theorem 5.2.10.

For that, let R be the radius of convergence of f. Suppose that $|z_0| < r < R$. We write

$$f(z) = S_N(z) + \rho_N(z)$$

where

$$S_N(z) = \sum_{n_0}^{N} a_n z^n \qquad \text{and} \qquad \rho_N(z) = \sum_{N+1}^{\infty} a_n z^n.$$

Then, if h is chosen so that $\left|z_0 + h\right| < r$, we have

$$\frac{f\left(z_0 + h\right) - f\left(z_0\right)}{h} - g\left(z_0\right) = \left(\frac{S_N\left(z_0 + h\right) - S_N\left(z_0\right)}{h} - S_N'\left(z_0\right)\right) + \left(S_N'\left(z_0\right) - g\left(z_0\right)\right) + \left(\frac{\rho\left(z_0 + h\right) - \rho_N\left(z_0\right)}{h}\right).$$

Since $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$, we obtain that

$$\begin{aligned}\left|\frac{\rho_N(z_0+h) - \rho_N(z_0)}{h}\right| &\le \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \\ &\le \sum_{n=N+1}^{\infty} |a_n| \sum_{j=0}^{n-1} \underbrace{|z_0+h|^j}_{<r} \underbrace{|z_0|^{n-j-1}}_{<r} \\ &\le \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}.\end{aligned}$$

The expression

$$\sum_{n=N+1}^{\infty} |a_n| n r^n$$

is the tail of a convergent series, since g converges absolutely on $|z| < R$. Therefore, given $\varepsilon > 0$, we can find N_1 such that $N > N_1$ implies

$$\left|\frac{\rho_N(z_0+h) - \rho_N(z_0)}{h}\right| < \varepsilon.$$

Also, as

$$\lim_{N\to\infty} S_N'(z_0) = g(z_0),$$

we can find N_2 such that $N > N_2$ implies

$$|S_N'(z_0) - g(z_0)| < \varepsilon.$$

If $N > max\{N_1, N_2\}$, then we can find $\delta > 0$ such that $|h| < \delta$ implies

$$\left|\frac{S_N(z_0+h) - S_N(z_0)}{h} - S'N(z_0)\right| < \varepsilon,$$

because the derivative of a polynomial is obtained by differentiating it term by term. Therefore,

$$\left|\frac{f(z_0+h) - f(z_0)}{h} - g(z_0)\right| < 3\varepsilon,$$

whenever $|h| < \delta$, thereby concluding the proof of the theorem. □

Corollary 5.2.13. *(Important!) A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.*

Remark 5.2.14. In fact, if $k(z) = \sum_{n=0}^{\infty} a_n z^n$, then f is obtained by translating k namely

$$f(z) = k(w), \text{ where } w = z - z_0.$$

As a consequence everything about k also holds for f after we made the appropriate translation.

By the chain rule

$$f'(z) = k'(w) = \sum_{n=0}^{\infty} n a_n \left(z - z_0\right)^{n-1}.$$

A function f defined on a open set Ω is said to have a power series expansion at a point $z_0 \in \Omega$, if there exists series $\sum a_n \left(z - z_0\right)^n$ centered at z_0 with a positive radius of convergence such that

$$f(z) = \sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n$$

for all z in a neighbourhood of z_0.

6 The exponential, sine and cosine functions

6.1 The exponential function

(1) Recall the real exponential function $e^x : \mathbb{R} \to \mathbb{R}_+ = (0, \infty)$ has the Taylor expression

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \text{for all } x \in \mathbb{R}.$$

(2) Recall the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is $R = \infty$. Thus, this series converges absolutely on the whole complex plane $\mathbb{C}$. Hence, by Theorem 5.2.10, the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ defines an analytic function $\mathbb{C} \to \mathbb{C}$ (in fact, $\mathbb{C} \to \mathbb{C} \setminus \{0\}$.)

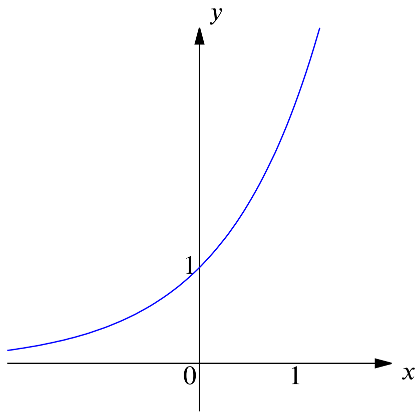


Figure 6.1: The graph of the real exponential function $e^x : \mathbb{R} \to \mathbb{R}_+$, here $y = e^x$.

Definition 6.1.1. We define the exponential function $\exp : \mathbb{C} \to \mathbb{C}$ by

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Remark. Since the Taylor expansion of the real exponential function is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \text{for all } x \in \mathbb{R},$$

we have $\exp(x) = e^x$ if $x \in \mathbb{R}$.

Remark. We may write e^z for $\exp(z)$.

6.1.2 Properties of the exponential function

(1) The function $\exp(z)$ is analytic in $\mathbb{C}$ and

$$\frac{\mathrm{d}}{\mathrm{d}z} \exp(z) = \exp'(z) = \exp(z), \qquad \text{for all } z \in \mathbb{C}.$$

Proof. By example 5.2.6 and Theorem 5.2.10,

$$\exp'(z) = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z),$$

for all $z \in \mathbb{C}$. □

(2) $\exp(z + \omega) = \exp(z)\exp(\omega)$, for all $z, \omega \in \mathbb{C}$.

Proof. By Mertens's theorem, the Cauchy product rule and the binomial formula give:

$$\begin{aligned}
\exp(z)\exp(\omega) &= \left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{\omega^k}{k!}\right) = \sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{1}{p!}z^p\frac{1}{(n-p)!}\omega^{n-p}\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{p=0}^{n}\frac{n!}{p!(n-p)!}z^p\omega^{n-p} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{p=0}^{n}\binom{n}{p}z^p\omega^{n-p}\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(z+\omega)^n = \exp(z+\omega).
\end{aligned}$$

□

(3) $\exp\big(0\big) = 1$.

Proof. It follows immediately from the definition. □

(4) $\exp(z) \neq 0$, for all $z \in \mathbb{C}$.

Proof. By (1) and (2), $1 = \exp\big(0\big) = \exp(z-z) = \exp(z)\exp(-z)$. □

(5) $\exp(-z) = \frac{1}{\exp(z)}$, for all $z \in \mathbb{C}$.

Proof. By (1) and (2), $\exp(z)\exp(-z) = 1$, so $\exp(-z) = \frac{1}{\exp(z)}$. □

6.1.3 The complex conjugate and the modulus of the exponential function

(1) $\exp\big(\overline{z}\big) = \overline{\exp(z)}$, for all $z \in \mathbb{C}$.

Proof. By our previous results,

$$\exp\big(\overline{z}\big) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{\overline{z}^k}{k!} = \lim_{n\to\infty}\overline{\sum_{k=0}^{n}\frac{z^k}{k!}} = \overline{\lim_{n\to\infty}\sum_{k=0}^{n}\frac{z^k}{k!}} = \overline{\exp(z)}.$$

□

(2) $\big|\exp(z)\big| \leq \exp\big(|z|\big)$, for all $z \in \mathbb{C}$, since the series $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ converges absolutely on the whole complex plane.

(3) $\big|\exp(z)\big| = \exp(\operatorname{Re} z) = e^{\operatorname{Re} z}$, for all $z \in \mathbb{C}$.

Proof. By our previous results, we obtain:

$$\begin{aligned}
\big|\exp(z)\big|^2 &= \exp(z)\overline{\exp(z)} \overset{(1)}{=} \exp(z)\exp\big(\overline{z}\big) \overset{6.1.2}{=} \exp\big(z+\overline{z}\big)\\
&= \exp\big(2\operatorname{Re} z\big) = \exp(\operatorname{Re} z + \operatorname{Re} z) = \exp(\operatorname{Re} z)\exp(\operatorname{Re} z) = \big(\exp(\operatorname{Re} z)\big)^2.
\end{aligned}$$

Since the numbers $\big|\exp(z)\big|$ and $\exp(\operatorname{Re} z)$ are both positive real numbers, so the claim follows. □

(4) $\big|\exp(iy)\big|^2 = 1$, for all $y \in \mathbb{R}$.

Proof. By (3), we have $\big|\exp(z)\big| = \exp(\operatorname{Re} z)$. But $\operatorname{Re}(iy) = 0$, hence $\big|\exp(iy)\big| = 1$, from which the result follows. □

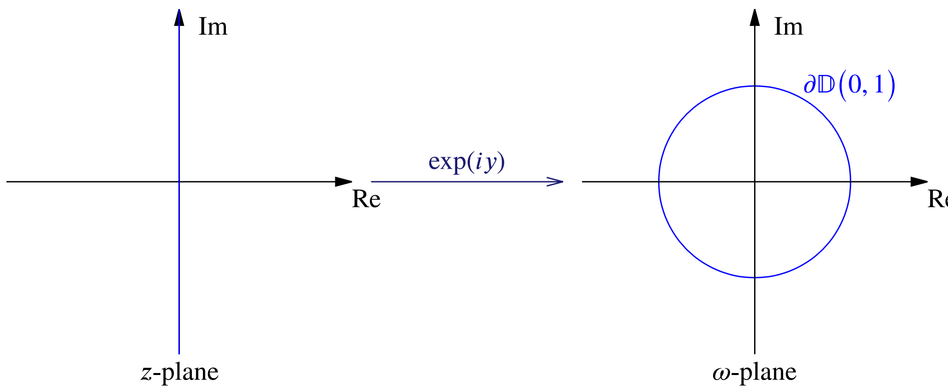


Figure 6.2: The complex exponential function maps a point iy, where $y \in \mathbb{R}$, to a point on the unit circle.

6.1.4 Euler's formula

Previously we defined the notation

$$e^{i\alpha} := \cos\alpha + i\sin\alpha, \qquad \text{where } \alpha \in \mathbb{R}.$$

We verify now that

$$\exp(i\alpha) = \cos\alpha + i\sin\alpha, \qquad \text{where } \alpha \in \mathbb{R}.$$

By the Taylor series of the real sine and cosine functions, we obtain

$$\exp(i\alpha) = \sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} = \sum_{k=0}^{\infty} \frac{i^k}{k!}\alpha^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\alpha^{2k} + i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\alpha^{2k+1} = \cos\alpha + i\sin\alpha.$$

6.2 Trigonometric functions

6.2.1 The sine and cosine functions of one complex variable

Recall that, by Euler's formula for every real number x,

$$\exp(ix) = \cos x + i\sin x \qquad \text{and} \qquad \exp(-ix) = \cos x - i\sin x.$$

Hence,

$$2i\sin x = \exp(ix) - \exp(-ix) \qquad \text{and} \qquad 2\cos x = \exp(ix) + \exp(-ix),$$

that is,

$$\sin x = \frac{1}{2i}\big(\exp(ix) - \exp(-ix)\big) \qquad \text{and} \qquad \cos x = \frac{1}{2}\big(\exp(ix) + \exp(-ix)\big).$$

It is, therefore, natural to define the sine and cosine functions of a complex variable as follows:

$$\begin{aligned} \sin : \mathbb{C} \to \mathbb{C} \quad & \sin z = \frac{1}{2i}\big(\exp(iz) - \exp(-iz)\big) \text{ and} \\ \cos : \mathbb{C} \to \mathbb{C} \quad & \cos z = \frac{1}{2}\big(\exp(iz) + \exp(-iz)\big). \end{aligned} \tag{6.2.2}$$

6.2.3. From this, we obtain Euler's formula for all $z \in \mathbb{C}$:

$$\cos z + i\sin z = \exp(iz).$$

Remark. By Euler's formula for $z = x + iy \in \mathbb{C}$, where $x, y \in \mathbb{R}$, we have

$$\exp(z) = \exp(x+iy) = e^x(\cos y + i\sin y). \tag{6.2.4}$$

Proposition 6.2.5. *The functions* $\sin : \mathbb{C} \to \mathbb{C}$ *and* $\cos : \mathbb{C} \to \mathbb{C}$ *are entire and*

$$\frac{\partial}{\partial z} \sin z = \cos z \qquad \textit{and} \qquad \frac{\partial}{\partial z} \cos z = -\sin z.$$

6.2.6 (Addition formulas)**.**

$$\sin\big(z_1 + z_2\big) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$$
$$\cos\big(z_1 + z_2\big) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

Proof. Homework 6.4. □

Remark 6.2.7.

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad \text{and} \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

The radius of convergence of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad \text{and} \qquad \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

is $R = \infty$.

We could have defined $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ as power series. They are entire, as

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad \text{and} \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Remark 6.2.8. The connections between

$$z \to \exp(z),$$
$$z \to \sin(z),$$
$$z \to \cos(z)$$

are

$$\exp(z) = \cos(z) + i\sin(z),$$
$$\sin(z) = \frac{1}{2i}\big(\exp(iz) - \exp(-iz)\big),$$
$$\cos(z) = \frac{1}{2}\big(\exp(iz) + \exp(-iz)\big).$$

6.3 Complex hyperbolic functions

We define the complex hyperbolic sine and cosine of one complex variable z,

$$\sinh : \mathbb{C} \to \mathbb{C} \qquad \text{and} \qquad \cosh : \mathbb{C} \to \mathbb{C},$$

as follows:

$$\sinh z = \frac{1}{2}\big(\exp(z) - \exp(-z)\big) \qquad \text{and} \qquad \cosh z = \frac{1}{2}\big(\exp(z) + \exp(-z)\big). \tag{6.3.1}$$

They are entire functions and

$$\frac{\partial}{\partial z} \sinh z = \cosh z, \qquad \frac{\partial}{\partial z} \cosh z = \sinh z.$$

We obtain the Osborn's rules:

$$\sin iz = i \sinh z \qquad \text{and} \qquad \cos iz = \cosh z. \tag{6.3.2}$$

6.3.3 The real and imaginary parts

By the addition formulas and Osborn's rules we obtain for $z = x + iy$

$$\cos(x + iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y$$

and

$$\sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y.$$

Remark. Recall the graphs of the real hyperbolic functions.

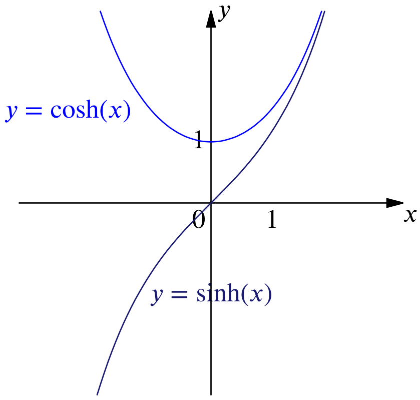


Figure 6.3: The graphs of the real hyperbolic functions cosh and sinh.

6.3.4 Unboundedness

Osborn's rules and the known behaviour of the real $\cosh : \mathbb{R} \to [1, \infty)$ and $\sinh : \mathbb{R} \to \mathbb{R}$ imply

$$|\cos iy| = |\cosh y| \to \infty, \quad \text{as } y \to \infty$$

and

$$|\sin iy| = |i \sinh y| \to \infty, \quad \text{as } y \to \infty.$$

Recall that the real sine and cosine functions are bounded:

$$|\cos x| \le 1 \quad \text{and} \quad |\sin x| \le 1, \qquad \text{for all } x \in \mathbb{R}.$$

6.4 Zeros

Recall that if f is analytic in some domain D, then $\frac{1}{f}$ is analytic in D provided $f(z) \neq 0$ for all $z \in D$. Thus we are often interested in finding out the zeroes of a particular analytic function. In the case of complex trigonometrical functions, which are defined by the exponential function, the fundamental equation is

$$\exp(2\pi i) = 1. \tag{6.4.1}$$

This equation comes from Euler's formula

$$\exp(i\alpha) = \cos\alpha + i \sin\alpha,$$

assuming that $\sin 2\pi = 0$ and $\cos 2\pi = 1$.

Proposition 6.4.2.

$$\exp(z) = 1 \quad \textit{if and only if} \quad z = 2k\pi i, \quad k \in \mathbb{Z}, \quad \textit{and}$$
$$\exp(z) = -1 \quad \textit{if and only if} \quad z = (2k+1)\pi i, \quad k \in \mathbb{Z}.$$

Remark 6.4.3. We have already proved that $\exp(z) \neq 0$ for any $z \in \mathbb{C}$. On the other hand, Proposition 6.4.2 tells us that

$$\exp(z) + 1 \qquad \text{and} \qquad \exp(z) - 1$$

both take value zero at infinitely many points. Contrast this with the real case:

$$e^x - 1 = 0 \quad \text{if and only if} \quad x = 0$$

and

$$e^x + 1 \neq 0, \qquad \text{for all } x \in \mathbb{R}.$$

The fundamental equation is

$$\exp\big(2\pi i\big) = 1.$$

Proposition 6.4.4.

$$\exp(z) = 1 \quad \textit{if and only if} \quad z = k2\pi i, \qquad k \in \mathbb{R}.$$

The complex exponential function is periodic:

$$\exp\big(z + 2\pi i\big) = \exp(z).$$

We say that the exponential function is periodic with a pure imaginary number $2\pi i$.

Remark 6.4.5. If $\exp(z) = \exp(w)$, then $z = w + k2\pi i$ for some $k \in \mathbb{Z}$.

6.4.6. Let $z \in \mathbb{C}$. Then

$$\begin{aligned}
\cos z = 0 &\quad \text{if and only if} \quad z = k\pi + \frac{\pi}{2}, && k \in \mathbb{Z},\\
\sin z = 0 &\quad \text{if and only if} \quad z = k\pi, && k \in \mathbb{Z},\\
\cosh z = 0 &\quad \text{if and only if} \quad z = i\left(k + \frac{1}{2}\right)\pi, && k \in \mathbb{Z}, \quad \text{and}\\
\sinh z = 0 &\quad \text{if and only if} \quad z = k\pi i, && k \in \mathbb{Z}.
\end{aligned}$$

Now we can define the complete tangent function

$$\tan z := \frac{\sin z}{\cos z} \quad \text{in the set} \quad \Omega = \{z \in \mathbb{C} : z \neq n\pi + \frac{\pi}{2},\ n \in \mathbb{Z}\}.$$

The tangent function is analytic in Ω. Similarly,

$$\cot z := \frac{\cos z}{\sin z} \quad \text{is analytic in} \quad \mathbb{C} \setminus \{n\pi : n \in \mathbb{Z}\}.$$

Remark 6.4.7. The exponential, sine and cosine and the hyperbolic functions are periodic:

$$\begin{aligned}
\exp(z+\alpha) = \exp(z), &\quad \text{for all } z \in \mathbb{C} \quad \text{if and only if} \quad \alpha = 2\pi i k, && k \in \mathbb{Z},\\
\sin(z+\alpha) = \sin(z), &\quad \text{for all } z \in \mathbb{C} \quad \text{if and only if} \quad \alpha = 2\pi k, && k \in \mathbb{Z},\\
\cos(z+\alpha) = \cos(z), &\quad \text{for all } z \in \mathbb{C} \quad \text{if and only if} \quad \alpha = 2\pi k, && k \in \mathbb{Z},\\
\sinh(z+\alpha) = \sinh(z), &\quad \text{for all } z \in \mathbb{C} \quad \text{if and only if} \quad \alpha = 2\pi i k, && k \in \mathbb{Z},\\
\cosh(z+\alpha) = \cosh(z), &\quad \text{for all } z \in \mathbb{C} \quad \text{if and only if} \quad \alpha = 2\pi i k, && k \in \mathbb{Z}.
\end{aligned}$$

7 Complex mappings

Recall that the properties of a real-valued function of a real variable are often expressed by the graph of the function. But complex functions $\omega = f(z)$ are difficult to graph directly, since $\omega \in \mathbb{C}$ and $z \in \mathbb{C}$. Because of this difficulty, it is common to display the z-plane and the ω-plane side by side and describe how the points on the z-plane map to the points in the ω-plane. When a function f is considered in this way, it is often referred to as a mapping, a map or a transformation.

More information is usually obtained by sketching the images of curves and domains than by simply indicating images of individual points.

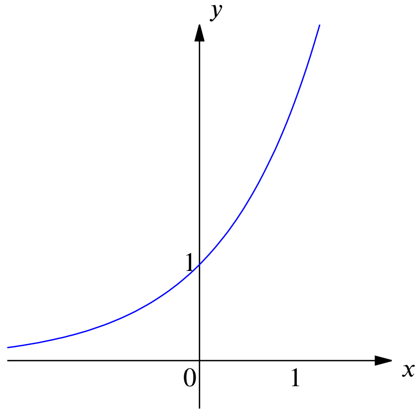


Figure 7.1: Example: the graph of the real exponential function $e^x : \mathbb{R} \to (0, \infty)$, $x \mapsto e^x$.

7.1 Some properties of the complex mappings

Recall from 6.1.3(4) that $\left|\exp(iy)\right|^2 = 1$. This means that the point iy, where $y \in \mathbb{R}$, maps to a point on the unit circle.

Proposition 7.1.1. *Let us consider the mapping $\omega = \exp(z)$.*

(1) The lines

$$\{z \in \mathbb{C} : \operatorname{Re} z = x_0\},$$

which are parallel to the imaginary axis in the z-plane are mapped into circles

$$\{\omega \in \mathbb{C} : |\omega| = e^{x_0}\},$$

with center at the origin of the ω-plane.

(2) The lines

$$\{z \in \mathbb{C} : \operatorname{Im} z = y_0\},$$

which are parallel to the real axis in the z-plane are mapped into rays

$$\{\omega \in \mathbb{C} : \arg \omega = y_0\}$$

emanating from the origin in the ω-plane.

Proof.

(1) Let $z = x_0 + iy$, where $x_0 \in \mathbb{R}$ is fixed and $y \in \mathbb{R}$ is a variable. Then

$$\exp(z) = \exp\big(x_0 + iy\big) = e^{x_0}\left(\cos y + i \sin y\right).$$

That is, the radius stays constant, while the argument $\cos y + i \sin y$ varies with y.

(2) Let $z = x + iy_0$, where $y_0 \in \mathbb{R}$ is fixed and $x \in \mathbb{R}$ varies. Then

$$\exp\big(x + iy_0\big) = e^x \left(\cos y_0 + i \sin y_0\right).$$

That is, the argument stays constant, while the radius e^x varies with x.

□

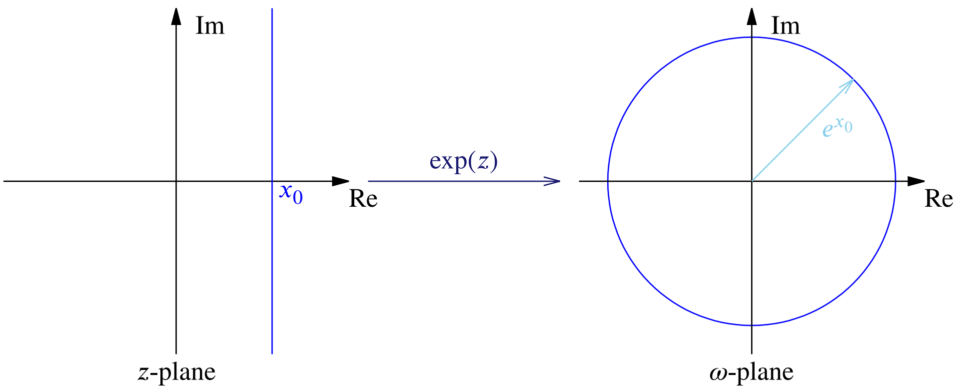


Figure 7.2: The mapping $\omega = \exp(z)$ in the proposition 7.1.1(1) maps lines $\{z \in \mathbb{C} : \operatorname{Re} z = x_0\}$, which are paraller to the imaginary axis in the z-plane, into origin centered cirles with radius e^{x_0} in the ω-plane.

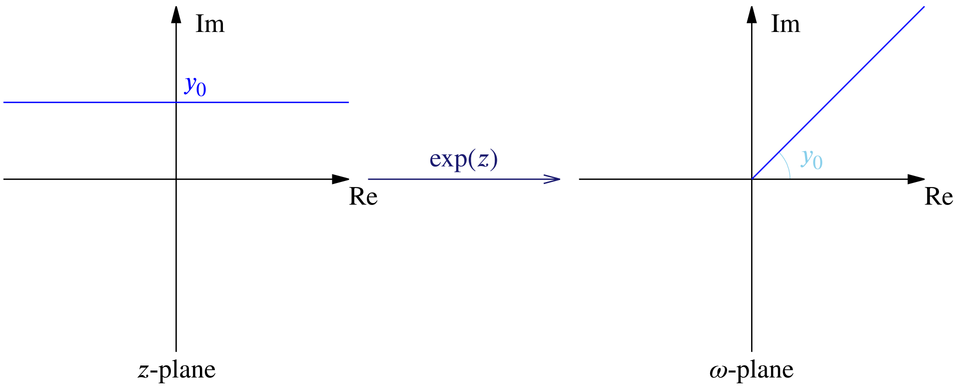


Figure 7.3: The mapping $\omega = \exp(z)$ in the proposition 7.1.1(2) maps lines $\{z \in \mathbb{C} : \operatorname{Im} z = y_0\}$, which are paraller to the real axis in the z-plane, into rays $\{\omega \in \mathbb{C} : \arg \omega = y_0\}$ emanating from the origin in the ω-plane.

Proposition 7.1.2. *The exponential mapping*

$$\Omega = \{z \in \mathbb{C} : 0 \le \operatorname{Im}(z) < 2\pi\} \to \mathbb{C} \setminus \{0\}, \qquad z \mapsto \exp(z),$$

is one-to-one and onto.

Proof. Note that $\exp(z) \neq 0$, for all $z \in \mathbb{C}$.

One-to-one: Let $z_1 = x_1 + iy_1 \in \Omega$ and $z_2 = x_2 + iy_2 \in \Omega$ be such that $\exp(z_1) = \exp(z_2)$. We have to show that $z_1 = z_2$. Since $\exp(z_1) = \exp(z_2)$, we have

$$e^{x_1} = \left|\exp(z_1)\right| = \left|\exp(z_2)\right| = e^{x_2}.$$

Hence, $x_1 = x_2$. Since

$$y_1 + 2\pi n = \arg\Big(\exp(z_1)\Big) = \arg\Big(\exp(z_2)\Big) = y_2 + 2\pi k, \quad n, k \in \mathbb{Z},$$

we have $y_1 - y_2 = 2\pi h$, where $h \in \mathbb{Z}$. By the definition of Ω, $0 \le y_1, y_2 < 2\pi$, we obtain $y_1 = y_2$. Thus $z_1 = z_2$.

Onto: Let $\omega \in \mathbb{C} \setminus \{0\}$ be given. We have to show that there exists $z \in \Omega$ for which $\exp(z) = \omega$. Choose $\alpha \in [0, 2\pi)$ such that $\alpha = \arg \omega$ and write $z = \ln|\omega| + i\alpha$. Then $z \in \Omega$ and

$$\exp(z) = \exp\big(\ln|\omega| + i\alpha\big) = e^{\ln|\omega|}\,\big(\cos\alpha + i\sin\alpha\big) = |\omega|\,\big(\cos\alpha + i\sin\alpha\big) = \omega.$$

□

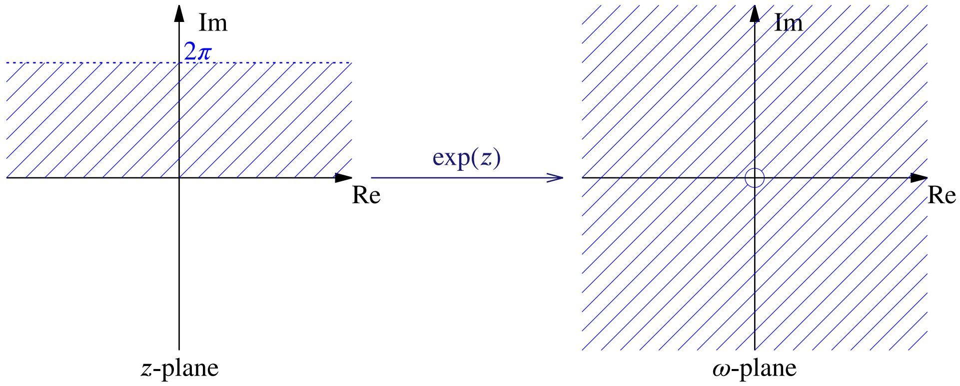


Figure 7.4: The mapping $\omega = \exp(z)$ in the proposition 7.1.2 maps the strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) < 2\pi\}$, into set $\mathbb{C} \setminus \{0\}$ in the ω-plane.

8 Complex logarithms

8.0.1 Argument

The identity

$$z = |z|\exp(i\alpha) = |z|\left(\cos\alpha + i\sin\alpha\right)$$

is not uniquely determined. This is the fundamental cause of many-valuedness in complex function theory.

Recall that for any $z \neq 0$ we defined the argument of z to be $\arg(z) = \alpha + 2\pi k$, where $k \in \mathbb{Z}$ and α is any fixed real number such that

$$\frac{z}{|z|} = \cos\alpha + i\sin\alpha.$$

So, $\arg(z)$ is not a single number, but all numbers of the form $\alpha + 2\pi k$, $k \in \mathbb{Z}$.

8.0.2 Complex logarithms: the inverse of the exponential function

Recall that for $e^x : \mathbb{R} \to \left(0,\infty\right)$ there exists an inverse function, which is $\ln : \left(0,\infty\right) \to \mathbb{R}$. That is, for each positive real number x, there exists a unique real solution $y = \ln x$ to the equation $e^y = x$. We work with logarithms to the base e and we write $\ln x$ for the real logarithm.

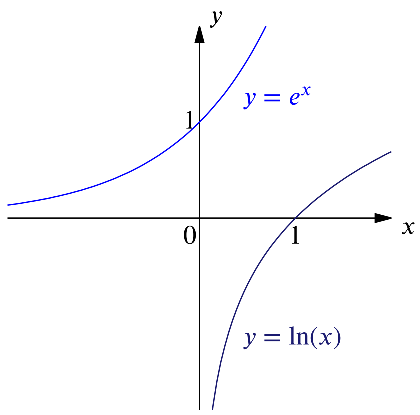


Figure 8.1: The exponential function $e^x : \mathbb{R} \to (0,\infty)$ and its inverse, the natural logarithm $\ln : (0,\infty) \to \mathbb{R}$.

In the complex plane we seek solutions to the equation $\exp(z) = \omega$. We write $\log z$ for the complex logarithm. Recall, however, that the exponential of a complex variable $\exp(z) : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is not bijective. We seek all possible values of $\log(z)$.

Suppose that $\omega \in \mathbb{C} \setminus \{0\}$. Let us write

$$\omega = \exp(z) = \exp(u + iv), \qquad \text{and} \qquad \arg\omega = v + 2\pi k, \quad u, v \in \mathbb{R} \text{ and } k \in \mathbb{Z}.$$

Note that $\arg(\omega)$ is the collection of all the numbers of the form $\alpha + 2\pi k$, $k \in \mathbb{Z}$.

For $z \neq 0$, we define the complex logarithm

$$\log z = \ln|z| + i\arg(z) + 2\pi i k, \qquad k \in \mathbb{Z},$$

where $\arg(z)$ is one of the values of the argument. Alternatively, we may say the multi-valued complex logarithm is given by

$$\log(z) = \ln|z| + i\arg(z),$$

where $\arg(z)$ is the multi-valued argument function.

Examples.

(1) $\log 1 = \ln\left|1\right| + i2\pi k = i2\pi k, \qquad k \in \mathbb{Z}.$

(2) $\log\left(1+i\right) = \ln\left(\sqrt{2}\right) + i\left(\frac{\pi}{4} + 2\pi k\right), \qquad k \in \mathbb{Z}.$

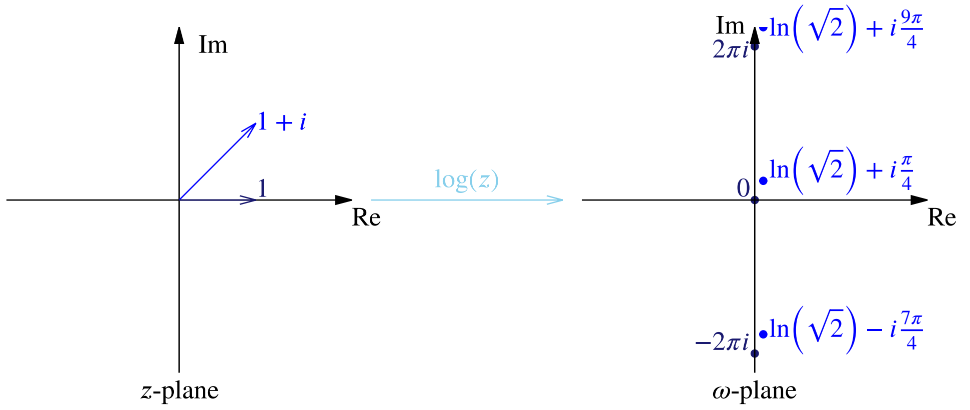


Figure 8.2: Some values of the multi-valued complex logarithms log 1 and $\log\left(1+i\right)$.

8.0.3. In order to get a single value for $\omega = \log(z)$ we need to make restrictions on ω; for example, we can restrict the imaginary part of ω to the interval $(-\pi, \pi]$ or $[0, 2\pi)$. Such a restriction is called a branch of the complex logarithm function.

8.0.4 Branches of the argument

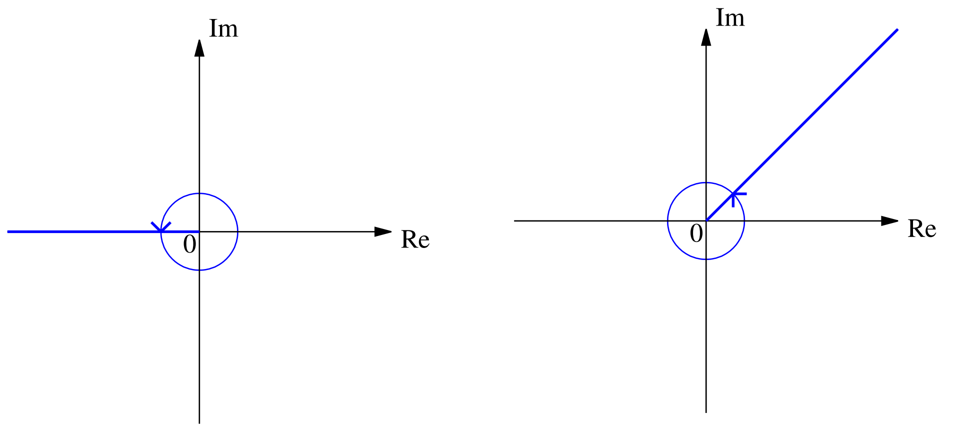
The multi-valued complex logarithm is given by

$$\log(z) = \ln|z| + i\arg(z).$$

So we need to find a branch for the multi-valued arg function.

The standard branch of the argument, $\mathrm{Arg}_{(-\pi,\pi]}$, which takes values in $(-\pi, \pi]$ is one of a branch of arg. In fact, for every half-open interval $(\alpha, \alpha+2\pi]$ we can create a branch of arg, $\mathrm{Arg}_{(\alpha,\alpha+2\pi]}(z)$, which takes values in that interval. That is, $\omega = \arg(z)$ such that $\alpha < \arg(z) \le \alpha + 2\pi$. The branch $\mathrm{Arg}_{(\alpha,\alpha+2\pi]}$ has a branch cut on the ray

$$\{z : \alpha = \arg(z)\}.$$



(a) $\mathrm{Arg}_{(-\pi,\pi]}(z)$ (b) $\mathrm{Arg}_{\left(\frac{\pi}{4},\frac{9\pi}{4}\right]}(z)$

Figure 8.3: The standard branch of the argument of complex numbers (left) and another branch of the argument.

8.0.5 Branches of the complex logarithm

Every branch $\mathrm{Arg}_{(\alpha,\alpha+2\pi]}$ gives rise to a branch $\mathrm{Log}_{(\alpha,\alpha+2\pi]}$ of the logarithm

$$\mathrm{Log}_{(\alpha,\alpha+2\pi]}(z) = \ln|z| + i\,\mathrm{Arg}_{(\alpha,\alpha+2\pi]}(z).$$

That is, $\omega = \log(z), \qquad \alpha < \arg(\omega) \le \alpha + 2\pi.$

Example.

$$\mathrm{Log}_{\left(\frac{\pi}{2},\frac{5\pi}{2}\right]}\left(1+i\right) = \ln\left(\sqrt{2}\right) + i\frac{9\pi}{4}.$$

8.0.6. The analyticity of a function has been defined in an open set. So we are interested in the branch of the logarithm $\mathrm{Log}_{(\alpha,\alpha+2\pi]}$ when it is defined in an open set

$$\mathbb{C} \setminus \{z = r\exp(i\alpha) : r \ge 0\},$$

and $\mathrm{Log}_{(\alpha,\alpha+2\pi]}$ maps the above domain onto the set

$$\{\omega \in \mathbb{C} : \alpha < \mathrm{Im}(\omega) < \alpha + 2\pi\}.$$

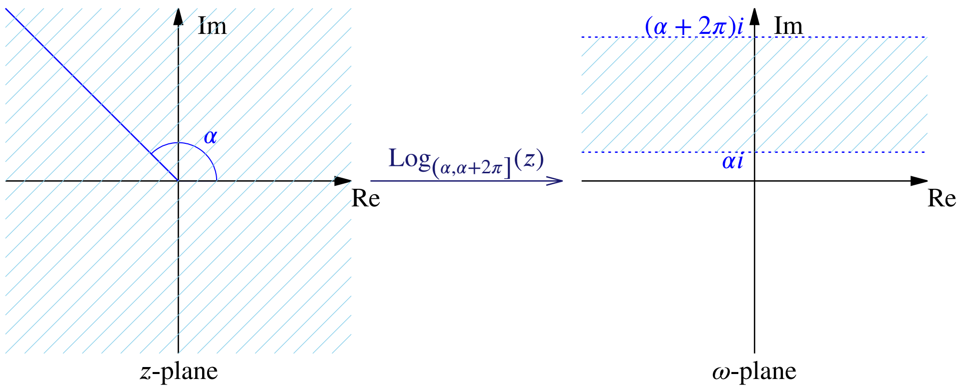


Figure 8.4: The branch $\mathrm{Log}_{(\alpha,\alpha+2\pi]}$ of the complex logarithm maps the complex plane, minus a ray $\{z = r\exp(i\alpha) : r \ge 0\}$, to a strip $\{\omega \in \mathbb{C} : \alpha < \mathrm{Im}(\omega) < \alpha + 2\pi\}$.

So we could have defined the branch of the logarithm as follows:

Let $\alpha \in \mathbb{R}$ be given. Write

$$\Omega_\alpha = \{z = r\exp(i\beta) : r > 0, \alpha < \beta < \alpha + 2\pi\}.$$

We define

$$\mathrm{Log}_{(\alpha,\alpha+2\pi]}(z) = \ln r + i\beta,$$

where (r, β) is a unique solution of the equation $z = r\exp(i\beta)$ such that $r > 0$ and $\alpha < \beta < \alpha + 2\pi$.

Remark. In the literature one might bump into the following definitions:

(1) Given $\alpha \in \mathbb{R}$, the half-open strip/belt $\{\omega \in \mathbb{C} : \alpha < \mathrm{Im}(\omega) \le \alpha + 2\pi\}$ is called a fundamental domain of the exponential function with respect to α, because the exponential function is invertible on this set.

(2) The principal branch of the complex logarithm is used when α is taken from the interval $[-2\pi, 0]$ and then fixed. Depending on the book or the lectures, α has been chosen, for example, as $\alpha = -\pi$ or $\alpha = 0$.

Some remarks

Remark. Note that

$$\exp\left(\log(z)\right) = z, \qquad \text{but}$$
$$\log\left(\exp(z)\right) = z + 2k\pi i, \ k \in \mathbb{R}.$$

Remark. All though

$$\log(zw) = \log(z) + \log(w)$$
$$\log\left(\frac{z}{w}\right) = \log(z) - \log(w).$$

This is not true for the branches of the logarithm.

Example 8.0.7.

$$0 = \log_{(\pi,\pi]}\left(-1 \cdot -1\right) = 2\pi.$$

Remark. The $\log_{(\alpha,\alpha+2\pi]}$ branch is complex differentiable everywhere except at the branch $\left\{z : \arg(z) = \alpha\right\}$ and at the origin where it is undefined.

Example 8.0.8. From the definition

$$\begin{aligned} i^{\pi} &= \exp\left(\pi \log(i)\right) \\ &= \exp\left(\pi^2 i \left(\frac{1}{2} + 2n\right)\right), \quad n \in \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} i^{i} &= \exp\left(i \log(i)\right) \\ &= \exp\left(-\pi \left(\frac{1}{2} + 2n\right)\right), \quad n \in \mathbb{Z}. \end{aligned}$$

Proposition 8.0.9. *Let $\alpha \in \mathbb{R}$ be given. Let*

$$\Omega_\alpha = \{z = r\exp(i\beta) : r > 0, \alpha < \beta < \alpha + 2\pi\}.$$

Now

(1) *The function* $\mathrm{Log}_{(\alpha,\alpha+2\pi]} : \Omega_\alpha \to \mathbb{C}$ *is analytic.*

(2) $\frac{\partial}{\partial z}\mathrm{Log}_{(\alpha,\alpha+2\pi]}(z) = \frac{1}{z}$, *for all* $z \in \Omega_\alpha$.

(3) $\mathrm{Log}_{(\alpha,\alpha+2\pi]}\left(\Omega_\alpha\right) = \{\omega \in \mathbb{C} : \alpha < \mathrm{Im}(\omega) < \alpha + 2\pi\}$.

(4) $\exp\Big(\mathrm{Log}_{(\alpha,\alpha+2\pi]}(z)\Big) = z$, *for all* $z \in \Omega_\alpha$.

(5) *If* $z_1, z_2 \in \Omega_\alpha$, *then*

$$\mathrm{Log}_{(\alpha,\alpha+2\pi]}\left(z_1 z_2\right) = \mathrm{Log}_{(\alpha,\alpha+2\pi]}\left(z_1\right) + \mathrm{Log}_{(\alpha,\alpha+2\pi]}\left(z_2\right) + i2\pi k, \quad \textit{for some } k \in \mathbb{Z}.$$

Proof.

(1) Homework.

(2) Let us write $\mathrm{Log}(z) := \mathrm{Log}_{(0,2\pi]}(z)$. Start with $\exp\big(\mathrm{Log}(z)\big) = z,\ z \neq 0$. Using the chain rule, we get

$$\exp\big(\mathrm{Log}(z)\big)\frac{\partial}{\partial z}\mathrm{Log}(z) = 1,$$

and so

$$z\frac{\partial}{\partial z}\mathrm{Log}(z) = 1, \quad \text{hence} \quad \frac{\partial}{\partial z}\mathrm{Log}(z) = \frac{1}{z}.$$

(3) Ok.

(4) Ok.

(5) Ok.

□

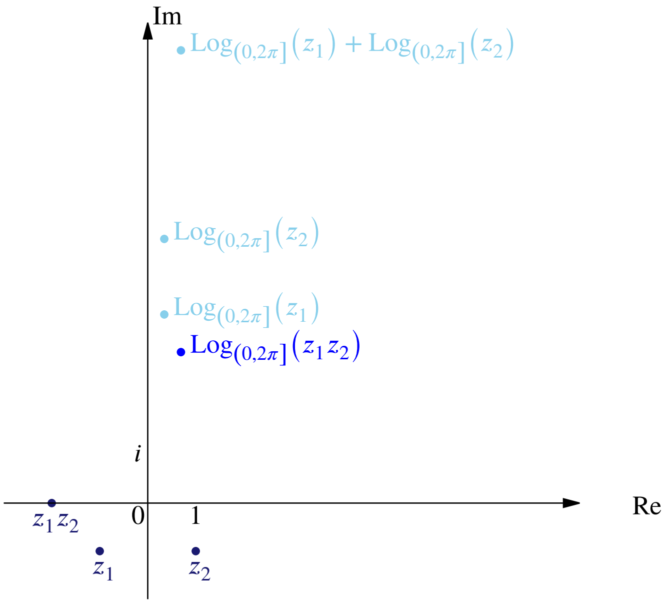


Figure 8.5: As shown in the example 8.0.10, the complex logarithm has the property 8.0.9(5). Here $z_1 = -1 - i$ and $z_2 = 1 - i$ so the product $z_1 z_2 = (-1 - i)(1 - i) = -2$. Values of the corresponding logarithms are also shown.

Example 8.0.10 (To emphasize (5))**.** Let $z_1 = -1 - i$, $z_2 = 1 - i$, thus $z_1 z_2 = -2$. Now we have

$$\begin{aligned}
\mathrm{Log}_{(0,2\pi]}(z_1 z_2) &= \mathrm{Log}_{(0,2\pi]}(-2) = \ln(2) + i\pi,\\
\mathrm{Log}_{(0,2\pi]}(z_1) &= \ln\sqrt{2} + i\frac{5\pi}{4},\\
\mathrm{Log}_{(0,2\pi]}(z_2) &= \ln\sqrt{2} + i\frac{7\pi}{4}.
\end{aligned}$$

Hence,

$$\mathrm{Log}_{(0,2\pi]}(z_1) + \mathrm{Log}_{(0,2\pi]}(z_2) = 2\ln\sqrt{2} + i3\pi = \ln(2) + i\pi + i2\pi = \mathrm{Log}_{(0,2\pi]}(z_1 z_2) + i2\pi.$$

Examples 8.0.11.

(1) Solve the equation $\exp(z) = 1 + i$.

<u>*Solution*</u>. $z = \log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right), \quad k \in \mathbb{Z}.$

(2) Find all solutions to $\cos(z) = 2$.

we rewrite this as
$$\frac{1}{2}\left(\exp(iz) + \exp(-iz)\right) = 2,$$
which is equivalent to
$$\left(\exp(iz) + \exp(-iz)\right) = 4.$$
Writing $\exp(iz) = \omega$, where $\omega \neq 0$, we obtain

$$\begin{aligned}
& \omega - 4 + \frac{1}{\omega} = 0,\\
\Leftrightarrow\quad & \omega^2 - 4\omega + 1 = 0,\\
\Leftrightarrow\quad & (\omega - 2)^2 = 3,\\
\Leftrightarrow\quad & \omega = 2 \pm \sqrt{3}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\exp(iz) = 2 \pm \sqrt{3}, \\
\Leftrightarrow\quad & iz = \log\left(2 \pm \sqrt{3}\right) = \ln\left(2 \pm \sqrt{3}\right) + i2\pi k, \\
\Leftrightarrow\quad & z = -i\ln\left(2 \pm \sqrt{3}\right) + 2\pi k = 2\pi k - i\ln\left(2 \pm \sqrt{3}\right), \quad k \in \mathbb{Z}.
\end{aligned}
$$

8.0.12 General power function

If $z \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{C}$, then we define

$$z^a = \exp\left(a \log z\right).$$

Since $\log z = \ln|z| + i \arg z + i2\pi k$, where $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
z^a &= \exp\left(a \log z\right) \\
&= \exp\Big(a\left(\ln|z| + i \arg z + i2\pi k\right)\Big) \\
&= e^{\ln|z|^a} \exp\Big(ia\left(\arg z + 2\pi k\right)\Big) \\
&= |z|^a \exp\Big(ia\left(\arg z + 2\pi k\right)\Big), \quad k \in \mathbb{Z}.
\end{aligned}
$$

9 Integrals

9.1 Complex-valued functions of real variables

If $f(t) = u(t) + iv(t)$ is a complex-valued function of a real variable t, where u and v are real-valued, the definite integral of $f(t)$ over an interval $a \le t \le b$ is defined as

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t, \tag{9.1.1}$$

provided that the individual integrals on the right-hand side exist. Thus

$$\begin{aligned} \operatorname{Re}\int_a^b f(t)\,\mathrm{d}t &= \int_a^b \operatorname{Re}\big(f(t)\big)\,\mathrm{d}t \\ \operatorname{Im}\int_a^b f(t)\,\mathrm{d}t &= \int_a^b \operatorname{Im}\big(f(t)\big)\,\mathrm{d}t. \end{aligned} \tag{9.1.2}$$

Recall that if u and v are piecewise continuous on the interval $\big[a, b\big]$, then the corresponding integrals exist.

The fundamental theorem of calculus can be extended to apply to the integrals of 9.1.2. Suppose that the functions

$$f(t) = u(t) + iv(t) \qquad \text{and} \qquad F(t) = U(t) + iV(t)$$

are continuous on $\big[a, b\big]$. If f is differentiable, and $F'(t) = f(t)$ when $t \in \big[a, b\big]$, then

$$U'(t) = u(t) \qquad \text{and} \qquad V'(t) = v(t).$$

Hence

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Example 9.1.3.

$$\int_0^{\frac{\pi}{4}} \exp(it)\,\mathrm{d}t = \frac{1}{i} \Big/_{0}^{\frac{\pi}{4}} \exp(it) = \frac{1}{i}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - 1\right) = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).$$

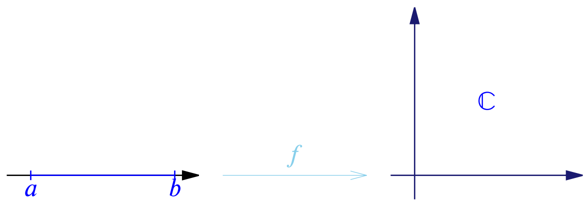


Figure 9.1: Function $f : [a, b] \subset \mathbb{R} \to \mathbb{C}$.

9.2 Curves and contours

Let $\big[\alpha, \beta\big]$, $-\infty < \alpha \le \beta < \infty$, be a closed and bounded interval in $\mathbb{R}$. A parametrized curve γ with parameter interval $\big[\alpha, \beta\big]$ is a continuous function $\gamma : \big[\alpha, \beta\big] \to \mathbb{C}$. The point $\gamma(\alpha)$ is the so called initial point of γ and $\gamma(\beta)$ is the final point of γ. If $\gamma(\alpha) = \gamma(\beta)$, then γ is said to be closed. The parametrized curve γ is smooth, if it is differentiable and $\gamma'(t) \ne 0$ for all $t \in \big[\alpha, \beta\big]$.

Figure 9.2: A parametrized curve $\gamma : [0, 1] \subset \mathbb{R} \to \mathbb{C}$, $\gamma(t) = a + t(b - a)$.

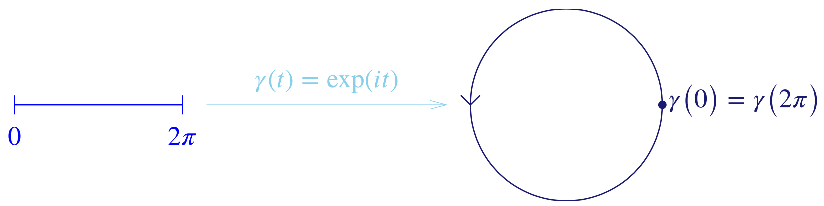


Figure 9.3: A closed parametrized curve $\gamma : [0, 2\pi] \subset \mathbb{R} \to \mathbb{C}$, $\gamma(t) = \exp(it)$.

Remark. At the points $t = \alpha$ and $t = \beta$, the quantities $\gamma'(\alpha)$ and $\gamma'(\beta)$ are the one-sided limits

$$\gamma'(\alpha) = \lim_{h\to 0^+} \frac{\gamma(\alpha+h) - \gamma(\alpha)}{h} \qquad \text{and} \qquad \gamma'(\beta) = \lim_{h\to 0^-} \frac{\gamma(\beta-h) - \gamma(\beta)}{h}, \qquad h > 0.$$

These quantities are called the right-hand derivative of $\gamma(t)$ at α and the left-hand derivative of $\gamma(t)$ at β, respectively.

The image $\gamma\left(\left[\alpha, \beta\right]\right) := \{\gamma(t) : t \in \left[\alpha, \beta\right]\}$ is denoted by $|\gamma|$. The curve is said to lie in a set $A \in \mathbb{C}$, if $|\gamma| \subset A$. A parametrized curve is piecewise smooth, if γ is continuous on $\left[\alpha, \beta\right]$ and if there exist points

$$\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_n = \beta,$$

where $\gamma(t)$ is smooth on the intervals $\left[\alpha_k, \alpha_{k+1}\right]$. Here, the right-hand derivative at α_k may differ from the left-hand derivative at α_k, $k = 1, 2, \ldots, n-1$.

Examples 9.2.1.

(1) The piecewise smooth curve in figure 9.4.

(2) Let

$$\gamma_1(t) = \exp\left(2\pi i t\right), \qquad t \in \left[0, 1\right]$$

and

$$\gamma_2(t) = \exp(it), \qquad t \in \left[0, 2\pi\right].$$

These two curves are two different parametrizations of the same curve.

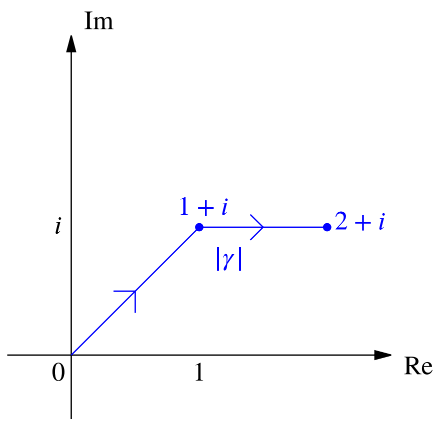


Figure 9.4: A piecewise smooth curve γ.

Definition 9.2.2. Two parametrizations

$$\gamma : \left[a, b\right] \to \mathbb{C} \qquad \text{and} \qquad \mu : \left[c, d\right] \to \mathbb{C}$$

are equivalent, if there exists a continuously differentiable bijection $s \mapsto t(s)$ from $\left[c, d\right] \to \left[a, b\right]$, so that $t'(s) > 0$ and

$$\mu(s) = \gamma\left(t(s)\right) = (\gamma \circ t)(s).$$

We say that μ is a reparametrization of γ.

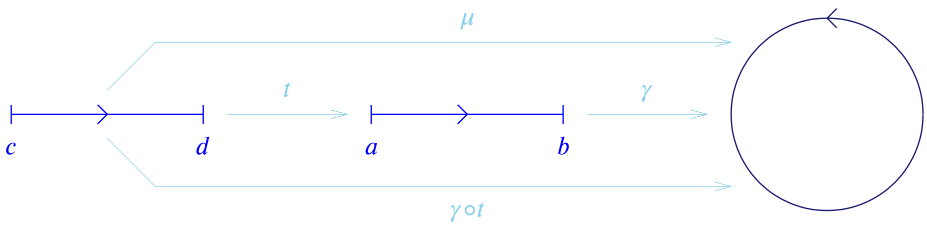


Figure 9.5: Two parametrizations $\gamma\colon \left[a,b\right] \to \mathbb{C}$ and $\mu\colon \left[c,d\right] \to \mathbb{C}$ are equivalent if there exists a continuously differentiable bijection $t\colon \left[c,d\right] \to \left[a,b\right]$ so that $t'(s) > 0$ and $\mu(s) = \gamma\left(t(s)\right) = (\gamma \circ t)\,(s)$.

9.2.3. The curve γ carries a built-in orientation determined by the direction in which $\gamma(t)$ traces out the image $|\gamma|$ as t increases from α to β. The positive orientation (counterclockwise) is the one that is given by the standard parametrization

$$\gamma(t) = z_0 + r\exp(it), \qquad t \in \left[0, 2\pi\right].$$

The negative orientation (clockwise) is given by

$$\gamma(t) = z_0 + r\exp(-it), \qquad t \in \left[0, 2\pi\right].$$

(a) counterclockwise

(b) clockwise

Figure 9.6: The positive or counterclockwise orientation and the negative or clockwise orientation of a curve.

Example 9.2.4. Let $f\colon \left[0,2\right] \to \mathbb{C}$, $f(t) = 2t + it^3$. Now

$$\int_0^2 f(t)\,\mathrm{d}t = \int_0^2 2t\,\mathrm{d}t + i\int_0^2 t^3\,\mathrm{d}t = 2\,\Big/_{0}^{2}\,\frac{1}{2}t^2 + i\,\Big/_{0}^{2}\,\frac{1}{4}t^4 = 4 + \frac{i}{4}\cdot 36 = 4 + i4.$$

Example 9.2.5. 1. Let $f(z) = z$, $\gamma(t) = 1 + i - t$, when $t \in \left[0,1\right]$. Now

$$\begin{aligned}\int_\gamma z\,\mathrm{d}z &= \int_0^1 \left(1 + i - t\right)\left(-1\right)\mathrm{d}t \\ &= \int_0^1 \left(1 + i - t\right)\mathrm{d}t + i\int_0^1 \left(-1\right)\mathrm{d}t \\ &= \Big/_{0}^{1}\left(\tfrac{1}{2}t^2 - t\right) + i\,\Big/_{0}^{1}\,(-t) = \tfrac{1}{2} - 1 - i = -\tfrac{1}{2} - i.\end{aligned}$$

2. Let $f(z) = |z|^2$, $\gamma(t) = t + i\frac{t^2}{2}$, when $t \in \left[0,1\right]$. Notice that $|\gamma(t)|^2 = \left(\sqrt{t^2 + \frac{t^4}{4}}\right)^2$ and

$\gamma'(t) = 1 + it$. Now

$$\int_\gamma |z|^2 \,\mathrm{d}z = \int_0^1 \left(\sqrt{t^2 + \tfrac{t^4}{4}}\right)^2 \big(1 + it\big)\,\mathrm{d}t$$
$$= \int_0^1 \left(t^2 + \tfrac{t^4}{4}\right)\mathrm{d}t + i\int_0^1 \left(t^3 + \tfrac{t^5}{4}\right)\mathrm{d}t$$
$$= \Big/_0^1 \left(\tfrac{1}{3}t^3 + \tfrac{1}{20}t^5\right) + i \Big/_0^1 \left(\tfrac{1}{4}t^4 + \tfrac{1}{24}t^6\right)$$
$$= \frac{1}{3} + \frac{1}{20} + i\left(\frac{1}{4} + \frac{1}{24}\right) = \frac{23}{60} + i\frac{7}{24}.$$

Example 9.2.6. 1. Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \cosh z = \frac{1}{2}\big(\exp(z) + \exp(-z)\big)$. Then

$$F : \mathbb{C} \to \mathbb{C}, \quad F(z) = \frac{1}{2}\big(\exp(z) - \exp(-z)\big) = \sinh z.$$

2. Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \frac{q}{\sin^2 z}$. Then

$$F : \mathbb{C} \setminus \{n\pi : n \in \mathbb{Z}\} \to \mathbb{C}), \quad F(z) = \frac{\cos z}{\sin z} + C, \quad C \in \mathbb{Z}.$$

9.2.7. Given γ, there exists a curve $-\gamma$ (or γ^-) with the same image set $|\gamma|$, but the opposite orientation:

$$-\gamma(t) = \gamma(\alpha + \beta - t), \qquad t \in \left[\alpha, \beta\right].$$

The curve $-\gamma$ is called the negative of γ.

Example. The negative of

$$\gamma(t) = \exp(it), \qquad t \in \left[0, 2\pi\right]$$

is

$$-\gamma(t) = \exp(-it) = \exp\Big(i\left(2\pi - t\right)\Big), \qquad t \in \left[0, 2\pi\right].$$

9.2.8. In the literature, one might bump into the term “an arc”. A set of points $z = (x, y)$ in the complex plane is said to be an arc if $x = x(t)$ and $y = y(t)$, where $\alpha \le t \le \beta$ and $x(t)$ and $y(t)$ are continuous functions of the real variable t. In this case we have $\gamma(t) = \big(x(t), y(t)\big)$.

9.2.9. Let $0 \le \alpha \le \alpha_1 \le \beta_1 \le \beta$ and let $\gamma : \left[\alpha, \beta\right] \to \mathbb{C}$ be a curve. By restricting the curve γ to $\left[\alpha_1, \beta_1\right]$, we obtain a new parametrized curve, namely $\gamma|_{[\alpha_1,\beta_1]}$. Suppose that $\alpha < s < \beta$ and $\gamma_1 = \gamma|_{[\alpha,s]}$ and $\gamma_2 = \gamma|_{[s,\beta]}$. The final point of γ_1 coincides with the initial point of γ_2 and $|\gamma|$ is traced by first tracing $|\gamma_1|$ and then tracing $|\gamma_2|$.

Let $\gamma_1 : \left[\alpha_1, \beta_1\right] \to \mathbb{C}$ and $\gamma_2 : \left[\alpha_2, \beta_2\right] \to \mathbb{C}$ be curves. As long as $\gamma_1\big(\beta_1\big) = \gamma_2\big(\alpha_2\big)$ we can form a join $\gamma_1 + \gamma_2$ (or $\gamma_1 \cup \gamma_2$):

$$\big(\gamma_1 + \gamma_2\big)(t) = \begin{cases} \gamma_1(t), & t \in \left[\alpha_1, \beta_1\right] \\ \gamma_2\big(t + \alpha_2 - \beta_1\big), & t \in \left[\beta_1, \beta_1 + \beta_2 - \alpha_2\right]. \end{cases}$$

A path is the join of finitely many smooth parametrized curves (H. A. Priestley). We, however, use the term contour.

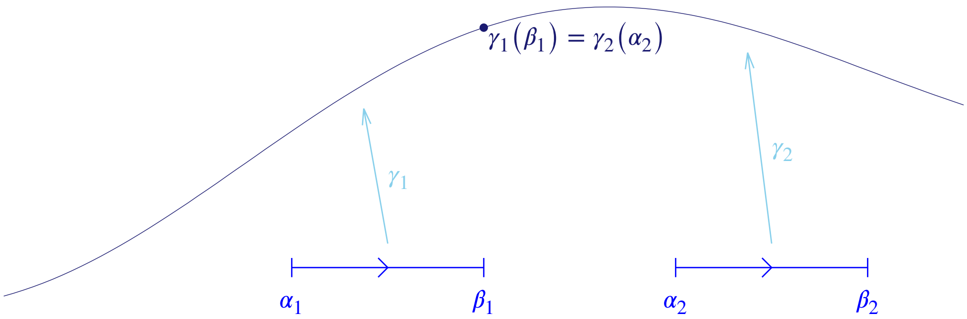


Figure 9.7: A contour (or a path) is the join of finitely many smooth parametrized curves.

Example 9.2.10. Let

$$\begin{aligned}
\gamma_1(t) &= t, && 0 \le t \le 1,\\
\gamma_2(t) &= 1 + it, && 0 \le t \le 1,\\
\gamma_3(t) &= 1 + i - t, && 0 \le t \le 1,\\
\gamma_4(t) &= i - it, && 0 \le t \le 1,
\end{aligned}$$

Then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is a contour.

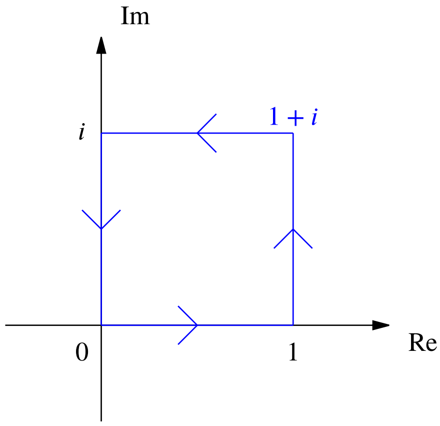


Figure 9.8: The contour $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ in the example 9.2.10.

9.2.11. A smooth or a piecewise smooth curve is said to be simple, if it is not self-intersecting, that is $\gamma(t) \neq \gamma(s)$ unless $s = t$. If the curve is closed to begin with, then it is simple if $\gamma(t) \neq \gamma(s)$, for all $s, t \in (\alpha, \beta)$ whenever $t \neq s$.

9.3 Integration along curves/contours

Let $f : A \to \mathbb{C}$ be continuous and $A \subset \mathbb{C}$. The integral along a smooth curve $\gamma : [\alpha, \beta] \to A$ is defined as

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\alpha^\beta f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t. \tag{9.3.1}$$

Example 9.3.2 (Important example)**.** Let $\gamma(t) = \omega + r\exp(it)$, $t \in [0, 2\pi]$ and $\omega \in \mathbb{C}$. What are the solutions to the integrals

$$\int_\gamma \frac{\mathrm{d}z}{z - \omega} \quad \text{and} \quad \int_\gamma (z - \omega)^n\,\mathrm{d}z, \quad n \in \mathbb{Z} \setminus \{-1\}\ ?$$

We begin by calculating the derivative of γ:

$$\gamma'(t) = ir\exp(it).$$

Now, for the first integral we have

$$\int_\gamma \frac{\mathrm{d}z}{z-\omega} = \int_0^{2\pi} \frac{ir\exp(it)}{\omega + r\exp(it) - \omega}\,\mathrm{d}t = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i,$$

and for the second integral we get

$$\begin{aligned}\int_\gamma (z-\omega)^n\,\mathrm{d}z &= \int_0^{2\pi} \left(\omega + r\exp(it) - \omega\right)^n ir\exp(it)\,\mathrm{d}t\\ &= ir^{n+1}\int_0^{2\pi} \exp\Big(i\left(n+1\right)t\Big)\,\mathrm{d}t\\ &= \frac{ir^{n+1}}{i\left(n+1\right)} \Big/_0^{2\pi} \exp\Big(i\left(n+1\right)t\Big) = 0.\end{aligned}$$

9.3.3 Basic properties

Proposition 9.3.4: Invariance. *The right-hand side integral of (9.3.1) is independent of the parametrization chosen for γ.*

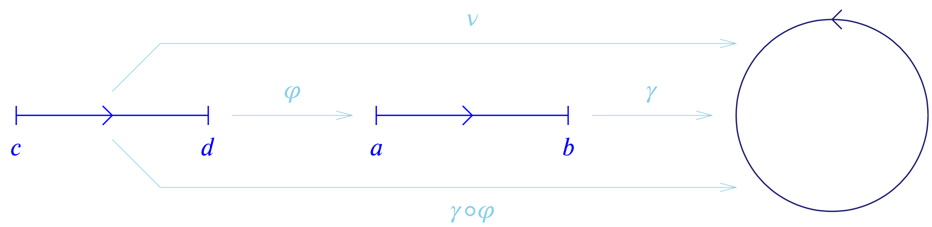


Figure 9.9: Reminder for the proof of theorem 9.3.4. Two parametrizations $\gamma : [a,b] \to \mathbb{C}$ and $\nu : [c,d] \to \mathbb{C}$ are equivalent if there exists a continuously differentiable bijection $\varphi : [c,d] \to [a,b]$ so that $\varphi'(s) > 0$ and $\nu(s) = \gamma\big(\varphi(s)\big) = (\gamma\circ\varphi)(s)$.

Proof. Let $\nu : [c,d] \to \mathbb{C}$ be a reparametrization of γ. Then, there exists a function $\varphi : [c,d] \to [a,b]$ such that φ is continuously differentiable and $\varphi'(t) > 0$ for all $t \in [c,d]$. The change of variables theorem and the chain rule imply that

$$\begin{aligned}\int_\nu f(z)\,\mathrm{d}z &= \int_c^d f\big(\nu(s)\big)\nu'(s)\,\mathrm{d}s\\ &\overset{\nu=\gamma\circ\varphi}{=} \int_c^d f\Big((\gamma\circ\varphi)(s)\Big)(\gamma\circ\varphi)'(s)\,\mathrm{d}s\\ &= \int_c^d f\Big(\gamma\big(\varphi(s)\big)\Big)\gamma'\big(\varphi(s)\big)\varphi'(s)\,\mathrm{d}s\\ &\overset{\varphi(s)=t}{=} \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t = \int_\gamma f(z)\,\mathrm{d}z.\end{aligned}$$

□

Proposition 9.3.5. *Integration of continuous functions along smooth curves is linear, that is, if $\lambda, \mu \in \mathbb{C}$ and f, g are continuous functions on a smooth curve γ, then*

$$\int_\gamma \big[(\lambda f)(z) + (\mu g)(z)\big]\,\mathrm{d}z = \lambda\int_\gamma f(z)\,\mathrm{d}z + \mu\int_\gamma g(z)\,\mathrm{d}z.$$

Proof. The claim follows from the definition and the linearity of the Riemann integral. We refer to (9.1). □

Proposition 9.3.6: The integral of the negative. *If γ^- is γ with the reverse orientation, then*

$$\int_{\gamma^-} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z.$$

Proof. By the invariance proposition, we may choose the parameter interval for γ as $[0, 1]$. Then, by the definition of the negative,

$$\gamma^- : \; \big[0, 1\big] \to \mathbb{C}, \quad \gamma^-(t) = \gamma\big(1 - t\big).$$

The change of variable formula with $s = 1 - t$ yields

$$\begin{aligned} \int_{\gamma^-} f(z)\,\mathrm{d}z &= \int_0^1 f\big(\gamma^-(t)\big)(\gamma^-)'(t)\,\mathrm{d}t \\ &= \int_0^1 f\Big(\gamma\big(1-t\big)\Big)\gamma'\big(1-t\big)\big(-1\big)\,\mathrm{d}t \\ &= \int_1^0 f\big(\gamma(s)\big)\gamma'(s)\big(-1\big)\big(-1\big)\,\mathrm{d}s \\ &= -\int_0^1 f\big(\gamma(s)\big)\gamma'(s)\,\mathrm{d}s = -\int_\gamma f(z)\,\mathrm{d}z. \end{aligned}$$

□

Proposition 9.3.7: Joining. *Let $\gamma : \big[\alpha, \beta\big] \to \mathbb{C}$ be a smooth curve. Let $\alpha < s < \beta$ and $\gamma_1 = \gamma|_{[\alpha,s]}$ and $\gamma_2 = \gamma|_{[s,\beta]}$. If f is a continuous function on γ, then*

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Definition 9.3.8. If γ is a piecewise smooth curve, then the integral of f over γ is the sum of the integrals of f over smooth parts of γ; so if $\gamma(t)$ is a piecewise smooth parametrization, then

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n-1} \int_{\alpha_k}^{\alpha_{k+1}} f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t.$$

Definition 9.3.9. The length of a smooth curve $\gamma : \big[\alpha, \beta\big] \to \mathbb{C}$ is

$$\mathrm{length}(\gamma) = \int_\alpha^\beta \Big|\gamma'(t)\Big|\,\mathrm{d}t.$$

This definition is independent of the parametrization chosen for γ (arguing as in the invariance proposition). If γ is only piecewise smooth, then its length is the sum of the lengths of its smooth parts.

Example 9.3.10. Let $\gamma_0 : \big[0, 2\pi\big] \to \mathbb{C}$, $\gamma_0(t) = \exp(it)$. The length of γ_0 is

$$\mathrm{length}\big(\gamma_0\big) = \int_0^{2\pi} \big|i\exp(it)\big|\,\mathrm{d}t = \int_0^{2\pi} \mathrm{d}t = 2\pi.$$

Let $\gamma_1 : \big[0, \pi\big] \to \mathbb{C}$, $\gamma_1(t) = \exp\big(4it\big)$. The length of γ_1 is

$$\mathrm{length}\big(\gamma_1\big) = \int_0^{\pi} \Big|4i\exp\big(4it\big)\Big|\,\mathrm{d}t = \int_0^{\pi} 4\,\mathrm{d}t = 4\pi.$$

9.3.11 Integrals along curves with respect to the arclength

Given a smooth curve γ in $\mathbb{C}$ parametrized by $\gamma : [\alpha, \beta] \to \mathbb{C}$ and a continuous function f on γ, we define the integral of f along γ with respect to the arclength by

$$\int_\gamma f\left|\mathrm{d}z\right| = \int_\gamma f(z)\left|\mathrm{d}z\right| = \int_\alpha^\beta f\left(\gamma(t)\right)\left|\gamma'(t)\right| \mathrm{d}t.$$

Remark. Invariance, linearity and joining property of this integral are as the properties of the integral (9.3.1). The difference is in determining the integral along the negative:

$$\int_{\gamma^-} f(z)\left|\mathrm{d}z\right| = \int_\gamma f(z)\left|\mathrm{d}z\right|.$$

Remark. If a curve γ is parametrized with respect to its arclength, it means that

$$\gamma : \left[0, \mathrm{length}(\gamma)\right] \to \mathbb{C}.$$

Lemma 9.3.12: Estimation lemma.

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \sup_{z\in\gamma}|f(z)|\,\mathrm{length}(\gamma).$$

Proof.

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\int_\alpha^\beta f\left(\gamma(t)\right)\gamma'(t)\,\mathrm{d}t\right| \\ &\le \sup_{t\in[\alpha,\beta]}\left|f\left(\gamma(t)\right)\right| \int_\alpha^\beta \left|\gamma'(t)\right| \mathrm{d}t \\ &= \sup_{z\in\gamma}|f(z)|\,\mathrm{length}(\gamma).\end{aligned}$$

□

10 Theorems related to an antiderivative

Definition 10.0.1. Let f be a function defined on an open set $\Omega \subset \mathbb{C}$. A primitive function (or an antiderivative) for f on Ω is a function F that is analytic on Ω and verifies that $F'(z) = f(z)$ for all $z \in \Omega$.

Example 10.0.2. Both functions $F(z) = \frac{z^4}{4}$ and $\tilde{F}(z) = \frac{z^4}{4} + 10^{10}$ are antiderivatives of the function $f(z) = z^3$.

Theorem 10.0.3. *If a continuous function f has an antiderivative F in Ω and γ is a smooth curve in Ω with the initial point ω_1 and the end point ω_2, then*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\omega_2) - F(\omega_1).$$

Proof. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a parametrization for γ. Then $\gamma(\alpha) = \omega_1$ and $\gamma(\beta) = \omega_2$. We have that

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t),$$

and hence

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}t}(F\circ\gamma)(t)\,\mathrm{d}t$$
$$= F(\gamma(\beta)) - F(\gamma(\alpha)) = F(\omega_2) - F(\omega_1),$$

where we have applied the Fundamental Theorem of Calculus. □

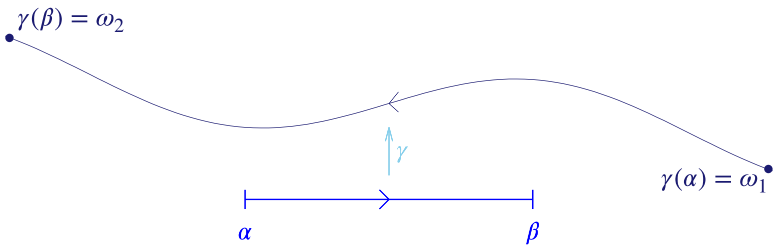


Figure 10.1: A parametrized smooth curve γ in the proof of Theorem 10.0.3.

Remark 10.0.4. If γ is piecewise smooth, we obtain, using Theorem (10.0.3) and definition (9.3.8), a telescoping sum and we have

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n-1} \int_{\alpha_k}^{\alpha_{k+1}} f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$
$$= \sum_{k=0}^{n-1} \Big(F\big(\gamma(\alpha_{k+1})\big) - F\big(\gamma(\alpha_k)\big)\Big)$$
$$= F\big(\gamma(\alpha_n)\big) - F\big(\gamma(\alpha_0)\big) = F(\gamma(\beta)) - F(\gamma(\alpha)).$$

Example 10.0.5. Let $\gamma : [0, \pi] \to \mathbb{C}$, $\gamma(t) = t + i\frac{t^2}{\pi}$, and $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, $f(z) = \exp(z)$. An antiderivative of f is $F(z) = \exp(z)$ and hence

$$\int_\gamma \exp(z)\,\mathrm{d}z = \exp\left(\pi + i\frac{\pi^2}{\pi}\right) - \exp(0) = e^\pi \exp(i\pi) - 1$$
$$= e^\pi(\cos(\pi) + i\sin(\pi)) - 1 = -e^\pi - 1.$$

Corollary 10.0.6. *If γ is a closed smooth curve in an open set Ω and the function f is continuous and has an antiderivative, then*

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0.$$

Remark. Since γ is closed, the initial point and the end point coincide.

Example 10.0.7 (Important example)**.** The function $f(z) = \frac{1}{z}$ does not have an antiderivative in the open set $\mathbb{C} \setminus \{0\}$.

Namely, if $\gamma(t) = \exp(it), t \in [0, 2\pi]$, we calculated that

$$\int_{\gamma} f(z)\,\mathrm{d}z = 2\pi i \neq 0.$$

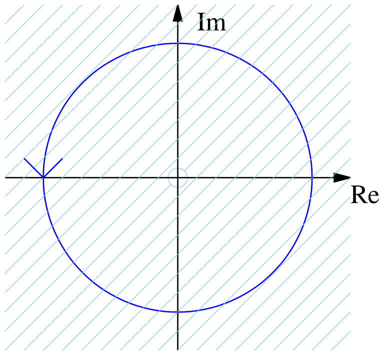


Figure 10.2: Curve $\gamma(t) = \exp(it), t \in [0, 2\pi]$ in the open set $\mathbb{C} \setminus \{0\}$ as in Example 10.0.7. This means that the function $f(z) = \frac{1}{z}$ does not have an antiderivative in this set.

Theorem 10.0.8: Fundamental theorem of Calculus II. *Suppose that f is a continuous function on a domain D in $\mathbb{C}$. If*

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0,$$

for all closed contours Γ lying entirely in D, then $f(z)$ has a primitive $F(z)$ throughout D.

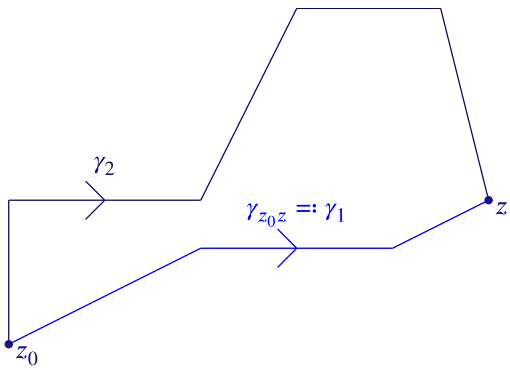


Figure 10.3: Two piecewise smooth curves γ_1 and γ_2 joining points z_0 and z in a domain D. As shown in the first part of the proof of the theorem 10.0.8, the integration is independent of a chosen piecewise smooth curve.

Proof. We start by showing that integration is independent of a piecewise smooth curve.

Let $z_0 \in D$ be arbitrarily chosen and then fixed. Let $z \in D$ be any other point in D.

Since D is a domain, D is open and connected, and in fact D is polygonally connected. Hence, there exists a polygonal curve in D from z_0 to z, that is

$$[z_0, z_1] \cup \cdots \cup [z_{n-1}, z_n = z] \subset D.$$

So we have $\gamma_{z_0 z} : [\alpha, \beta] \to D$, which is a piecewise smooth curve whose initial point is $\gamma_{z_0 z}(\alpha) = z_0$ and end point is $\gamma_{z_0 z}(\beta) = z$. Write $\gamma_{z_0 z} = \gamma_1$.

If there exists another piecewise smooth curve γ_2 joining z_0 to z, we have

$$0 = \int_{\gamma_1 * \gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2^-} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Thus $\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z$ and therefore the integration is independent of the piecewise smooth curve in D under the assumptions of this theorem.

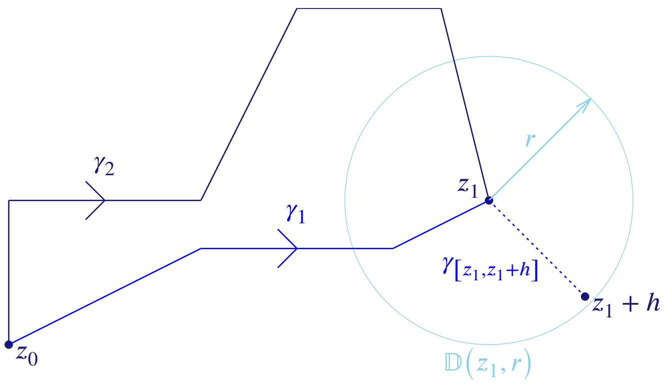


Figure 10.4: The second part of the proof of the theorem 10.0.8. Two piecewise smooth curves γ_1 and γ_2 joining fixed points z_0 and z_1 in a domain D and a smooth curve $\gamma_{[z_1, z_1+h]}$ joining point z_1 to any other point $h \in \mathbb{D}(z_1, r)$.

Now, we can define the function

$$F(z) = \int_{\gamma_{z_0 z}} f(\omega)\,\mathrm{d}\omega, \quad \text{for all } z \in D.$$

We have to show that $F'(z) = f(z)$ where $z \in D$.

We fixed the starting point $z_0 \in D$ in the beginning of the proof. Now we fix the end point as well, and label it $z_1 \in D$. We let $z_1 + h$ be any point distinct from z_1, which lies in some disc $\mathbb{D}(z_1, r)$, $r > 0$ that is small enough so that $\mathbb{D}(z_1, r) \subset D$.

We have $\gamma_{z_0 z_1}$ joining z_0 and z_1. Let $[z_1, z_1 + h]$ be the line segment joining z_1 and $z_1 + h$. The parametrized curve in this case is

$$\gamma_{[z_1, z_1+h]}(s) = z_1 + hs, \quad s \in [0, 1].$$

Because the integration is independent of the piecewise smooth curve, we have

$$F(z_1 + h) = \int_{\gamma_{z_0 z_1 + h}} f(\omega)\,\mathrm{d}\omega = \int_{\gamma_{z_0 z_1} * \gamma_{[z_1, z_1+h]}} f(\omega)\,\mathrm{d}\omega.$$

Then,

$$\frac{1}{h}\left(F(z_1 + h) - F(z_1)\right) - f(z_1) = \frac{1}{h}\left(\int_{\gamma_{z_0 z_1} * \gamma_{[z_1, z_1+h]}} f(\omega)\,\mathrm{d}\omega - \int_{\gamma_{z_0 z_1}} f(\omega)\,\mathrm{d}\omega\right) - f(z_1).$$

Note that since

$$\int_{\gamma_{z_0 z_1} * \gamma_{[z_1, z_1+h]}} f(\omega)\,\mathrm{d}\omega = \int_{\gamma_{z_0 z_1}} f(\omega)\,\mathrm{d}\omega + \int_{\gamma_{[z_1, z_1+h]}} f(\omega)\,\mathrm{d}\omega,$$

we are left with

$$\frac{1}{h}\Big(F\big(z_1+h\big) - F\big(z_1\big)\Big) - f\big(z_1\big) = \frac{1}{h}\left(\int_{\gamma_{[z_1,z_1+h]}} f(\omega)\,\mathrm{d}\omega\right) - f\big(z_1\big).$$

Recall that

$$\int_{\gamma_{[z_1,z_1+h]}} \mathrm{d}\omega = \int_0^1 h\,\mathrm{d}s = h,$$

and hence $\int_{\gamma_{[z_1,z_1+h]}} f\big(z_1\big)\,\mathrm{d}\omega = f\big(z_1\big)h$. With this, we have

$$\frac{1}{h}\Big(F\big(z_1+h\big) - F\big(z_1\big)\Big) - f\big(z_1\big) = \frac{1}{h}\left(\int_{\gamma_{[z_1,z_1+h]}} f(\omega) - f\big(z_1\big)\,\mathrm{d}\omega\right).$$

By the estimation lemma 9.3.12, we have

$$\begin{aligned}\left|\frac{F\big(z_1+h\big) - F\big(z_1\big)}{h} - f\big(z_1\big)\right| &= \left|\frac{1}{h}\right|\left|\int_{\gamma_{[z_1,z_1+h]}} \big(f(\omega) - f\big(z_1\big)\,\mathrm{d}\omega\right| \\ &\le \frac{1}{|h|}\sup_{\omega\in[z_1,z_1+h]}\left|f(\omega) - f\big(z_1\big)\right| \underbrace{\mathrm{length}\Big(\gamma_{[z_1,z_1+h]}\Big)}_{=|h|}.\end{aligned}$$

Since f is continuous, we have $\left|f(\omega) - f\big(z_1\big)\right| \to 0$ as $\omega \to z_1$, that is $h \to 0$.

This means that $F'\big(z_1\big) = f\big(z_1\big)$. □

So far, we have proved the following

Theorem 10.0.9. *Suppose that f is a continuous function on a domain D in $\mathbb{C}$. Then, f has a primitive on D if and only if*

$$\int_\Gamma f(z)\,\mathrm{d}z = 0$$

for all closed contours Γ in D.

Examples 10.0.10.

1. Let $\gamma : \big[0, 2\pi\big] \to \mathbb{C}$, $\gamma(t) = \exp(it)$. Then

$$\int_\gamma \overline{z}\,\mathrm{d}z = \int_0^{2\pi} \overline{\exp(it)} i\exp(it)\,\mathrm{d}t = i\int_0^{2\pi} \underbrace{\big|\exp(it)\big|^2}_{=1}\,\mathrm{d}t = 2\pi i \neq 0.$$

   Hence, the function $z \mapsto \overline{z}$ does not have a primitive on any domain which contains the unit circle.

2. Since the function $f(z) = z^2$ has a primitive, namely $F(z) = \frac{1}{3}z^3$, we know that

$$\int_\Gamma z^2\,\mathrm{d}z = 0$$

   for every closed contour in $\mathbb{C}$.

Remarks.

- The notation $\oint$ is often used instead of $\int$ when integrating over a closed contour.
- By convention, if an orientation of a closed contour is not specified, it is assumed to be counterclockwise.
- It is not always easy to check whether a function has a primitive or not.

Example. Let $f(z) = \exp\left(z^2\right)$.

11 Integral theorems

Theorem 11.0.1: Goursat's theorem. *Let R be a closed rectangle whose sides are parallel to the coordinate axes. Suppose that $R \subset \mathbb{D}(0,r) = \{z \in \mathbb{C} : |z| < r\}$ and that f is analytic in $\mathbb{D}(0,r)$. Then*

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0.$$

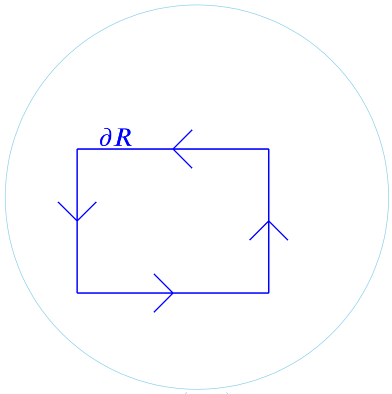


Figure 11.1: The Goursat's theorem 11.0.1 states that given a closed rectangle $R \subset \mathbb{D}(0,r) \subset \mathbb{C}$, with sides parallel to coordinate axes, and an analytic function f in $\mathbb{D}(0,r)$ then $\int_{\partial R} f(z)\,\mathrm{d}z = 0$.

Remark. This is one of the key results in this Complex Analysis I course.

Proof. The proof is based on the method of bisection.

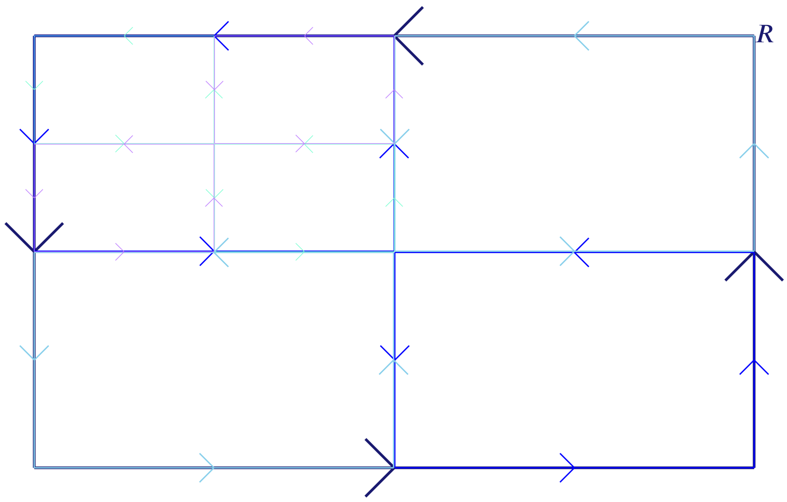


Figure 11.2: Bisections of the rectangle R and bisections of the bisected parts of the R.

Let us write $\mathcal{I} := \int_{\partial R} f(z)\,\mathrm{d}z$. Let us divide R into four congruent rectangles R_k, $k = 1,2,3,4$. Suppose that the boundary of R and the boundaries of R_k are all oriented counterclockwise. Hence

$$\int_{\partial R} f(z)\,\mathrm{d}z = \sum_{k=1}^{4} \int_{\partial R_k} f(z)\,\mathrm{d}z.$$

By the triangle inequality for some R_k, we have

$$|\mathcal{I}| \leq 4\left|\int_{\partial R_k} f(z)\,\mathrm{d}z\right|.$$

This R_k is denoted by R^1 and we set

$$\mathcal{I}_1 = \int_{\partial R^1} f(z)\,\mathrm{d}z.$$

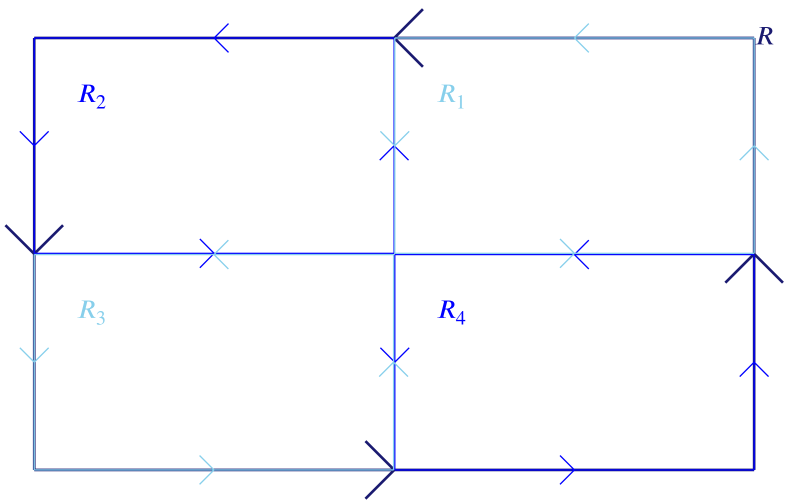


Figure 11.3: The rectangle R divided into four congruent rectangles $R_k, k = 1, 2, 3, 4$.

Now we divide R^1 into four congruent rectangles R^1_k. Again, for some $k = 1, 2, 3, 4$, we denote $R^1_k = R^2$ and get

$$|\mathcal{I}_1| \leq 4\left|\int_{\partial R^2} f(z)\,\mathrm{d}z\right|.$$

This process can be repeated indefinitely and we obtain a sequence of nested rectangles

$$R \supset R^1 \supset R^2 \supset \cdots \supset R^n \supset \cdots.$$

If we write $\mathcal{I}_j = \int_{\partial R^j} f(z)\,\mathrm{d}z$, we have the following property for these rectangles

$$\left|\mathcal{I}_j\right| \leq 4\left|\mathcal{I}_{j+1}\right|.$$

Hence $|\mathcal{I}| \leq 4^j\left|\mathcal{I}_j\right|$.

If the sidelengths of R are L_1 and L_2, then the sidelengths of R^j are $2^{-j}L_1$ and $2^{-j}L_2$. We show that the set $\bigcap_{j=1}^{\infty} R^j$ contains exactly one point.

Since $\operatorname{diam}(R^j) = 2^{-j}\sqrt{L_1^2 + L_2^2} \to 0$ as $j \to \infty$, the set $\bigcap_{j=1}^{\infty} R^j$ contains at most one point.

Let $z_j \in R^j$, $j = 1, 2, \ldots$ be arbitrarily chosen points. Since $z_k \in R^j$ whenever $k \geq j$, the sequence (z_k) is a Cauchy sequence and there exists

$$z^* = \lim_{k\to\infty} z_k.$$

Since each R^k is closed and z^* is in each R^k, we have $z^* \in \bigcap_{k=1}^{\infty} R^k$.

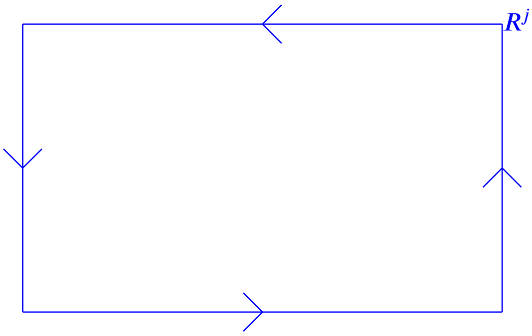


Figure 11.4: The integral $\mathcal{I}_j = \int_{\partial R^j} f(z)\,\mathrm{d}z$ is the integration of the function f over the boundary of the rectangle R^j counterclockwise.

Let $\varepsilon > 0$ be arbitrarily chosen and then fixed. We can choose $\delta > 0$ to be small enough so that f is analytic in $\mathbb{D}(z^*, \delta)$. Then

$$\left|f(z) - f(z^*) - f'(z^*)(z - z^*)\right| < \varepsilon |z - z^*| \tag{11.0.2}$$

whenever $|z - z^*| < \delta$.

We may choose j to be large enough so that $R^j \subset \mathbb{D}(z^*, \delta)$. So j is fixed now.

Since the constant function 1 has a primitive, i.e. z, and the function $g(z) = z$ has a primitive, i.e. $\frac{1}{2}z^2$, we know that

$$\int_{\partial R^j} \mathrm{d}z = 0$$

and

$$\int_{\partial R^j} z\,\mathrm{d}z = 0.$$

Hence,

$$\left|\mathcal{I}_j\right| = \left|\int_{\partial R^j} f(z)\,\mathrm{d}z\right| = \left|\int_{\partial R^j} \left(f(z) - f(z^*) - f'(z^*)(z - z^*)\right)\mathrm{d}z\right|.$$

By the estimation lemma 9.3.12 and 11.0.2 we have

$$\left|\mathcal{I}_j\right| \le \varepsilon |z - z^*| \operatorname{length}(\partial R^j).$$

Here,

$$\operatorname{length}(\partial R^j) = 2\left(2^{-j}L_1 + 2^{-j}L_2\right) = 2^{-j+1}(L_1 + L_2) \quad \text{and} \quad |z - z^*| \le 2^{-j}\sqrt{L_1^2 + L_2^2},$$

since $z, z^* \in R^j$. Hence,

$$\left|\mathcal{I}_j\right| \le \varepsilon\left(2^{-j}\sqrt{L_1^2 + L_2^2}\right)\left(2^{-j+1}(L_1 + L_2)\right) \quad \text{and} \quad |\mathcal{I}| \le 4^j\left|\mathcal{I}_j\right| \le 2\varepsilon (L_1 + L_2)^{\frac{3}{2}}.$$

Since $\varepsilon > 0$ was arbitrarily chosen, we have shown that $|\mathcal{I}| = \left|\int_{\partial R} f(z)\,\mathrm{d}z\right| = 0$. □

As the following, very important theorem shows, the assumption in Goursat's theorem can be weakened.

Theorem 11.0.3. *Let $\omega_0 \in \mathbb{C}$. Let $r > 0$ be given. If f is analytic in $\mathbb{D}(0, r) \setminus \{\omega_0\}$ and R is a closed rectangle in $\mathbb{D}(0, r)$ such that $\omega_0 \in R$ and f is continuous in $\mathbb{D}(0, r)$, then*

$$\int_{\partial R} f = 0.$$

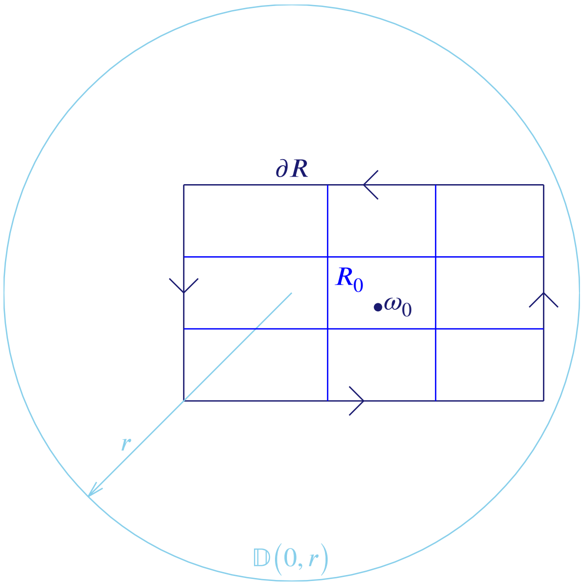


Figure 11.5: For the proof of the theorem 11.0.3 we divide the rectangle R into nine parts.

Proof. Suppose that $\omega_0 \in \operatorname{int} R$. We divide R into nine rectangles as shown in figure 11.5 and apply Goursat's theorem to all but the rectangle R^0. If the corresponding equations

$$\int_{\partial R} f = 0, \qquad j = 1, 2, \ldots, 8$$

are added, we obtain, after cancellations,

$$\int_{\partial R} f(z)\,\mathrm{d}z = \int_{\partial R^0} f(z)\,\mathrm{d}z.$$

Suppose that $\operatorname{length}\left(\partial R^0\right) = \varepsilon > 0$. By the estimation lemma 9.3.12 we have

$$\left|\int_{\partial R} f\right| = \left|\int_{\partial R^0} f\right| \le M\varepsilon, \quad \text{for some } M \in \mathbb{R}.$$

Since $\varepsilon > 0$ was arbitrarily chosen, the claim of the theorem follows.

In the case that $\omega_0 \in \partial R$, a similar procedure gives the claim. □

Remark. Later on we will find that, in fact, the assumptions that f is analytic in $\mathbb{D}(0, r) \setminus \{\omega_0\}$ and continuous on $\mathbb{D}(0, r)$ imply that f is analytic in $\mathbb{D}(0, r)$.

First we will prove the existence of antiderivatives in a disc as a consequence of Goursat's theorem.

Theorem 11.0.4. *An analytic function in an open disc has an antiderivative in that disc.*

Proof. Fix $z_0 \in \mathbb{C}$. Our goal is to show that there exists an analytic function F such that

$$F'(z) = f(z), \qquad \text{for all } z \in \mathbb{D}(z_0, R),$$

i.e. an antiderivative of f.

Denote $z_0 = x_0 + iy_0$ and, for each $z = x + yi \in \mathbb{D}(z_0, R)$, let

$$\gamma_z = \gamma_{[z_0, x+iy_0]} * \gamma_{[x+iy_0, z]},$$

i.e. the join of the line segments connecting the points z_0, $x + iy_0$ and z (as demonstrated in Figure 11.6). Note that the point $x + iy_0$ and the contour γ_z are included in $\mathbb{D}(z_0, R)$ for any

$z \in \mathbb{D}(z_0, R)$; this can be seen by drawing the line segment between z_0 and z and noticing that it is the hypotenuse of a right-angled triangle whose catheti form the contour γ_z. The hypotenuse is always longer than either cathetus, and here its length is also less than R because $z \in \mathbb{D}(z_0, R)$.

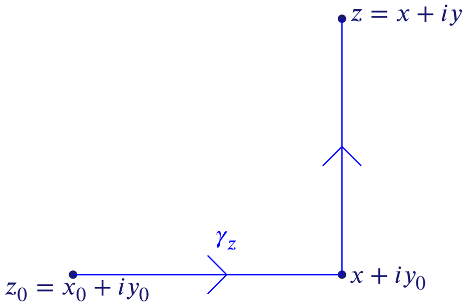


Figure 11.6: The contour γ_z is a path between two points z_0 and z in the open disc $\mathbb{D}(z_0, R)$.

Let us then define a function $F : \mathbb{D}(z_0, R) \to \mathbb{C}$ such that

$$F(z) = \int_{\gamma_z} f(u)\, \mathrm{d}u.$$

To show that this is the primitive of f, it is enough to show that

$$\frac{\partial}{\partial y} F(z) = i f(z) \quad \text{and} \quad \frac{\partial}{\partial x} F(z) = f(z). \tag{$*$}$$

The reason for this is the following: if $F = U + iV$ and $f = u + iv$, where $U, V, u, v : \mathbb{R}^2 \to \mathbb{R}$, we then have

$$\frac{\partial}{\partial y} F(z) = \frac{\partial}{\partial y} U + i \frac{\partial}{\partial y} V = i f(z) = i(u + iv) = -v + iu, \qquad \text{and}$$
$$\frac{\partial}{\partial x} F(z) = \frac{\partial}{\partial x} U + i \frac{\partial}{\partial x} V = f(z) = u + iv.$$

Hence,

$$\frac{\partial U}{\partial x} = u, \qquad \frac{\partial U}{\partial y} - v, \qquad \frac{\partial V}{\partial x} = v \quad \text{and} \quad \frac{\partial V}{\partial y} = u,$$

and thus the Cauchy–Riemann equations hold for F. Since f is continuous, also u and v, which are the partial derivatives of F, are continuous. Hence U and V are differentiable in the real sense and satisfy the Cauchy–Riemann equations, so by Theorem 4.3.10, F is differentiable and $F' = f$.

We proceed to show that $(*)$ holds.

(I) Let $\varepsilon > 0$, and let $h \in \mathbb{R} \setminus \{0\}$. It is straightforward to check that

$$ih = \int_{\gamma_{[z, z+ih]}} \mathrm{d}u.$$

Also notice that $\gamma_{z+ih} = \gamma_z * \gamma_{[z,z+ih]}$ (see Figure 11.7). Then

$$\left| \frac{F(z + ih) - F(z)}{h} - i f(z) \right| = \left| \frac{1}{h} \left(\int_{\gamma_{z+ih}} f(u)\, \mathrm{d}u - \int_{\gamma_z} f(u)\, \mathrm{d}u \right) - \frac{ih}{h} f(z) \right|$$
$$= \left| \frac{1}{h} \left(\int_{\gamma_z} f(u)\, \mathrm{d}u + \int_{\gamma_{[z,z+ih]}} f(u)\, \mathrm{d}u - \int_{\gamma_z} f(u)\, \mathrm{d}u \right) - \frac{1}{h} f(z) \int_{\gamma_{[z,z+ih]}} \mathrm{d}u \right|$$

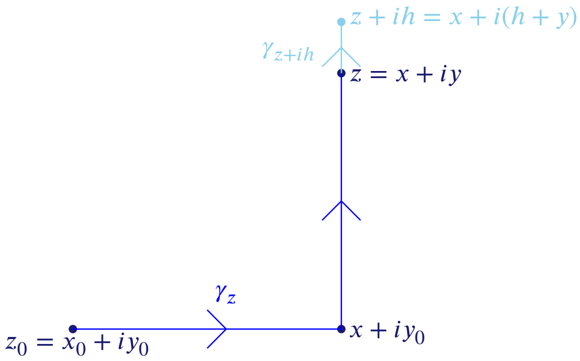


Figure 11.7: Extension of the path γ_z to the path γ_{z+ih} in the open disc $\mathbb{D}(z_0, R)$. The path is now between points z_0 and $z+ih$.

$$= \left| \frac{1}{h} \int_{\gamma_{[z,z+ih]}} f(u)\,\mathrm{d}u - \frac{1}{h} \int_{\gamma_{[z,z+ih]}} f(z)\,\mathrm{d}u \right|$$
$$= \left|\frac{1}{h}\right| \left| \int_{\gamma_{[z,z+ih]}} (f(u)-f(z))\,\mathrm{d}u \right|.$$

Noting that length$(\gamma_{[z,z+ih]}) = |h|$ and using the estimation lemma 9.3.12, we get that

$$\begin{aligned}\frac{1}{|h|}\left| \int_{\gamma_{[z,z+ih]}} (f(u)-f(z))\,\mathrm{d}u \right| &\le \frac{1}{|h|} \sup_{u\in\gamma_{[z,z+ih]}} \left|f(u)-f(z)\right| \cdot \mathrm{length}(\gamma_{[z,z+ih]}) \\ &= \frac{1}{|h|} \sup_{u\in\gamma_{[z,z+ih]}} \left|f(u)-f(z)\right||h| \\ &= \sup_{u\in\gamma_{[z,z+ih]}} \left|f(u)-f(z)\right|.\end{aligned}$$

Since f is continuous, there exists $\delta > 0$ such that whenever $|u-z| < \delta$, we have $|f(u) - f(z)| < \varepsilon$. Thus if we let $|h| < \delta$, we get that $|u-z| \le |h| < \delta$ for all $u \in \mathbb{C}$ such that $u - z \in \gamma_{[z,z+ih]}$ and hence $\sup_{u\in\gamma_{[z,z+ih]}} \left|f(u)-f(z)\right| < \varepsilon$.

To conclude the proof so far, we have found a δ such that for our fixed ε, we have

$$\begin{aligned}\left| \frac{F(z+ih)-F(z)}{h} - if(z) \right| &= \left|\frac{1}{h}\right| \left| \int_{\gamma_{[z,z+ih]}} (f(u)-f(z))\,\mathrm{d}u \right| \\ &\le \sup_{u\in\gamma_{[z,z+ih]}} \left|f(u)-f(z)\right| \\ &< \varepsilon,\end{aligned}$$

whenever $|h| < \delta$. Hence $\frac{\partial}{\partial y} F(z) = if(z)$.

(II) Let $h \in \mathbb{R} \setminus \{0\}$. We first make some observations about contours:

(1) $\gamma_z = \gamma_{[z_0,x+iy_0]} * \gamma_{[x+iy_0,z]}$ by definition, and

(2) $\gamma_{z+h} = \gamma_{[z_0,x+iy_0]} * \gamma_{[x+iy_0,z+h+iy_0]} * \gamma_{[x+h+iy_0,z+h]}$.

Then

$$\left| \frac{F(z+h)-F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \left(\int_{\gamma_{z+h}} f(u)\,\mathrm{d}u - \int_{\gamma_z} f(u)\,\mathrm{d}u \right) - f(z) \right|$$
$$= \left| \frac{1}{h} \left(\int_{\gamma_{[z_0,x+iy_0]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[x+iy_0,z+h+iy_0]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[x+h+iy_0,z+h]}} f(u)\,\mathrm{d}u \right.\right.$$

$$- \int_{\gamma_{[z_0, x+iy_0]}} f(u)\,\mathrm{d}u - \int_{\gamma_{[x+iy_0, z]}} f(u)\,\mathrm{d}u \Bigg) - f(z) \Bigg|$$

$$= \left| \frac{1}{h} \left(\int_{\gamma_{[x+iy_0, z+h+iy_0]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[x+h+iy_0, z+h]}} f(u)\,\mathrm{d}u - \int_{\gamma_{[x+iy_0, z]}} f(u)\,\mathrm{d}u \right) - f(z) \right|$$

$$= \left| \frac{1}{h} \left(\int_{\gamma_{[x+iy_0, z+h+iy_0]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[x+h+iy_0, z+h]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[z, x+iy_0]}} f(u)\,\mathrm{d}u \right) - f(z) \right|.$$

Goursat's theorem gives us (see Figure 11.8) the identity

$$\int_{\gamma_{[x+iy_0, z+h+iy_0]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[x+h+iy_0, z+h]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[z+h, z]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[z, x+iy_0]}} f(u)\,\mathrm{d}u = 0,$$

whence we get that

$$- \int_{\gamma_{[z+h, z]}} f(u)\,\mathrm{d}u = \int_{\gamma_{[x+iy_0, z+h+iy_0]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[x+h+iy_0, z+h]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[z, x+iy_0]}} f(u)\,\mathrm{d}u.$$

Using the above and the fact that

$$\int_{\gamma_{[z, z+h]}} \mathrm{d}u = h,$$

we can continue:

$$\left| \frac{1}{h} \left(\int_{\gamma_{[x+iy_0, z+h+iy_0]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[x+h+iy_0, z+h]}} f(u)\,\mathrm{d}u + \int_{\gamma_{[z, x+iy_0]}} f(u)\,\mathrm{d}u \right) - f(z) \right|$$

$$= \left| -\frac{1}{h} \int_{\gamma_{[z+h, z]}} f(u)\,\mathrm{d}u - f(z)\frac{h}{h} \right|$$

$$= \left| \frac{1}{h} \int_{\gamma_{[z, z+h]}} f(u)\,\mathrm{d}u - \frac{1}{h} f(z) \int_{\gamma_{[z, z+h]}} \mathrm{d}u \right|$$

$$= \left| \frac{1}{h} \left(\int_{\gamma_{[z, z+h]}} f(u)\,\mathrm{d}u - \int_{\gamma_{[z, z+h]}} f(z)\,\mathrm{d}u \right) \right|$$

$$= \left| \frac{1}{h} \right| \left| \int_{\gamma_{[z, z+h]}} (f(u) - f(z))\,\mathrm{d}u \right|.$$

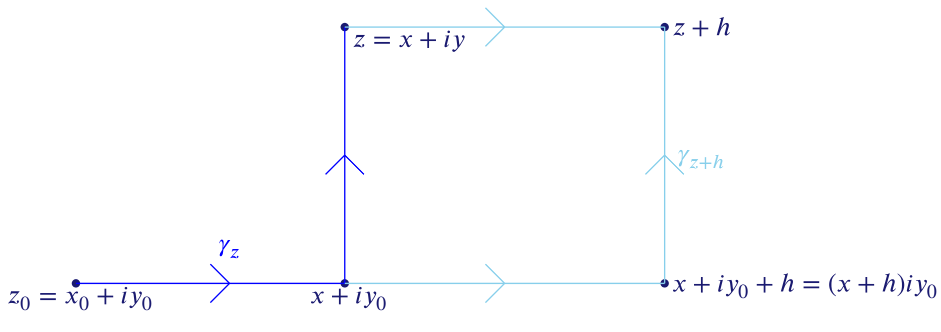


Figure 11.8: Extension of the path γ_z to the path γ_{z+h} in the open disc $\mathbb{D}(z_0, R)$. The path is now between points z_0 and $z + h$.

Now exactly the same way as in (I), we can estimate the above to be at most

$$\sup_{u \in \gamma_{[z,z+h]}} \left| f(u) - f(z) \right|,$$

which can be made sufficiently small whenever $|h|$ is small enough. This shows that

$$\frac{\partial}{\partial x} F(z) = f(z).$$

Now we have shown that $(*)$ holds, finishing the proof of the theorem. □

Theorem 11.0.5: The Cauchy–Goursat theorem. *If f is analytic in a disc, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0,$$

for any closed smooth curve in that disc.

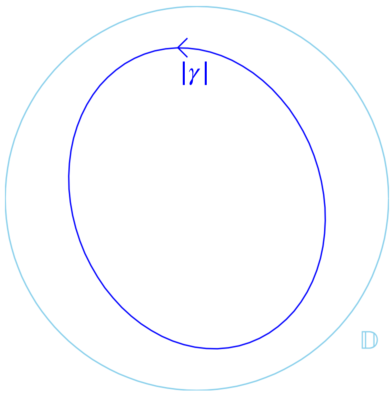


Figure 11.9: A closed smooth curve γ in a disc $\mathbb{D}$.

Proof. By 11.0.4, since f has an antiderivative, we can apply corollary 10.0.6 of the Fundamental Theorem of Calculus. □

12 Cauchy's integral formulas

Integral representation formulas are important in mathematics.

Theorem 12.0.1: Cauchy's local integral formula. *Suppose that f is analytic in a domain Ω and that $\mathbb{D} = \mathbb{D}(z_0, r)$ is a disc such that $\overline{\mathbb{D}} \subset \Omega$. If $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$ with positive orientation, then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(u)}{u - z}\, \mathrm{d}u, \qquad \textit{for any } z \in \mathbb{D}.$$

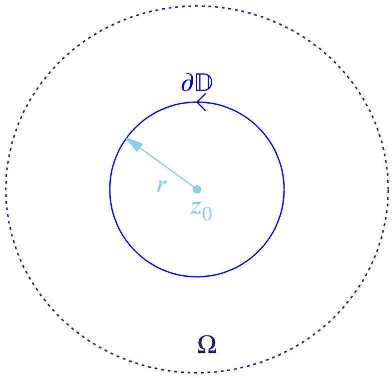


Figure 12.1: The domain Ω, a disc $\mathbb{D} = \mathbb{D}(z_0, r)$ such that the closure $\overline{\mathbb{D}} \subset \Omega$ and the boundary $\partial\mathbb{D}$ with positive orientation. The value of an analytical function f in this disc can be found by the Theorem 12.0.1.

Remark.

(1) Now we have an integral formula expressing the value of f inside a disc in terms of the values of f on the boundary of the disc.

(2) Any analytic function in a small disc can be expressed as an integral.

Examples 12.0.2. Let $\partial\mathbb{D}$ denote the boundary of the unit disc $\mathbb{D}(0, 1)$ with positive orientation. Evaluate the following integrals:

(1)

$$\oint_{\partial\mathbb{D}} \frac{\exp(z)}{z}\, \mathrm{d}z = \oint_{\partial\mathbb{D}} \frac{\exp(z)}{z - 0}\, \mathrm{d}z = 2\pi i \exp(0) = 2\pi i.$$

(2)

$$\oint_{\partial\mathbb{D}} \frac{\exp(z)}{z\,(z - 2)}\, \mathrm{d}z = \oint_{\partial\mathbb{D}} \frac{\frac{\exp(z)}{z-2}}{z - 0}\, \mathrm{d}z = 2\pi i \frac{\exp(0)}{0 - 2} = -\pi i,$$

since $z \mapsto \frac{\exp(z)}{z-2}$ is analytic in $\mathbb{D}\left(0, \frac{3}{2}\right)$.

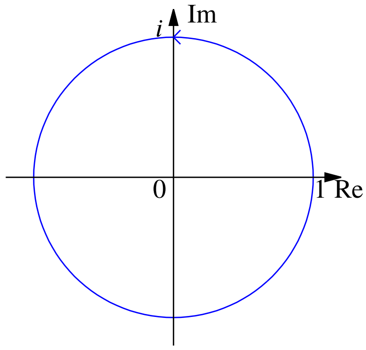


Figure 12.2: The boundary (with a positive orientation) of the unit disc $\mathbb{D}(0, 1)$.

Before we prove the Cauchy local integral formula, we give its corollary. We have a second remarkable fact about analytic functions, namely their regularity:

Theorem 12.0.3: Cauchy's second formula. *If f is analytic in a domain Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $\overline{\mathbb{D}} \subset \Omega$ is a disc so that the boundary $\partial\mathbb{D}$ is positively oriented, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(u)}{(u-z)^{n+1}} \, \mathrm{d}u, \qquad \text{for all } z \in \operatorname{int} \mathbb{D}.$$

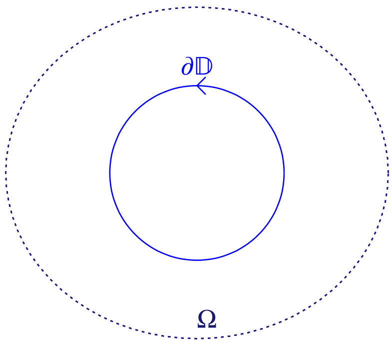


Figure 12.3: The positively oriented boundary $\partial\mathbb{D}$ of the disc $\mathbb{D}$ and $\overline{\mathbb{D}} \subset \Omega$, where Ω is a domain.

Proof of the Cauchy local integral formula 12.0.1. Let $z \in \mathbb{D}(z_0, r)$ be fixed. Let us define the function

$$F(u) = \begin{cases} \frac{f(u)-f(z)}{u-z}, & \text{when } u \in \Omega \setminus \{z\}, \\ f'(z), & \text{when } u = z. \end{cases}$$

Then F is analytic in $\Omega \setminus \{z\}$ and continuous in Ω.

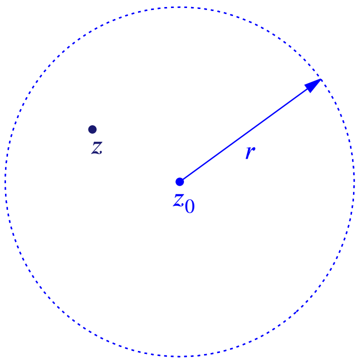


Figure 12.4: The function F in the proof of the Cauchy local integral formula 12.0.1 is defined in the disc $\mathbb{D}(z_0, r)$ with a fixed point z.

Since $\overline{\mathbb{D}}(z_0, r) \subset \Omega$ and Ω is a domain, there exists an open disc $\mathbb{D}_1$ such that $\overline{\mathbb{D}}(z_0, r) \subset \mathbb{D}_1 \subset \Omega$. By the weakened form of Goursat's theorem we know that

$$\begin{aligned} 0 &= \int_{\partial\mathbb{D}(z_0,r)} F(u)\,\mathrm{d}u = \int_{\partial\mathbb{D}(z_0,r)} \frac{f(u)}{u-z}\,\mathrm{d}u - \int_{\partial\mathbb{D}(z_0,r)} \frac{f(z)}{u-z}\,\mathrm{d}u \\ &= \int_{\partial\mathbb{D}(z_0,r)} \frac{f(u)}{u-z}\,\mathrm{d}u - f(z)\int_{\partial\mathbb{D}(z_0,r)} \frac{\mathrm{d}u}{u-z}, \end{aligned}$$

since F is analytic in $\mathbb{D}_1 \setminus \{z\}$ and continuous in $\mathbb{D}_1$ and $\partial\mathbb{D}(z_0, r)$ is a closed contour in $\mathbb{D}_1$.

It remains to show that

$$\int_{\partial\mathbb{D}(z_0,r)} \frac{f(z)}{u-z}\,\mathrm{d}u = \int_{\partial\mathbb{D}(z_0,r)} \frac{\mathrm{d}u}{u-z} = 2\pi i,$$

from which the claim follows.

Let $\gamma_0 : [0,1] \to \Omega, \quad \gamma_0(t) = z_0 + r\exp(2\pi i t)$. Then

$$\begin{aligned} \int_{\gamma_0} \frac{\mathrm{d}u}{u-z} &= \int_0^1 \frac{r2\pi i \exp(2\pi i t)}{z_0 + r\exp(2\pi i t) - z}\,\mathrm{d}t \\ &= 2\pi i \int_0^1 \frac{r\exp(2\pi i t)}{z_0 - z + r\exp(2\pi i t)}\,\mathrm{d}t \\ &= 2\pi i \int_0^1 \frac{\mathrm{d}t}{\frac{z_0-z}{r\exp(2\pi i t)} + 1} \\ &= 2\pi i \int_0^1 \frac{1 + \frac{z_0-z}{r}\exp(-2\pi i t)}{1 + \frac{z_0-z}{r}\exp(-2\pi i t)}\,\mathrm{d}t - 2\pi i \int_0^1 \frac{\frac{z_0-z}{r}\exp(-2\pi i t)}{1 + \frac{z_0-z}{r}\exp(-2\pi i t)}\,\mathrm{d}t \\ &= 2\pi i - 2\pi i \int_0^1 \frac{\frac{z_0-z}{r}\exp(-2\pi i t)}{1 + \frac{z_0-z}{r}\exp(-2\pi i t)}\,\mathrm{d}t. \end{aligned}$$

Now, let us write

$$\gamma_1 : [0,1] \to \mathbb{D}(0,1), \quad \gamma_1(t) = \frac{z_0 - z}{r}\exp(-2\pi i t),$$

and hence

$$\gamma_1'(t) = -2\pi i \frac{z_0 - z}{r}\exp(-2\pi i t).$$

Thus, we get

$$\int_0^1 \frac{\gamma_1'(t)}{1 + \gamma_1(t)}\,\mathrm{d}t = \int_{\gamma_1} \frac{\mathrm{d}u}{1+u}.$$

Since $|z_0 - z| < r$, we have $|\gamma_1| \subset \mathbb{D}(0,1)$. Since $g(u) = \frac{1}{1+u}$ is analytic in $\mathbb{D}(0,1)$, we obtain by Theorem 11.0.5 that

$$\int_{\gamma_1} \frac{\mathrm{d}u}{1+u} = 0.$$

Thus

$$0 = \int_{\partial\mathbb{D}(z_0,r)} \frac{f(u)}{u-z}\,\mathrm{d}u - f(z)\left(2\pi i + 0\right),$$

and the claim follows. □

Proof of the Cauchy's second formula 12.0.3. We assume that the boundary $\partial\mathbb{D}$ has positive orientation. The proof is by induction on n. The case $n = 1$ is the Cauchy integral formula.

Suppose that f has up to $n-1$ complex derivatives and

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(u)}{(u-z)^n}\,\mathrm{d}u.$$

Now, for h small, the difference quotient for $f^{(n-1)}$ takes the form:

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_{\partial\mathbb{D}} f(u)\frac{1}{h}\left(\frac{1}{(u-z-h)^n} - \frac{1}{(u-z)^n}\right)\mathrm{d}u.$$

Recall that $a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}\right)$.

With $a = \frac{1}{u-z-h}$ and $b = \frac{1}{u-z}$, we have

$$\frac{1}{(u-z-h)^n} - \frac{1}{(u-z)^n} = \frac{1}{(u-z-h)(u-z)}\left(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}\right).$$

Since h is small enough, the points $z+h$ and z stay at a finite distance from the boundary. Hence, as $h \to 0$, the quotient converges to

$$\frac{(n-1)!}{2\pi i} \int_{\partial\mathbb{D}} f(u)\frac{1}{(u-z)^2}\frac{n}{(u-z)^{n-1}}\,\mathrm{d}u = \frac{n!}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(u)}{(u-z)^{n+1}}\,\mathrm{d}u,$$

which completes the induction argument and proves the theorem. □

Example 12.0.4. Evaluate in a disc $\mathbb{D}(0,r)$

$$\int \frac{\sin z}{(z-2)\exp(z)}\,\mathrm{d}z.$$

1. The case $r = 1$:

$$f(z) = \frac{\sin z}{(z-2)\exp(z)}$$

   is analytic in $\mathbb{D}\left(0, \frac{3}{2}\right)$. So the C-G theorem implies that $\int f(z)\,\mathrm{d}z = 0$.

2. The case $r = 3$:

$$f(\omega) = \frac{\sin\omega}{\exp(\omega)}$$

   is analytic in $\mathbb{D}\left(0, 4\frac{1}{2}\right)$ and $2 \in \mathbb{D}(0,3)$ and $\mathbb{D}(0,3) \subset \mathbb{D}\left(0,4\frac{1}{2}\right)$. The Cauchy local integral formula gives

$$\int_{\partial\mathbb{D}(0,3)} \frac{\frac{\sin(\omega)}{\exp(\omega)}}{\omega-2}\,\mathrm{d}\omega = 2\pi i\frac{\sin(2)}{e^2}.$$

13 Corollaries of the Cauchy integral formulas

Corollary 13.0.1. *Let Ω be an open set in $\mathbb{C}$. If $f : \Omega \to \mathbb{C}$ has an antiderivative, then f is analytic.*

Proof. Let F be an antiderivative of f. Then, by Definition 10.0.1, F is analytic and $F' = f$. By Theorem 12.0.3, f is analytic. □

Theorem 13.0.2: Morera's theorem. *Let Ω be an open set in $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ is continuous and*

$$\int_\Gamma f(z)\,\mathrm{d}z = 0$$

along every closed contour in Ω. Then f is analytic.

Proof. Theorem 10.0.9 and Corollary 13.0.1. □

13.0.3. Cauchy's inequalities

Suppose that f is analytic in an open set Ω and $\overline{\mathbb{D}(z_0, R)} \subset \Omega$. Then:

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{R^n} \|f\|_{\partial\mathbb{D}(z_0,R)},$$

where $\|f\|_{\partial\mathbb{D}(z_0,R)} = \sup_{z\in\partial\mathbb{D}(z_0,R)}|f(z)|$.

Proof. We apply the Cauchy integral formula for $f^{(n)}(z_0)$:

$$\begin{aligned}\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_{\partial\mathbb{D}(z_0,R)} \frac{f(u)}{(u-z_0)^{n+1}}\,\mathrm{d}u\right| = \frac{n!}{2\pi i}\left|\int_0^{2\pi} \frac{f\left(z_0 + R\exp(i\phi)\right) iR\exp(i\phi)}{\left(R\exp(i\phi)\right)^{n+1}}\,\mathrm{d}\phi\right| \\ &= \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f\left(z_0 + R\exp(i\phi)\right)}{\left(R\exp(i\phi)\right)^{n}}\,\mathrm{d}\phi\right| \\ &\le \frac{n!}{2\pi} 2\pi \frac{\|f\|_{\partial\mathbb{D}(z_0,R)}}{R^n} = \frac{n!}{R^n}\|f\|_{\partial\mathbb{D}(z_0,R)}.\end{aligned}$$

□

Theorem 13.0.4. *Suppose that f is analytic in an open set Ω and $\overline{\mathbb{D}(z_0, R)} \subset \Omega$. Then f has a power series expansion at z_0*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

for all $z \in \mathbb{D}(z_0, R)$ and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \qquad \textit{for all } n = 0, 1, 2 \ldots$$

<u>*Note 13.0.5*</u>. This is a local result.

Proof. Let $0 < r < R$ and $z \in \mathbb{D}(z_0, R)$. By the Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}(z_0,R)} \frac{f(u)}{u - z}\,\mathrm{d}u, \qquad z \in \mathbb{D}(z_0, R).$$

For each $u \in \partial\mathbb{D}\left(z_0, r\right)$ we can write

$$\frac{1}{u-z} = \frac{1}{u-z_0-(z-z_0)} = \frac{1}{u-z_0}\frac{1}{1-\frac{z-z_0}{u-z_0}}$$
$$= \frac{1}{u-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{u-z_0}\right)^k,$$

since

$$\left|\frac{z-z_0}{u-z_0}\right| = \frac{\left|z-z_0\right|}{r} < 1.$$

Hence,

$$f(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}(z_0,r)}\sum_{k=0}^{\infty}\frac{f(u)}{u-z_0}\left(\frac{z-z_0}{u-z_0}\right)^k \mathrm{d}u.$$

Since f is continuous on the compact set $\partial\mathbb{D}\left(z_0, r\right)$, f is bounded there and $|f(u)| \le M \le \infty$ for all $u \in \partial\mathbb{D}\left(z_0, r\right)$. Thus

$$\left|\frac{f(u)}{u-z_0}\cdot\left(\frac{z-z_0}{u-z_0}\right)^k\right| \le \frac{M}{r}\frac{\left|z-z_0\right|^k}{r^k}.$$

Here $\sum\limits_{k=0}^{\infty}\frac{M}{r}\left(\frac{|z-z_0|}{r}\right)^k$ converges as a geometric series (with $q < 1$). By the Weiestrass criteria

$$\sum_{k=0}^{\infty}\frac{f(u)}{u-z_0}\left(\frac{z-z_0}{u-z_0}\right)^k$$

converges uniformly on $\partial\mathbb{D}\left(z_0, r\right)$. Hence, the order of the summation and integration can me changed. By changing the order of summation and integration, we get

$$f(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\left(\int_{\partial\mathbb{D}(z_0,r)}\frac{f(u)}{(u-z)^{k+1}}\mathrm{d}u\right)\left(z-z_0\right)^k.$$

By the Cauchy's 2$^{\text{nd}}$ local formula

$$\int_{\partial\mathbb{D}(z_0,r)}\frac{f(u)}{(u-z)^{k+1}}\mathrm{d}u = \frac{2\pi i}{k!}f^{(k)}\left(z_0\right).$$

Thus

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}\left(z_0\right)}{k!}\left(z-z_0\right)^k, \quad \text{where } a_k = \frac{f^{(k)}\left(z_0\right)}{k!}.$$

Note that r can be chosen as close as we like near R and since the coefficients are unique, we get always the same series. Hence

$$f(z) = \sum_{k=0}^{\infty}a_k\left(z-z_0\right)^k, \quad \text{with } a_k = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}(z_0,r)}\frac{f(u)}{\left(u-z_0\right)^{k+1}}\mathrm{d}u, \ \ 0 < r < R,$$

is valid in the whole $\mathbb{D}\left(z_0, R\right)$. □

Remark. This theorem gives another proof that an analytic function is automatically indefinitely differentiable.

Remark. In particular, if f is entire (that is, f is analytic on whole $\mathbb{C}$), the theorem implies that f has a power series expansion around 0, $f(z) = \sum_{n=0}^{\infty}a_n z^n$, that converges in all of $\mathbb{C}$.

Corollary 13.0.6: Liouville's theorem. *If f is entire and bounded, then f is constant.*

Remark 13.0.7. In the proof, we use the following lemma: If f is analytic in a domain Ω and $f' = 0$, then f is constant.

Proof of remark 13.0.7. Fix $\omega_0 \in \Omega$. It suffices to show that $f(\omega) = f(\omega_0)$ for all $\omega \in \Omega$. Since Ω is connected, for any $\omega \in \Omega$, there exists a suitable curve γ_ω which joins ω_0 to ω. Since f is an antiderivative of f', we have, by Fundamental Theorem of Calculus,

$$\int_{\gamma_\omega} f'(z)\,\mathrm{d}z = f(\omega) - f(\omega_0).$$

By assumption, $f' = 0$, so the integral on the left hand side is 0, and we obtain $f(\omega) = f(\omega_0)$. □

Proof of Liouville's theorem 13.0.6. It suffices, because of Remark 13.0.7, to prove that $f' = 0$, since $\mathbb{C}$ connected. For each $z_0 \in \mathbb{C}$ and $R > 0$, the Cauchy inequality yields

$$\left|f'(z_0)\right| \leq \frac{M}{R}$$

where M is a bound for f.

Letting $R \to \infty$, gives that $\left|f'\right| = 0$. This implies (a small lemma is needed) that $f' = 0$. □

We can give an elegant proof for the Fundamental Theorem of Algebra.

Corollary 13.0.8. *Every non-constant polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ with complex coefficients has a root in $\mathbb{C}$.*

Proof. If p has no root, then $\frac{1}{p}$ is a bounded analytic function. To see this, let us assume that $a_n \neq 0$ and write

$$\frac{p(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}\right),$$

whenever $z \neq 0$. Since every term in the brackets goes to 0 as $|z| \to \infty$, there exists $R > 0$ so that $\left|p(z)\right| \geq \frac{|a_n|}{2}|z|^n$ whenever $|z| > R$. Hence p is bounded from below when $|z| > R$.

Since p is continuous and has no roots in the disc $|z| \leq R$, it is bounded from below in the disc $\mathbb{D}(0, R)$ as well. So $\frac{1}{p}$ is indeed a bounded analytic function. By Liouville's theorem, $\frac{1}{p}$ is constant. But this contradicts our assumption that p is non-constant and proves this corollary. □

Corollary 13.0.9. *Every polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $\omega_1, \omega_2, \ldots, \omega_n$, then*

$$p(z) = a_n(z - \omega_1)(z - \omega_2)\ldots(z - \omega_n).$$

Remark 13.0.10. The following claims are essentially equivalent in an open disk $\mathbb{D}$:

1. f is analytic,
2. f has a primitive and
3. for all closed $\mathbb{D}$
$$\int_\gamma f(u)\,\mathrm{d}u = 0.$$

   The link from the 1. to 2. comes from Cauchy-Goursat theorem. The link from the 2. to 3. comes from The fundamental theorem of calculus. The link from the 2. to 1. comes from Cauchy's 2nd theorem. The link from the 3. to 2. comes from The 2nd fundamental theorem of calculus.

13.0.11 Analytic continuation

In the following theorem, Ω is assumed connected.

Theorem 13.0.12: Identity theorem. *Suppose that f is an analytic function in a domain Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then f is identically 0.*

Proof. Suppose that $z_0 \in \Omega$ is a limit point for the sequence $(\omega_k)_{k=1}^{\infty}$ and $f(\omega_k) = 0$.

First, we show that $f \equiv 0$ in a small disc containing z_0. For that, we choose a disc $\mathbb{D}\big(z_0, r\big) \subset \Omega$ with $r > 0$ fixed. We consider the power series expansion of f in this disc, $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$.

If we assume that f is not identically 0 in $\mathbb{D}\big(z_0, r\big)$, then there exists a smallest integer m such that $a_m \neq 0$. But then

$$f(z) = a_m(z - z_0)^m(1 + g(z - z_0)),$$

where $g(z-z_0)$ converges to 0 as $z \to z_0$. Taking $z = \omega_k \neq z_0$ for a sequence of points converging to z_0, we get

$$a_m(\omega_k - z_0) \neq 0 \quad \text{and} \quad (1 + g(z - z_0)) \neq 0,$$

which is a contradiction, since $f(\omega_k) = 0$.

Now, we apply the fact that Ω is connected. Let

$$\Omega_1 = \text{int}\{z \in \Omega : \ f(z) = 0\}.$$

Then Ω_1 is open by definition and Ω is non-empty by the first part. However, Ω_1 is also closed: if $z_n \in \Omega_1$ and $z_n \to z$, then $f(z) = 0$, since f is continuous and $f \equiv 0$ in a small open disc containing z by the first part. Hence $z \in \Omega_1$.

Since Ω is connected and $z_0 \in \Omega_1$, and hence $\Omega_1 \neq \emptyset$, we have $\Omega_1 = \Omega$. □

Remark 13.0.13. In other words, if the zeros of an analytical function f in a domain Ω accumulates in Ω, then $f = 0$.

Remark 13.0.14. Note that in this theorem the points should accumulate in Ω.

Example 13.0.15. The function $\sin z$ is entire and

$$\sin z = 0 \Leftrightarrow z = n\pi, n = 0, \pm 1, \pm 2, \dots, .$$

Let $f(z) = \sin \frac{1}{z}$. Then $f(z_k) = 0$ at the points $z_k = \frac{1}{k\pi}$, where $k = \pm 1, \pm 2, \dots$.

But 0 is the accumulation point for

$$\left\{ \frac{1}{n\pi} : \ n = \pm 1, \pm 2, \dots \right\}.$$

However, 0 is not in the domain of analyticity of f.

Note that $\sin \frac{1}{z}$ is analytic in $\mathbb{C} \setminus \{0\}$.

Example 13.0.16. Let $r(z) = \exp\left(\frac{1}{z}\right) - 1$, $z \in \mathbb{C} \setminus \big\{0\big\}$. Then g is analytic in $\mathbb{C} \setminus \big\{0\big\}$. Note that

$$g(z) = 0 \quad \Leftrightarrow \quad z_n = \frac{1}{2\pi i n} = -\frac{i}{2\pi n}, \text{ when } n \in \mathbb{Z} \setminus \big\{0\big\}.$$

Here the points z_n accumulate to the origin, but $0 \notin \mathbb{C} \setminus \big\{0\big\}$. And g is not identically zero.

Corollary 13.0.17. *Suppose that f and g are analytic in a domain Ω and $f(z) = g(z)$ for all z in some sequence of distinct points with a limit point in Ω. Then $f(z) = g(z)$ throughout Ω.*

Corollary 13.0.18: Corollary of the Identity theorem. *The Uniqueness theorem. Suppose that f and g are analytic in a domain Ω and $f(z) = g(z)$ for all f in some non-empty open subset of Ω. Then*

$$f(z) = g(z), \quad \textit{throughout } \ \Omega.$$

Remark 13.0.19. Suppose that we are given a pair of functions f and g which are analytic in domains Ω and $\tilde{\Omega}$, respectively with $\Omega \subset \tilde{\Omega}$ (Figure 13.1.) If f and g agree on the smaller set Ω, we say that g is an analytic continuation of f into domain $\tilde{\Omega}$. The corollary 13.0.18 guarantees that there can be only one such continuation, since g is uniquely determined by f.

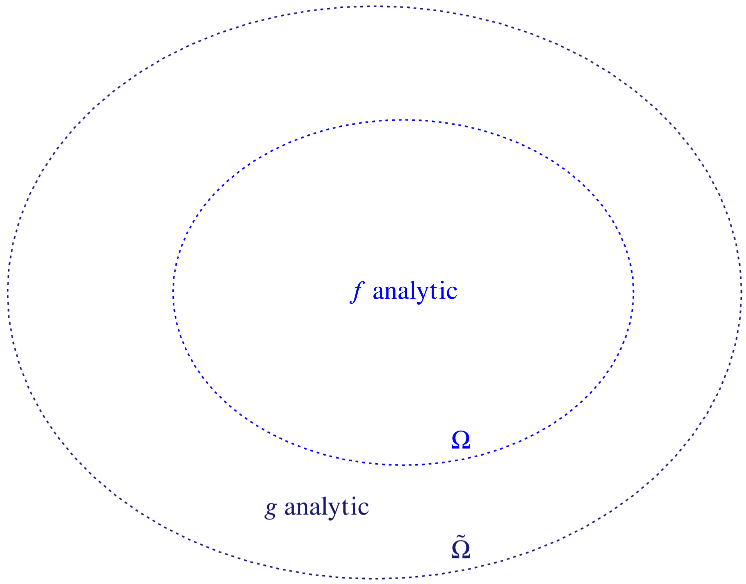


Figure 13.1: If analytic functions f and g agree on domain $\Omega \subset \tilde{\Omega}$, then g is an analytic continuation of f and as g is uniquely determined by f there can be only one such continuation. (Remark 13.0.19.)

13.0.20 (The Maximum modulus principle)**.** If f is a non-constant analytic function in a domain Ω, then $|f|$ can not attain a maximum in Ω.

Theorem 13.0.21: Local maximum modulus theorem. *Suppose that f is analytic in $\mathbb{D}(z_0, R)$. If $|f|$ has a local maximum point in $\mathbb{D}(z_0, R)$, ie.*

$$|f(z)| \le |f(z_0)| \quad \textit{for all } z \in \mathbb{D}(z_0, R). \tag{13.0.22}$$

Then f is constant.

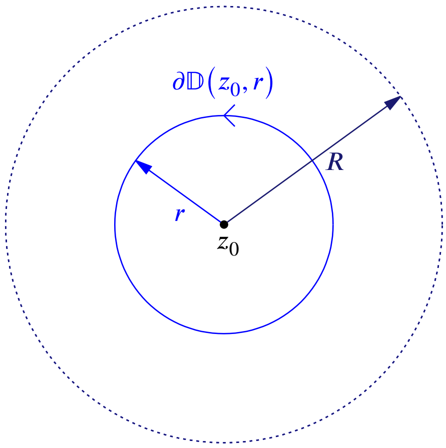


Figure 13.2: In the proof of Theorem 13.0.21, the integration path around the point z_0 is $\gamma(t) = \partial\mathbb{D}\big(z_0, r\big) = z_0 + r\exp(it)$, $t \in \big[0, 2\pi\big]$.

Proof. Fix r such that $0 < r < R$. (Figure 13.2.) By the Cauchy integral formula, we can write

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}(z_0,r)} \frac{f(u)}{u - z_0}\,\mathrm{d}u \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + r\exp(it)) i r\exp(it)}{r\exp(it)}\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\exp(it))\,\mathrm{d}t. \end{aligned}$$

Hence

$$\big|f(z_0)\big| \le \frac{1}{2\pi}\int_0^{2\pi} \big|f(z_0 + r\exp(it))\big|\,\mathrm{d}t \le \big|f(z_0)\big|,$$

and therefore

$$\big|f(z_0)\big| = \frac{1}{2\pi}\int_0^{2\pi} \big|f(z_0 + r\exp(it))\big|\,\mathrm{d}t.$$

We now have

$$\int_0^{2\pi} \big|f(z_0)\big| - \big|f(z_0 + r\exp(it))\big|\,\mathrm{d}t = 0,$$

where $\big|f(z_0)\big| - \big|f(z_0 + r\exp(it))\big| \ge 0$ by our assumption and the continuity of f.

Hence

$$\big|f(z_0)\big| - \big|f(z_0 + r\exp(it))\big| \equiv 0$$

for all t. This is true for all $r < R$. Thus, $|f|$ is constant in $\mathbb{D}\big(z_0, R\big)$, meaning that f is constant there as well. □

Corollary 13.0.23. *Suppose that f is analytic in a domain Ω. If $|f|$ has a local maximum point in Ω, then f is constant.*

Proof. Apply Theorem 13.0.21 and the identity theorem 13.0.12. □

Theorem 13.0.24: Maximum modulus theorem. *Suppose that Ω is a domain with a compact closure $\overline{\Omega}$. If f is analytic in Ω and continuous on $\overline{\Omega}$, then*

$$\sup_{z\in\Omega}\big|f(z)\big| \le \sup_{z\in\overline{\Omega}\setminus\Omega}\big|f(z)\big|.$$

Proof. Since $\overline{\Omega}$ is bounded and closed, and f is continuous on $\overline{\Omega}$, $|f|$ attains its supremum, M, at some point of $\overline{\Omega}$.

Now, assume that $|f|$ does not attain the value M on $\overline{\Omega}\setminus\Omega$. Then $\left|f(z_m)\right| = M$ for some $z_m \in \Omega$. Since Ω is open, there exists $R > 0$ such that $\mathbb{D}\big(z_m, R\big) \subset \Omega$ and

$$\left|f(z)\right| \leq \left|f(z_m)\right| \quad \text{for all } z \in \mathbb{D}\big(z_m, R\big).$$

Thus, by the local maximum modulus theorem 13.0.21, f is constant on $\mathbb{D}\big(z_m, R\big)$. Using the identity theorem 13.0.12 gives us that f is constant on Ω. Finally, by the continuity of f, f is also constant on $\overline{\Omega}$.

So f attains its supremum, M at every point of $\overline{\Omega} = \Omega \cup \partial\Omega$, contrary to hypothesis. □

Example 13.0.25. Let Ω be the open 1st quadrant bounded by the positive half-line $x \geq 0$ and the corresponding imaginary line $y \geq 0$. Let us consider the function

$$F(z) = \exp\left(-iz^2\right).$$

Then F is entire and F is continuous on $\overline{\Omega}$, and $|F(x)| = 1$ on the two boundary lines $z = x$ and $z = iy$. However, $F(z)$ is unbounded in Ω, since for example when we pick up

$$z = r\exp\left(i\frac{\pi}{4}\right) = r\left(1+i\right)\frac{\sqrt{2}}{2}, \quad r > 0,$$

then $F(z) = \exp\left(r^2\right)$.

Lemma 13.0.26: The Schwarz lemma. *Suppose that f is analytic in $\mathbb{D}\big(0, R\big)$, $f(0) = 0$ and*

$$\left|f(z)\right| \leq M \quad \textit{for all } z \in \overline{\mathbb{D}\big(0, R\big)}.$$

Then

$$\left|f(z)\right| \leq \frac{M}{R}|z| \quad \textit{for all } |z| \leq R.$$

Proof. Since $f(0) = 0$, there exists an analytic function g on $\mathbb{D}\big(0, R\big)$ such that

$$f(z) = zg(z) \quad \text{for all } z \in \mathbb{D}\big(0, R\big).$$

On $0 < |z| = r < R$ (see Figure 13.3), we have

$$\left|g(z)\right| \leq \frac{f(z)}{|z|} \leq \frac{M}{r}.$$

By the maximum modulus theorem 13.0.24 for g, we get

$$\left|g(z)\right| \leq \frac{M}{r} \quad \text{for all } |z| \leq r.$$

Now, we let $r \to R$ and we obtain

$$\left|g(z)\right| \leq \frac{M}{R} \quad \text{for all } |z| < R \text{ and } z \neq 0.$$

At $z = 0$, we had $f(z) = 0$ and therefore the required inequality holds for all $|z| \leq R$. □

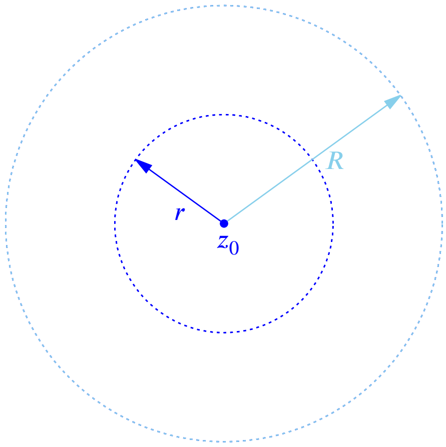


Figure 13.3: In the proof of Schwarz Lemma (13.0.26), choose r such that $0 < r < R$.

Remark. If equality occurs in

$$\left|f(z)\right| \leq \frac{M}{R}|z|$$

for some z with $|z| < R$, then there exists $\lambda \in \mathbb{R}$ such that

$$f(z) = \frac{M}{R} z \exp(i\lambda) \quad \text{for } z \in \mathbb{D}\left(0, R\right).$$

This means that f is a rotation around the origin. The proof follows the proof of the maximum modulus theorem 13.0.24.

Theorem 13.0.27: Minimum modulus theorem. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and $f : \Omega \to \mathbb{C}$ an analytic function with $f(z) \neq 0$ for all $z \in \Omega$. Assume that f is a non-constant function and that $f : \overline{\Omega} \to \mathbb{C}$ is continuous.*

Then

$$\left|f(z)\right| \geq \min_{\xi \in \partial\Omega}\left|f(\xi)\right|, \quad \textit{for all } z \in \Omega.$$

Proof. If $f(z_0) = 0$ for some $z_0 \in \partial\Omega$, the claim is true. Therefore, we can assume that $f(z_0) \neq 0$ for all $z_0 \in \partial\Omega$.

Now $\frac{1}{f(z)}$ is analytic, because $f(z) \neq 0$ for all $z \in \Omega$ and f is analytic. In addition, $\frac{1}{f(z)}$ is non-constant, because f was non-constant. Also, $\frac{1}{f(z)}$ is continuous on $\overline{\Omega}$, because f was continuous and non-zero on $\overline{\Omega}$.

Because Ω is a bounded domain, by the maximum modulus theorem 13.0.24, we have

$$\frac{1}{f(z_0)} > \frac{1}{f(z)}$$

for some $z_0 \in \partial\Omega$ and all $z \in \Omega$. Therefore $\left|f(z)\right| > f(z_0)$ for all $z \in \Omega$. □

14 Global Cauchy theorem

14.1 Cycles

Definitions 14.1.1. Let $\gamma_1, \ldots, \gamma_k$ be closed, piecewise smooth (C^1) curves in the complex plane. Let $m_1, \ldots, m_k$ be integers.

- A *cycle* is a formal sum
$$\sigma = m_1\gamma_1 + m_2\gamma_2 + \cdots m_k\gamma_k.$$
- The *trace* of a cycle σ is
$$|\sigma| = |\gamma_1| \cup |\gamma_2| \cup \cdots \cup |\gamma_k|,$$
where the trace of a parametrized curve $\gamma_j : [a, b] \to \mathbb{C}$ is its image
$$|\gamma_j| = \{\gamma_j(t) \mid a \le t \le b\}.$$

A cycle in 14.1.1 is a collection of curves where the parametrized curve γ_j appears m_j times if $m_j > 0$, and the negative of γ_j, denoted by

$$\gamma_j^- = -\gamma_j,$$

appears $-m_j$ times if $m_j < 0$.

If f is continuous in an open set Ω and $|\sigma| \subset \Omega$, then

$$\int_\sigma f(z)\,dz = \sum_{j=1}^{k} m_j \int_{\gamma_j} f(z)\,dz.$$

Definition. The *length of the cycle* σ is

$$\text{length}(\sigma) = \sum_{j=1}^{k} |m_j| \,\text{length}(\gamma_j).$$

Remarks.

1. For a curve γ,
$$\int_{\gamma^-} f(z)\,dz = -\int_\gamma f(z)\,dz,$$
where $\gamma^- = -\gamma$ is the negative of γ.
2. The negative of a cycle σ is
$$-\sigma = -m_1\gamma_1 - m_2\gamma_2 - \cdots - m_k\gamma_k$$
and $\int_{-\sigma} f(z)\,dz = -\int_\sigma f(z)\,dz$.

Definition 14.1.2: Winding number. Suppose Ω is an open set in $\mathbb{C}$ and γ is a closed piecewise smooth curve in Ω and $a \in \Omega \setminus |\gamma|$. Then

$$n(\gamma; a) := \frac{1}{2\pi i}\int_\gamma \frac{du}{u-a}$$

is called the winding number around/about a.

Remark. $a \mapsto n(\gamma, a)$ is an integer-valued function.

Lemma 14.1.3: Winding number lemma. *For any closed piecewise smooth curve γ and $a \notin |\gamma|$, the number $n(\gamma, a)$ is an integer.*

Proof. Define a function $h\colon [0,1] \to \mathbb{C}$ by letting

$$h(x) = \int_0^x \frac{\gamma'(t)}{\gamma(t)-a}\,\mathrm{d}t, \qquad \text{for all } x \in \left[0,1\right].$$

Then

$$h(1) = \int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}\,\mathrm{d}t = \int_\gamma \frac{\mathrm{d}z}{z-a} = 2\pi i\, n(\gamma;a). \tag{$*$}$$

Since γ is piecewise smooth, there are only a finite number of points where γ' is not continuous. Thus, for any x such that γ' is continuous at x,

$$h'(x) = \frac{\gamma'(x)}{\gamma(x)-a},$$

so $\gamma'(x) - h'(x)(\gamma(x)-a) = 0$. Define $\varphi\colon [0,1] \to \mathbb{C}$ by $\varphi(x) = (\gamma(x)-a)\exp(-h(x))$. Then

$$\begin{aligned}
\varphi'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\big((\gamma(x)-a)\exp(-h(x))\big)\\
&= \gamma'(x)\exp(-h(x)) + (\gamma(x)-a)(-h'(x))\exp(-h(x))\\
&= \exp(-h(x))(\gamma'(x) - h'(x)(\gamma(x)-a))\\
&= \exp(-h(x))\cdot 0\\
&= 0
\end{aligned}$$

for all but a finite number of $x \in [0,1]$.

Since $\varphi'(x) = 0$ almost everywhere, it is constant almost everywhere. But because γ and h are continuous, φ is continuous and, as such, must be a constant function, since $[0,1]$ is connected. Notice that

$$\begin{aligned}
\varphi(0) &= (\gamma(0)-a)\exp(-h(0))\\
&= (\gamma(0)-a)\exp(0) = (\gamma(0)-a)\cdot 1\\
&= \gamma(0)-a,
\end{aligned}$$

so because φ is constant, $\gamma(0)-a = \varphi(0) = \varphi(1) = (\gamma(1)-a)\exp(-h(1))$, i.e.

$$\frac{\gamma(0)-a}{\gamma(1)-a}\exp(-h(1)) = 1.$$

Because γ is a closed curve, $\gamma(0) = \gamma(1)$ and so $\exp(-h(1)) = 1$. This means that $-h(1) = 2\pi i m$ for some $m \in \mathbb{Z}$. But then by $(*)$,

$$n(\gamma;a) = \frac{h\big(1\big)}{2\pi i} = -m \in \mathbb{Z},$$

which completes the proof. □

Definition (Winding number of a cycle)**.** The winding number of a cycle σ is

$$n(\sigma;a) = \frac{1}{2\pi i}\int_\sigma \frac{\mathrm{d}u}{u-a}$$

where $a \notin |\sigma|$.

Examples 14.1.4.

1. If γ represents the boundary of the disc $\mathbb{D}\big(z_0, R\big)$ (traversed counter clockwise), then

$$n(\gamma;a) = \begin{cases} 1 & \text{if } a \in \mathbb{D}\big(z_0, R\big)\\ 0 & \text{if } a \in \mathbb{C}\setminus\overline{\mathbb{D}\big(z_0, R\big)}\end{cases}$$

2. Let $\gamma_m(t) = \exp(i2\pi mt), t \in [0,1], m \in \mathbb{Z} \setminus \{0\}$. Then,

$$n(\gamma_m; 0) = \frac{1}{2\pi i} \int_{\gamma_m} \frac{\mathrm{d}z}{z} = \frac{1}{2\pi} \int_0^1 \frac{2\pi m \exp(i2\pi mt)}{\exp(i2\pi mt)} \,\mathrm{d}t = m.$$

This explains the name "winding number".

Remarks 14.1.5.

1. $\sigma_m = m\gamma_1$ is a cycle where $\gamma_1(t) = \exp(i2\pi t), t \in [0,1]$; $\quad n(\sigma_m; 0) = m \in \mathbb{Z} \setminus \{0\}$.
2. $\sigma = \gamma_m$ is a cycle.

14.1.6. Notice that an open set $\Omega \setminus |\sigma|$ in the complex plane is the disjoint union of domains. These distinct components are the maximal connected open sets.

Remark 14.1.7. Since $|\sigma|$ is bounded, there exists $R > 0$ such that

$$\{z \in \mathbb{C} : |z| > R\} \cap |\sigma| = \emptyset.$$

This means that we have one unbounded component and the other components are in the disc $\overline{\mathbb{D}(0, R)}$.

Lemma 14.1.8: The component lemma. *Suppose that σ is a cycle in the complex plane. The mapping $a \mapsto n(\sigma; a)$ is a constant on each connected component of $\mathbb{C} \setminus |\sigma|$. Moreover the number $n(\sigma; a) = 0$ in the unbounded component of $\mathbb{C} \setminus |\sigma|$.*

Proof. Homework. □

14.2 Cauchy global integral theorem

Theorem 14.2.1: Cauchy global integral theorem. *Suppose that σ is a cycle in the complex plane, Ω is a domain and $f : \Omega \to \mathbb{C}$ is an analytic function. If σ is a cycle in Ω such that $n(\sigma; b) = 0$ for all $b \in \mathbb{C} \setminus \Omega$, then:*

$$n(\sigma; z) f(z) = \frac{1}{2\pi i} \int_\sigma \frac{f(u)}{u - z} \,\mathrm{d}u \quad \textit{for all } z \in \Omega \setminus |\sigma| \tag{14.2.2}$$

and

$$\int_\sigma f(z) \,\mathrm{d}z = 0. \tag{14.2.3}$$

Remark. It is not difficult to see that the first equation 14.2.2 implies the second. Let $z_0 \in \Omega \setminus |\sigma|$ be arbitrary. Since f is analytic in Ω, also the function $G : \Omega \to \mathbb{C}$, $G(z) = (z - z_0) f(z)$ is analytic in Ω. Thus we may apply equation 14.2.2 to it to get that

$$\frac{1}{2\pi i} \int_\sigma f(z) \,\mathrm{d}z = \frac{1}{2\pi i} \int_\sigma \frac{(z - z_0) f(z)}{z - z_0} \,\mathrm{d}z = n(\sigma; z_0) G(z_0) = n(\sigma; z_0)(z_0 - z_0) f(z_0) = 0.$$

Note that because the integration variable $z \in |\sigma|$ is never equal to the point $z_0 \in \Omega \setminus |\sigma|$, multiplying and dividing by $z - z_0$ in the integral is justified. □

Remark. If $\Omega = \mathbb{C} \setminus \{0\}$, then it is enough to require $n(\sigma; 0) = 0$.

Definition 14.2.4. A domain Ω in $\mathbb{C}$ is simply connected if $n(\gamma; \omega) = 0$ for each closed curve γ in Ω and each $\omega \in \mathbb{C} \setminus \Omega$. This means there are no holes in Ω.

14.3 Three corollaries of Cauchy global integral theorem

Corollary 14.3.1. *Suppose that Ω is a simply connected domain in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is an analytic function. Then, every cycle σ in Ω satisfies the following*

$$n(\sigma; z)f(z) = \frac{1}{2\pi i}\int_\sigma \frac{f(u)}{u-z}\,\mathrm{d}u \text{ for all } z \in \Omega \setminus |\sigma| \text{ and } \int_\sigma f(z)\,\mathrm{d}z = 0.$$

Proof. Since Ω is simply connected, $n(\sigma; b) = 0$ for all $\mathbb{C} \setminus \Omega$. □

Corollary 14.3.2. *If $\Omega = \mathbb{D}(z_0, R)$, then we recover the Cauchy local integral formula in the disc $\mathbb{D}(z_0, r)$, $r < R$, then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}(z_0,r)} \frac{f(u)}{u-z}\,\mathrm{d}u \ \forall\ z \in \mathbb{D}(z_0, r).$$

Corollary 14.3.3: The case $\Omega = \mathbb{C} \setminus \{0\}$.
If $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is analytic and σ is cycle in $\mathbb{C} \setminus \{0\}$ such that $n(\sigma; 0) = 0$, then

$$n(\sigma; z)f(z) = \frac{1}{2\pi i}\int_\sigma \frac{f(u)}{u-z}\,\mathrm{d}u$$

for all $z \notin |\sigma| \cup \{0\}$ and $\int_\sigma f(u)\,\mathrm{d}u = 0$.

14.4 Examples

1. An important example. Let $\gamma(t) = 3\exp(-it), t \in [0, 4\pi]$. Evaluate the integral $\int_\gamma \frac{\sin z}{4z-\pi}\,\mathrm{d}z$.
   (a) By Cauchy global integral formula, since $f(z) = \sin z$ is entire, that is $z \mapsto \sin z$ is analytic in the whole $\mathbb{C}$ and γ winds the point $\frac{\pi}{4}$ twice clockwise.
   (b) Also $\mathbb{C}$ is simply connected, we have with $\sigma = -2\gamma_0$, where $\gamma_0(t) = 3\exp(it), t \in [0, 2\pi]$,

   Now

$$\int_\gamma \frac{\sin z}{4z-\pi}\,\mathrm{d}z = \frac{1}{4}\int_\sigma \frac{\sin z}{z-\frac{\pi}{4}}\,\mathrm{d}z = \frac{1}{4}\cdot 2\pi i \cdot n(\sigma; \frac{\pi}{4})\sin\frac{\pi}{4} = \frac{1}{4}\cdot 2\pi i(-2)\frac{\sqrt{2}}{2} = -\frac{\pi\sqrt{2}}{2}i.$$

   Another way to solve the problem. By the definition of the integral along a cycle $\sigma = -2\gamma_0$, which equals to γ, and where $\gamma_0(t) = 3\exp(it)$, $t \in [0, 2\pi]$, and Cauchy local integral formula,

$$\int_\sigma \frac{\sin z}{4z-\pi}\,\mathrm{d}z = \frac{1}{4}\int_\sigma \frac{\sin z}{z-\frac{\pi}{4}}\,\mathrm{d}z \overset{*}{=} \frac{1}{4}(-2)\int_{\gamma_0} \frac{\sin z}{z-\frac{\pi}{4}}\,\mathrm{d}z \overset{**}{=} -\frac{1}{2}\cdot 2\sin\pi 4 = -\frac{\sqrt{2}}{2}\pi i.$$

   *The definition of the integral along the cycle γ. **Cauchy local formula, $\gamma_0(t) = \exp(it)$, $t \in [0, 2\pi]$.

2. Let Ω be a domain and let $\overline{\mathbb{D}(z_0, R)} \subseteq \Omega$. If $\sigma = 8\partial\mathbb{D}(z_0, R)$, where $\gamma_0(t) = z_0 + R\exp(it)$, $t \in [0, 2\pi]$, then by the definition of the integral along σ and by the Cauchy local integration formula,

$$\frac{1}{2\pi i}\int_\sigma \frac{f(u)}{u-z}\,\mathrm{d}u \overset{*}{=} \frac{1}{2\pi i}8\int_{\partial\mathbb{D}(z_0,R)} \frac{f(u)}{u-z}\,\mathrm{d}u = 8\frac{1}{2\pi i}\int_{\partial\mathbb{D}(z_0,R)} \frac{f(u)}{u-z}\,\mathrm{d}u$$
$$\overset{**}{=} 8f(z), \quad z \in \mathbb{D}(z_0, R).$$

   *The definition of the integral along the cycle γ. **Cauchy local formula.

   Hence, in Cauchy global integral formula,

$$\frac{1}{2\pi i}\int_\sigma \frac{f(u)}{u-z}\,\mathrm{d}u = n(\sigma; z)f(z),$$

   the winding number is needed!

Remark. Let $\Omega = \mathbb{C} \setminus \{0\}$. The condition $n(\sigma; 0) = 0$ is necessary. Let $\gamma(t) = \exp(it), t \in [0, 2\pi]$. Then $n(\gamma; 0) = 1$ and $\int_\gamma \frac{\mathrm{d}z}{z} = 2\pi \neq 0$, although f is analytic in $\mathbb{C} \setminus \{0\}$.

14.5 Deformation theorem

Still one easy corollary of Cauchy global integration theorem. This corollary is very useful when evaluating given integrals.

Theorem 14.5.1. *Suppose that Ω is a domain in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is an analytic function. If σ_1 and σ_2 are cycles in Ω such that*

$$n(\sigma_1, b) = n(\sigma_2, b), \quad \textit{for all } b \in \mathbb{C} \setminus \Omega, \tag{14.5.2}$$

then

$$\int_{\sigma_1} f(u)\,\mathrm{d}u = \int_{\sigma_2} f(u)\,\mathrm{d}u. \tag{14.5.3}$$

Proof. Let

$$\sigma_1 = \sum_{j=1}^{k} \eta_j \gamma_j \qquad \text{and} \qquad \sigma_2 = \sum_{j=1}^{l} \mu_j \eta_j.$$

We write $\sigma := \sigma_1 - \sigma_2$. Then $n(\sigma; b) = n(\sigma_1; b) - n(\sigma_2, b) = 0$ by (14.5.2) for all $b \in \mathbb{C} \setminus \Omega$. By Cauchy global integral theorem,

$$0 = \int_\sigma f(z)\,\mathrm{d}z = \int_{\sigma_1} f(u)\,\mathrm{d}u - \int_{\sigma_2} f(u)\,\mathrm{d}u.$$

□

Example 14.5.4. Evaluate the integral $\int_\Gamma \frac{\cos z}{z^2 - 2z}\,\mathrm{d}z$, where

$$\Gamma = \gamma_{[-1-i,1-i]} * \gamma_{[1-i,1+i]} * \gamma_{[1+i,-1+i]} * \gamma_{[-1+i,-1-i]}$$

is the edge of the rectangle in $\mathbb{C}$.

- First method. The Cauchy global integral formula:

  $$\int_\Gamma \frac{\frac{\cos z}{z-2}}{z}\,\mathrm{d}z = 2\pi i f\left(0\right) = -\pi i,$$

  as $n\left(\Gamma; 2\right) = 0$.

- Second method. By the deformation theorem and using two different cycles Γ and γ:

  $\gamma(t) = \frac{1}{2}\exp(it), t \in [0, 2\pi]$

  $f : z \mapsto \frac{\cos z}{z(z-2)}$ is analytic in $\mathbb{C} \setminus \{0, 2\}$

  $n(\Gamma; 0) = n(\gamma; 0) = 1$ and $n(\Gamma; 2) = n(\gamma; 2) = 0$

  $g : z \mapsto \frac{\cos z}{z-2}$ is analytic in $\mathbb{D}\left(0, \frac{3}{2}\right)$ and $g(0) = -\frac{1}{2}$.

  Then:

  $$\int_\Gamma \frac{\cos z}{z(z-2)}\,\mathrm{d}z \overset{*}{=} \int_\gamma \frac{\cos z}{z(z-2)}\,\mathrm{d}z = \int_\gamma \frac{\frac{\cos z}{z-2}}{z}\,\mathrm{d}z \overset{**}{=} 2\pi i g(0) = -\pi i.$$

  *Deformation theorem. **Cauchy local formula, as $g(z) = \frac{\cos z}{z-2}$ is analytic in $\mathbb{C} \setminus \left\{2\right\}$.

Example 14.5.5. Let $z_1, z_2 \in \mathbb{C}$ and $r > 0$, such that $|z_1| < r < |z_2|$. Let $\partial = 7\gamma$ be a cycle where $\gamma : [0, 2\pi] \mapsto \mathbb{C}$ and $\gamma(t) = r\exp(it)$. Show that

$$\int_\partial \frac{\mathrm{d}z}{(z - z_1)(z - z_2)} = \frac{14\pi i}{z_1 - z_2}.$$

Let us define $f : \mathbb{C} \setminus z_2 \mapsto \mathbb{C}$, $f(z) = \frac{1}{z-z_2}$. Then f is analytic in $\mathbb{C} \setminus z_2$. Now $n(\partial; z_2) = 0$. Then

$$\begin{aligned}\int_\partial \frac{\mathrm{d}z}{(z-z_1)(z-z_2)} &= \int_\partial \frac{\frac{1}{z-z_2}}{z-z_1}\,\mathrm{d} = 2\pi i n(\partial; z_1) f(z_1)\\ &= 2\pi i \cdot 7 \cdot \frac{1}{z_1 - z_2} = \frac{14\pi i}{z_1 - z_2}.\end{aligned}$$

14.6 Integrating rational functions on the real line

Recall that if the limits

$$\lim_{a\to\infty} \int_0^a f(x)\,\mathrm{d}x \quad \text{and} \quad \lim_{b\to-\infty} \int_b^0 f(x)\,\mathrm{d}x$$

exist, then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{b\to-\infty} \int_b^0 f(x)\,\mathrm{d}x + \lim_{a\to\infty} \int_0^a f(x)\,\mathrm{d}x, \tag{14.6.1}$$

and the integral

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x \tag{14.6.2}$$

exists and is equal to 14.6.1. On the other hand, the existence of 14.6.2 does not imply the existence of 14.6.1.

Example. Let $R > 0$ and $f(x) = x$. Then

$$\int_{-R}^{R} x\,\mathrm{d}x = \frac{1}{2} \Big/_{-R}^{R} x^2 = 0.$$

But the integral $\int_{-\infty}^{\infty} x\,\mathrm{d}x$ does not exist:

$$\lim_{a\to\infty} \int_0^a x\,\mathrm{d}x + \lim_{b\to\infty} \int_b^0 x\,\mathrm{d}x = \infty - \infty.$$

Remark. Let $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials such that $\deg Q \geq 2 + \deg P$ and Q does not have real roots. Then $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ exists and we can integrate

$$\int_{-\infty}^{\infty} \frac{P(z)}{Q(z)}\,\mathrm{d}x$$

with the help of complex analysis.

Example. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 - 2x + 5}.$$

Note that $x^2 - 2x + 5$ does not have real roots. Because this integral exists, and we can write

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 - 2x + 5} = \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{d}x}{x^2 - 2x + 5}.$$

Assume that $R \gg 10$. Let us write

$$f(z) = \frac{1}{z^2 - 2z + 5} = \frac{1}{\Big(z - \big(1 + 2i\big)\Big)\Big(z - \big(1 - 2i\big)\Big)};$$

f is analytic in $\mathbb{C} \setminus \{1+2i, 1-2i\}$. We may write

$$\int_{\gamma_{[-R,R]}} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z \overset{*}{=} \int_{\gamma_{[-R,R]} * C_R} f(z)\,\mathrm{d}z.$$

*Definition of integration and using the Deformation theorem and the Estimation lemma for $\int_{C_R}$.

Let $\gamma(t) = 1 + 2i + \exp\left(i2\pi t\right)$, $t \in [0,1]$. By the Deformation theorem

$$\int_{\gamma_{[-R,R]} * C_R} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z,$$

because f is analytic in $\mathbb{C} \setminus \{1+2i, 1-2i\} = \Omega$ and $\gamma_{[-R,R]}$ and γ are cycles in Ω and

$$n\left(\gamma_{[-R,R]} * C_R; 1+2i\right) = 1 = n\left(\gamma; 1+2i\right) \quad \text{and}$$
$$n\left(\gamma_{[-R,R]} * C_R; 1-2i\right) = 0 = n\left(\gamma; 1-2i\right).$$

By the Cauchy global integral formula we get

$$\int_{\gamma_{[-R,R]} * C_R} \frac{\frac{1}{z-(1-2i)}}{z-\left(1+2i\right)}\,\mathrm{d}z = 2\pi i n\left(\gamma_{[-R,R]} * C_R; 1+2i\right) g(1+2i)$$
$$= 2\pi i \frac{1}{1+2i-\left(1-2i\right)} = \frac{\pi}{2},$$

where $g : z \mapsto \frac{1}{z-(1-2i)}$ is an analytic function in the simply connected domain

$$\Omega = \left]-R-1, R+1\right[\times \left]-i, i\left(R+1\right)\right[.$$

By the estimation lemma 9.3.12

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| \le \pi R \frac{1}{R^2 - 2R + 5} = \frac{\pi}{r - 2\frac{5}{R}} \to 0, \quad \text{as } R \to \infty.$$

Hence,

$$\frac{\pi}{2} = \int_{\gamma_{[-R,R]} * C_R} f(z)\,\mathrm{d}z = \int_{\gamma_{[-R,R]}} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z,$$

where the last term approaches to zero when R approaches to infinity thus

$$\lim_{R\to\infty} \int_{-R}^{R} f(z)\,\mathrm{d}z = \frac{\pi}{2} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 - 2x + 5} = \frac{\pi}{2}.$$

14.7 The proof for the Cauchy global integral formula

The proof given here is based on a proof by Dixon in [10]. The proof of Theorem 14.2.1, more specifically equation (14.2.2), proceeds as follows:

1. Define $F : \Omega \times \Omega \to \mathbb{C}$ by
$$F(z,\xi) = \frac{f(z) - f(\xi)}{z - \xi} \qquad \text{when } \xi \neq z \tag{14.7.1}$$
and $F(z,\xi) = f'(\xi) = f'(z)$ when $\xi = z$. We will prove that F is continuous.

2. Define $h : \Omega \to \mathbb{C}$ by
$$h(z) = \frac{1}{2\pi i} \int_\sigma F(z,u)\,\mathrm{d}u. \tag{14.7.2}$$
We will prove that h is continuous and analytic in Ω. Notice that for all $z \in \Omega \setminus |\sigma|$, we have
$$h(z) = \frac{1}{2\pi i} \int_\sigma F(z,u)\,\mathrm{d}u = -f(z) n(\sigma; z) + \frac{1}{2\pi i} \int_\sigma \frac{f(u)}{u - z}\,\mathrm{d}u, \tag{14.7.3}$$
so our aim is to prove that $h(z) = 0$. This is done via Liouville's theorem, for which we need to extend h to all of $\mathbb{C}$.

3. Define $g : \mathbb{C} \setminus |\sigma| \to \mathbb{C}$ by
$$g(z) = \frac{1}{2\pi i} \int_\sigma \frac{f(u)}{u-z} \,\mathrm{d}u. \tag{14.7.4}$$
We will show that g is analytic in $\mathbb{C} \setminus |\sigma|$.

4. Putting h and g together, define $H : \mathbb{C} \to \mathbb{C}$ by
$$H(z) = \begin{cases} h(z), & z \in \Omega, \\ g(z), & z \in \mathbb{C} \setminus |\sigma|,\ n(\sigma; z) = 0. \end{cases} \tag{14.7.5}$$
We show that that H is well-defined and analytic in all of $\mathbb{C}$, so it is our desired extension of h.

5. Lastly, we see that H is bounded, so it is constant by Liouville's theorem. The constant value is seen to be 0, and so the claim follows.

To properly begin the proof, define F as in equation (14.7.1). Outside of the diagonal
$$\{(\omega, \omega) : \omega \in \Omega\} \subseteq \Omega \times \Omega,$$
the function F is clearly continuous. Thus it remains to verify continuity on the diagonal: let $\omega \in \Omega$ and $\varepsilon > 0$ be fixed.

Since the function f is analytic in Ω, it is infinitely differentiable in Ω by Cauchy's second formula 12.0.3. In particular, f' is continuous in Ω, so there exists a $\delta > 0$ so that $|f'(u) - f'(v)| < \varepsilon$ whenever $u, v \in \mathbb{D}(\omega, \delta)$.

Take any two points $z, \xi \in \mathbb{D}(\omega, \delta)$. If $\xi = z$ we immediately have
$$\left|F(z, \xi) - F(\omega, \omega)\right| = \left|f'(z) - f'(\omega)\right| < \varepsilon$$
which verifies continuity. Suppose then that $\xi \neq z$ and let $\gamma_{[z,\xi]} : [0,1] \to \mathbb{C}$ be the straight line $t \mapsto \xi + t(z - \xi)$ from z to ξ, which has a constant derivative $\gamma'_{[z,\xi]} \equiv z - \xi$. Notice that since the disc $\mathbb{D}(\omega, \delta)$ is convex, $|\gamma_{[z,\xi]}|$ is entirely contained in the disc. By the Fundamental Theorem of Calculus, we get that
$$f(z) - f(\xi) = \int_{\gamma_{[z,\xi]}} f'(u) \,\mathrm{d}u = (z - \xi) \int_0^1 f'(\xi + t(z - \xi)) \,\mathrm{d}t.$$
This allows us to estimate
$$\begin{aligned}
\left|F(z, \xi) - F(\omega, \omega)\right| &= \left| \frac{f(z) - f(\xi)}{z - \xi} - f'(\omega) \right| \\
&= \left| \int_0^1 f'(\xi + t(z - \xi)) \,\mathrm{d}t - f'(\omega) \int_0^1 \mathrm{d}t \right| \\
&\leq \int_0^1 \left| f'(\xi + t(z - \xi)) - f'(\omega) \right| \mathrm{d}t \\
&\leq \int_0^1 \varepsilon \,\mathrm{d}t \\
&= \varepsilon,
\end{aligned}$$
where we use the fact that $\omega, \xi + t(z - \xi) \in \mathbb{D}(\omega, \delta)$ for all $0 \leq t \leq 1$. Thus F is continuous.

For the second step, define h as in equation (14.7.2). Since F is continuous, the integral exists and h is well defined. To see that h is continuous, let $\omega \in \Omega$ and let $\omega_n \to \omega$ be a sequence such that $\omega_n \in \Omega$ for all n.

Since the set $\overline{\mathbb{D}}(\omega, \delta) \times |\sigma|$ is compact (as it is closed and bounded), the restriction of F to that set is uniformly continuous. Given any $\xi \in |\sigma|$, we have that $(\omega_n, \xi) \to (\omega, \xi)$, so by uniform continuity of F we have that $F(\omega_n, \xi) \to F(\omega, \xi)$ uniformly with respect to $\xi \in |\sigma|$. To be more specific, this means that the sequence of functions $\xi \mapsto F(\omega_n, \xi)$ converges uniformly to the function $\xi \mapsto F(\omega, \xi)$.

Since integration commutes with uniform convergence, we get that

$$h(\omega_n) = \frac{1}{2\pi i} \int_\sigma F(\omega_n, u)\,\mathrm{d}u \to \frac{1}{2\pi i} \int_\sigma F(\omega, u)\,\mathrm{d}u = h(\omega),$$

so h is indeed continuous. Verifying equation (14.7.3) takes just a calculation: given a $z \in \mathbb{C}\setminus|\sigma|$, z is never equal to the integration variable $u \in |\sigma|$, so

$$\begin{aligned} h(z) &= \frac{1}{2\pi i} \int_\sigma F(z, u)\,\mathrm{d}u \\ &= \frac{1}{2\pi i} \int_\sigma \frac{f(z) - f(u)}{z - u}\,\mathrm{d}u \\ &= \frac{1}{2\pi i} \int_\sigma \frac{f(z)}{z - u}\,\mathrm{d}u - \frac{1}{2\pi i} \int_\sigma \frac{f(u)}{z - u}\,\mathrm{d}u \\ &= -f(z) \frac{1}{2\pi i} \int_\sigma \frac{\mathrm{d}u}{u - z} + \frac{1}{2\pi i} \int_\sigma \frac{f(u)}{u - z}\,\mathrm{d}u \\ &= -f(z)\, n(\sigma; z) + \int_\sigma \frac{f(u)}{u - z}\,\mathrm{d}u, \end{aligned}$$

as claimed. Thus, if we show that $h \equiv 0$, it follows that

$$n(\sigma; z) f(z) = \int_\sigma \frac{f(u)}{u - z}\,\mathrm{d}u,$$

which is the claim of the theorem.

Next, we wish to show that h is analytic, not just continuous. For this, fix $\omega \in \Omega$ and fix a $\delta > 0$ so that $\mathbb{D}(\omega, \delta) \subseteq \Omega$. We will show that f is analytic in $\mathbb{D}(\omega, \delta)$ by Morera's theorem 13.0.2.

Let γ be a closed contour in $\mathbb{D}(\omega, \delta)$. Now, since $z \mapsto F(z, \xi)$ is analytic in $\mathbb{D}(\gamma, \omega) \setminus \{\xi\}$ and continuous in the whole disc for any $\xi \in \Omega$, we have that

$$\int_\gamma F(z, \xi)\,\mathrm{d}x = 0$$

by (a slightly strengthened version of) the Cauchy–Goursat Theorem 11.0.5. For the function h, we then get that

$$\int_\gamma h(z)\,\mathrm{d}z = \int_\gamma \left(\int_\sigma F(z, \xi)\,\mathrm{d}\xi \right) \mathrm{d}z = \int_\sigma \left(\int_\gamma F(z, \xi)\,\mathrm{d}z \right) \mathrm{d}\xi = \int_\sigma 0\,\mathrm{d}\xi = 0.$$

Since the integral of h over any closed contour in $\mathbb{D}(\omega, \delta)$ is 0, h is analytic in the disc (and especially at ω) by Morera's theorem 13.0.2. Since $\omega \in \Omega$ was arbitrary, h is analytic in Ω.

(Note: it is true in general that if σ and γ are continuously differentiable parametrized curves from closed, bounded intervals to $\mathbb{R}^2$ and F is a continuous function of two real variables, then the order of integration over the paths can be exchanged, i.e.

$$\int_\sigma \int_\gamma F(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_\gamma \int_\sigma F(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

This follows from the theory of iterated integrals, by expanding both path integrals into usual integrals over closed intervals.)

The next step is to define g as in equation (14.7.4) and to show it is analytic. Notice that since $z \in \mathbb{C}\setminus|\sigma|$ and $\xi \in |\sigma|$, the denominator in the integrand never vanishes and so g is well defined. We are going to show that g is analytic by finding a power series representation for it.

Fix a $z_0 \in \mathbb{C} \setminus |\sigma|$ to serve as the center, and let $\delta > 0$ be such that $\mathbb{D}(z_0, 2\delta) \subseteq \mathbb{C} \setminus |\sigma|$. Also, for the time being, fix a $z \in \mathbb{D}(z_0, \delta)$.

Consider then a $\xi \in |\sigma|$. By the above, we have $|\xi - z_0| \geq 2\delta > 2|z - z_0|$. We can expand $\frac{1}{\xi - z}$ into a geometric sum as follows:

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 - (z - z_0)} = \frac{1}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}},$$

and the sum converges uniformly regardless of $\xi \in |\sigma|$, since $\left|\frac{z-z_0}{\xi - z_0}\right| \leq \frac{|z-z_0|}{2|z-z_0|} = \frac{1}{2}$ which holds because of how we chose z. Now we can write the function g as

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \int_\sigma \frac{f(\xi)}{\xi - z} \, \mathrm{d}\xi \\ &= \frac{1}{2\pi i} \int_\sigma f(\xi) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} \, \mathrm{d}\xi \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\sigma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, \mathrm{d}\xi \right) (z - z_0)^n. \end{aligned}$$

Exchanging the order of integration and summation is justified, since the sum converges uniformly with respect to ξ. Since the power series for g above converges, as is seen by reading the above chain of equations backwards, we have that g is analytic in $\mathbb{D}(z_0, \delta)$, and as $z_0 \in \mathbb{C} \setminus |\sigma|$ was arbitrary, g is analytic in all of $\mathbb{C} \setminus |\sigma|$.

The fourth step is to define H as in equation (14.7.5). Let $\Upsilon = \{z \in \mathbb{C} \setminus |\sigma| \,:\, n(\sigma; z) = 0\}$, that is, Υ is the set in the lower condition in the definition of H.

Suppose thus that $z \in \Omega \cap \Upsilon$, so $z \in \Omega$, $z \in \mathbb{C} \setminus |\sigma|$ and $n(\sigma; z) = 0$. It is convenient to use equation (14.7.3) for the value of h: we get that

$$h(z) = -f(z) n(\sigma; z) + \frac{1}{2\pi i} \int_\sigma \frac{f(u)}{u - z} \, \mathrm{d}u = 0 + \int_\sigma \frac{f(u)}{u - z} \, \mathrm{d}u = g(z),$$

so H is indeed well defined. We see that the domain of H is all of $\mathbb{C}$ by virtue of the assumption that $n(\sigma; b) = 0$ for all $b \in \mathbb{C} \setminus \Omega$. Since both g and h are analytic on their domains of definition, which are both open, also the function H is analytic.

The last step is to see that the entire function H is bounded. Towards that goal, let $R > 0$ be large enough so that $|\sigma| \subseteq \mathbb{D}(0, R)$. By the component lemma 14.1.8, it holds that $n(\sigma; z) = 0$ when z is in the unbounded component U of $\mathbb{C} \setminus |\sigma|$, and the complement of the disc $\mathbb{D}(0, R)$ is a subset of U.

Also, let $M = \sup \{|f(\xi)| \,:\, \xi \in |\sigma|\}$: this supremum exists, since $|\sigma|$ is compact, and a continuous function obtains a maximum in a compact set. Lastly, for any $z \notin \mathbb{D}(0, R)$ and $\xi \in |\sigma|$, we have the following estimate from the reverse triangle inequality:

$$|z - \xi| \geq |z| - |\xi| \geq |z| - R \geq 0.$$

Now, suppose that $z \in \mathbb{C} \setminus \mathbb{D}(0, R)$, that is $|z| > R$. Then z is in the unbounded component U, so $n(\sigma; z) = 0$, and $z \notin |\sigma|$, so the second piecewise definition of H applies. Using the estimation lemma 9.3.12, we get the following upper bound:

$$|H(z)| = \left| \frac{1}{2\pi i} \int_\sigma \frac{f(\xi)}{\xi - z} \, \mathrm{d}\xi \right| \leq \frac{1}{2\pi} \cdot \sup_{\xi \in |\sigma|} \left| \frac{f(\xi)}{\xi - z} \right| \cdot \operatorname{length}(\sigma) \leq \frac{\operatorname{length}(\sigma)}{2\pi} \frac{M}{|z| - R}.$$

Since the last bound goes to 0 as $|z| \to \infty$, we have that H is bounded outside of the disc $\overline{\mathbb{D}}(0, R)$. But since H is analytic and thus continuous, and the disc $\overline{\mathbb{D}}(0, R)$ is compact, H attains a maximum there, so H is bounded within the disc.

Thus H is bounded, so by Liouville's theorem 13.0.6, H is constant. But since $|H(z)| \to 0$ as $|z| \to \infty$, it must be that $H \equiv 0$. In particular, $h \equiv 0$ in Ω, which concludes the proof. □

15 The extended complex plane

15.1 The Riemann sphere and the extended complex plane

Let us consider $\mathbb{C}$ as embedded in the Euclidean 3-space $\mathbb{R}^3$ by identifying $x + iy$ with $(x, y, 0)$. Let

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \right\}.$$

This is a sphere which touches the plane $\mathbb{C}$ at the point $(0, 0, 0) \in S$ as shown in Figure 15.1.

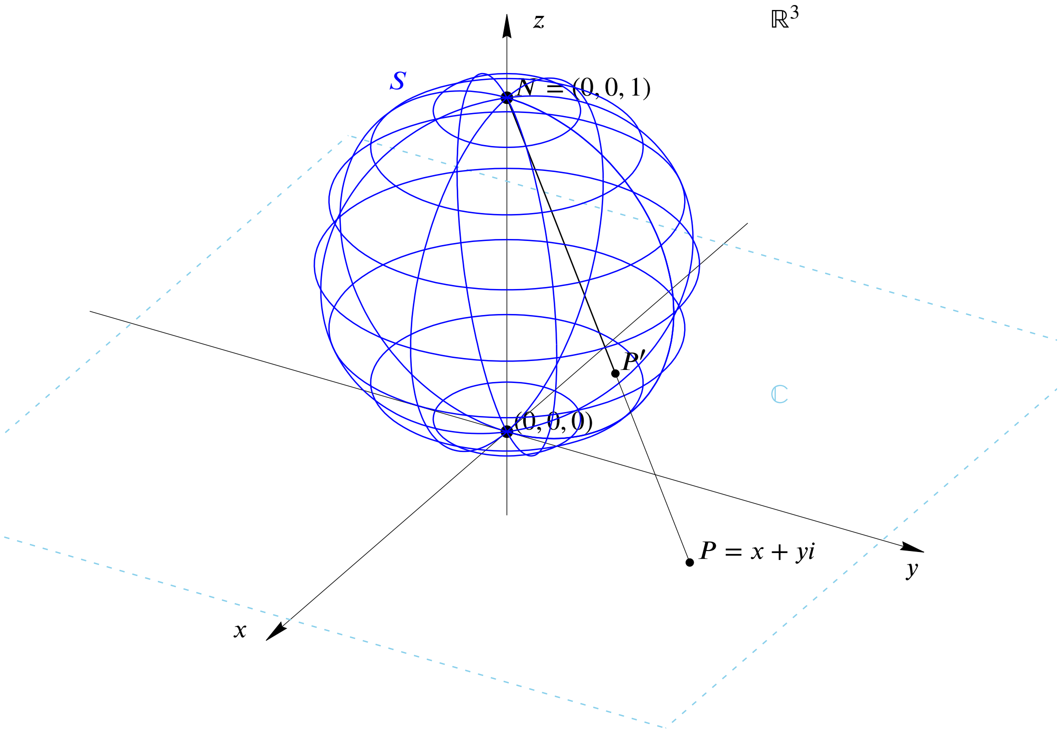


Figure 15.1: The Riemann sphere S and the complex plane $\mathbb{C}$ embedded in the Euclidean 3-space $\mathbb{R}^3$. A line segment between points $N \in S$ and $P \in \mathbb{C}$ intersects the set $S \setminus \{N\}$ at exactly one point P'. This way a one-to-one correspondence between the complex plane and the Riemann sphere (the extended complex plane $\overline{\mathbb{C}}$) can be constructed.

Stereographic projection allows us to set up a one-to-one correspondence between points in $\mathbb{C}$ and the points of S, excluding the north pole of S, namely the point $N = (0, 0, 1)$.

The line from any point $P \in \mathbb{C}$ to N intersects the set $S \setminus \{N\}$ at exactly one point P', and for every point P' in $S \setminus \{N\}$, the line through N and P' meets the plane $\mathbb{C}$ in a unique point P. We add to $\mathbb{C}$ an extra point $\infty \notin \mathbb{C}$ and we define the extended complex plane $\mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}} = \tilde{\mathbb{C}} = \dot{\mathbb{C}}$.

We have a correspondence between $\overline{\mathbb{C}}$ and S given by

$$P = x + iy = r\exp(i\phi) \leftrightarrow P' = \left(\frac{x}{1+r^2}, \frac{y}{1+r^2}, \frac{r^2}{1+r^2}\right),$$
$$\infty \leftrightarrow (0, 0, 1).$$

Remark. If we let $P = r\exp(i\phi)$ with ϕ fixed, and allow r to become arbitrarily large, then P' will approach N regardless of the value of ϕ.

Remark. We adopt the following conventions when working with $\overline{\mathbb{C}}$:

$$a \pm \infty = \pm\infty + a = \infty, \qquad \text{for all } a \in \mathbb{C},$$

$$
\begin{aligned}
&\frac{a}{\infty} = 0, && \text{for all } a \in \mathbb{C},\\
&a \cdot \infty = \infty \cdot a = \infty, && \text{for all } a \in \mathbb{C} \setminus \{0\},\\
&\frac{a}{0} = \infty, && \text{for all } a \in \mathbb{C} \setminus \{0\},\\
&\infty + \infty = \infty \cdot \infty = \overline{\infty} = \infty.
\end{aligned}
$$

Remark. The operations $\infty - \infty$, $0 \cdot \infty$ and $\frac{\infty}{\infty}$ are not defined.

15.2 Circlines

The benefits of moving to $\overline{\mathbb{C}}$ become clear when we consider lines and circles. Now we can treat lines and circles in a unified way.

15.3 The extended complex plane

15.3.1 Behavior of sets at ∞

Let us define

$$\mathbb{D}(\infty, r) = \{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}, r > 0.$$

The set $\mathbb{D}(\infty, r)$ is a "disc" centered on ∞ with radius $r > 0$. Using $\mathbb{D}(\infty, r)$, we can define a topology in $\overline{\mathbb{C}}$. Now $\Omega \subset \overline{\mathbb{C}}$ is an open set, if for every $a \in \Omega$ there exists $r(a) > 0$, such that $\mathbb{D}(a, r) \subset \Omega$.

Remark. Let (z_n) be a sequence in $\overline{\mathbb{C}}$. Then $z_n \to \infty$, if and only if, for each $r > 0$, there exists N_r such that

$$z_n \in \mathbb{D}(\infty, r) \text{ as soon as } n > N_r.$$

This means $|z_n| > r$ or $z_n = \infty$.

15.4 Behavior of functions at ∞

Let $a, b \in \overline{\mathbb{C}}$. Now

$$\lim_{z \to a} f(z) = b$$

if and only if for each $r > 0$, there exists $s > 0$ such that $f(z) \in \mathbb{D}(b, r)$ when $z \in \mathbb{D}(a, s) \setminus \{a\}$.

We use the inverse map $z \mapsto \frac{1}{z}$ to analyze what happens at ∞ or near ∞. We extend this mapping to $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by recalling that $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

Remark 15.4.1. Let $a, b \in \mathbb{C}$. Since

$$z \in \mathbb{D}(\infty, r) \text{ if and only if } \frac{1}{z} \in \mathbb{D}\left(0, \frac{1}{r}\right),$$

we have

$$
\begin{aligned}
&\lim_{z \to \infty} f(z) = b && \text{if and only if} && \lim_{z \to 0} f\left(\frac{1}{z}\right) = b\\
&\lim_{z \to a} f(z) = \infty && \text{if and only if} && \lim_{z \to a} \frac{1}{f(z)} = 0\\
&\lim_{z \to \infty} f(z) = \infty && \text{if and only if} && \lim_{z \to \infty} \frac{1}{f\left(\frac{1}{z}\right)} = 0.
\end{aligned}
$$

Proof. "$\Rightarrow$": Let $\varepsilon > 0$ be given. There exists $R > 0$ such that

$$|f(z)| \geq \frac{1}{\varepsilon} \text{ as soon as } |z| > R.$$

If $|z| = |z - 0| < \frac{1}{R}$, then $\left|\frac{1}{z}\right| > R$ and therefore

$$\left|\frac{1}{f\left(\frac{1}{z}\right)}\right| = \frac{1}{\underbrace{\left|f\left(\frac{1}{z}\right)\right|}_{\geq \frac{1}{\varepsilon}}} \leq \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

"$\Leftarrow$": Let $R > 0$ be given. There exists δ_r such that

$$\left|\frac{1}{f\left(\frac{1}{z}\right)}\right| < \frac{1}{R} \text{ as } |z| < \delta_r.$$

If $|z| > \frac{1}{\delta_r}$, then $\delta_r > \frac{1}{|z|}$. Then we have

$$|f(z)| = \frac{1}{\left|\frac{1}{f\left(\frac{1}{z}\right)}\right|} > \frac{1}{\frac{1}{R}} = R.$$

□

Definition 15.4.2. Suppose that f is defined in $\mathbb{D}(\infty, r)$ and its values are in $\mathbb{C}$. Then f is analytic at ∞ if $z \mapsto f\left(\frac{1}{z}\right)$ is analytic at the origin.

Definition 15.4.3. The case $f(z) = \infty$ for some $z \in \overline{\mathbb{C}}$. Suppose that f is defined in $\mathbb{D}(z_0, r)$ and $f(z_0) = \infty$.

1. If $z_0 \in \mathbb{C}$, then f is *meromorphic* at z_0, if the function $z \mapsto \frac{1}{f(z)}$ is analytic at z_0.
2. If $z_0 = \infty$, then f is meromorphic at $z_0 = \infty$, if the function $z \mapsto \frac{1}{f\left(\frac{1}{z}\right)}$ is analytic at 0.

The point z_0 in both cases is called *the pole*.

Definition 15.4.4. Let $\Omega \subset \overline{\mathbb{C}}$ be open and $f : \Omega \to \overline{\mathbb{C}}$. Then f is meromorphic in Ω, if f is meromorphic or analytic at each point of Ω.

Remarks.

1. When we consider analytic functions, we assume that their values are in $\mathbb{C}$. It might happen that ∞ is in the domain/set where the function is defined.
2. When we consider meromorphic functions, we allow the functions to have poles. The values of the functions are in $\overline{\mathbb{C}}$.

Examples.

1. Let $f(z) = \frac{z}{z^2-4}$.

   The function f is analytic in $\mathbb{C} \setminus \{\pm 2\}$. We extend f as a continuous function to $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by defining $f(2) = \infty$, $f(-2) = \infty$ and $f(\infty) = 0$. Now

$$f\left(\frac{1}{z}\right) = \frac{\frac{1}{z}}{\left(\frac{1}{z}\right)^2 - 4} = \frac{z^2}{\left(1 - 4z^2\right) z} = \frac{z}{1 - 4z^2} =: g(z).$$

The function $g(z)$ is analytic at 0. Thus, f is also analytic at ∞. What about $+2$ and -2? Now

$$\frac{1}{f(z)} = \frac{z^2 - 4}{z} =: h(z),$$

which is analytic at $+2$ and -2. These points are poles and thus f is meromorphic at them.

2. Let $f(z) = z^2 + z$ and $f(\infty) = \infty$. Now

$$\frac{1}{f\left(\frac{1}{z}\right)} = \frac{1}{\left(\frac{1}{z}\right)^2 + \frac{1}{z}} = \frac{z^2}{1+z} =: g(z).$$

The function $g(z)$ is analytic at 0 and thus f is meromorphic at ∞.

More generally, set $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $n \geq 1$ and $P(\infty) = \infty$. Since $\frac{1}{P\left(\frac{1}{z}\right)}$ is analytic at 0, P is meromorphic at ∞.

16 Möbius transformations

The extended complex plane is the right setting for studying Möbius transformations.

Definition. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $f(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$.

If $c \neq 0$, then $f\left(\frac{-d}{c}\right) = \infty$ and $f(\infty) = \frac{a}{c}$.

If $c = 0$, then $f(\infty) = \infty$.

Then, f is continuous, $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, and meromorphic. f is called a Möbius transformation of the complex plane.

Examples 16.0.1. Let α, β be given complex numbers and k and t_0 real-valued constants. The following are Möbius transformations:

1. Translation by β: $\omega = z + \beta$.
2. Rotation through t_0: $\omega = z(\exp(it_0))$. If $t_0 > 0$, the rotation is counterclockwise. If $t_0 < 0$, the rotation is clockwise.
   (a) $z \mapsto \exp\left(-i\frac{\pi}{2}\right)z = -iz$ is a rotation clockwise by $\frac{\pi}{2}$.
3. Dilation by a factor $k > 0$: $\omega = kz$. If $k > 1$, it is said to be stretching, and if $0 < k < 1$, it is said to be shrinking.
   (a) $f(z) = 2iz = 2\left(\exp\left(i\frac{\pi}{2}\right)\right)z$. Now f is a stretching by 2 and a counterclockwise rotation by $\frac{\pi}{2}$.
   (b) $f(z) = (1+i)z - i = \sqrt{2}\left(\exp\left(i\frac{\pi}{4}\right)\right)z - i$. Now f is a stretching by $\sqrt{2}$, a counterclockwise rotation by $\frac{\pi}{4}$ and a translation by $-i$.
4. Inversion: $\omega = \frac{1}{z}$.

Remarks.

1. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$, $z \neq 0$.
2. In polar coordinates: $z = r\exp(i\theta)$. Now $\frac{1}{z} = \frac{1}{r\exp(i\theta)} = r^{-1}\exp(-i\theta)$. Also, remember that $\left|\frac{1}{z}\right| = \frac{1}{|z|}$.

Proposition 16.0.2. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a Möbius transformation. Then f is one-to-one and onto, and the inverse of f is also a Möbius transformation, namely*

$$g(\omega) = \frac{d\omega - b}{-c\omega + a}.$$

Proposition 16.0.3. *Möbius transformations form a group, $(M, \circ)$, under the operation of composition of maps.*

Remark. The group $(M, \circ)$ is not commutative.

Remarks.

1. August Möbius (1790-1868). Möbius transformations are also known as the fractional linear transformations, the linear fractional transformations, and the bilinear transformations.
2. These mappings are useful for converting bounded domains into unbounded domains and vice versa.

3. Translations, rotations, dilations and inversions are all special cases of Möbius transformations.

Proposition 16.0.4. *Every Möbius transformation can be obtained by composing a translation, an inversion, a stretching/dilation, a rotation, and another translation, missing some out if necessary.*

Proof. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$. That is, $f = f_2 \circ f_1$, where

$$f_1(z) = \frac{a}{d}z,$$
$$f_2(z) = z + \frac{b}{d}.$$

If $c \neq 0$, we choose

$$f_1(z) = z + \frac{d}{c},$$
$$f_2(z) = \frac{1}{z},$$
$$f_3(z) = \frac{bc - ad}{c^2}z,$$
$$f_4(z) = z + \frac{a}{c}.$$

Then $f = f_4 \circ f_3 \circ f_2 \circ f_1$ and

$$f(z) = \frac{bc - ad}{c^2}\frac{1}{z + \frac{d}{c}} + \frac{a}{c}.$$

□

<u>*Remark*</u>. This composition is not unique.

16.1 The image of a circline is a circline

Möbius transformations map circles and lines to circles and lines. The image of a circle under a Möbius transformation is always either a circle or a line!

The general equation of a circle in the z-plane is

$$Az\overline{z} + Bz + \overline{B}\overline{z} + C = 0,$$

where $A, C \in \mathbb{R}$ and $B \in \mathbb{C}$. If $A = 0$ and $B \in \mathbb{C} \setminus \{0\}$, the circline reduces to a straight line.

Under the transformation of inversion, $\omega = \frac{1}{z}$, the above equation becomes

$$C \underbrace{\omega\overline{\omega}}_{=|\omega|^2} + \overline{B}\omega + B\overline{\omega} + A = 0,$$

a circline in the ω-plane.

Under the transformations of rotation, and stretching, $\omega = az$, the equation becomes

$$A\omega\overline{\omega} + \left(B\overline{a}\right)\omega + \left(\overline{B}a\right)\overline{\omega} + Ca\overline{a} = 0.$$

Also, under the transformation of translation, circlines are transformed into circlines.

Since a Möbius transformation can be obtained as a combination of translations, rotations, stretchings, and inversions, the claim follows.

16.2 Finding the image of a circline

If we are interested in finding the image of some circline under a Möbius transformation, we could use the fact that Möbius transformations map circlines to circlines and the fact that there is one and only one circline through any triplet of three distinct points in $\overline{\mathbb{C}}$.

Example 16.2.1. The real axis is the unique circline through 0, 1 and ∞, so its image under $z \mapsto \frac{1}{z-1}$ is a unique circline through -1, ∞ and 0, that is, the real axis.

Example 16.2.2. Let $f(z) = \frac{z+i}{z-i}$. Find the image of the boundary of the unit circle $\partial\mathbb{D}(0,1)$.

Solution. We choose three points from the boundary. Let us pick $-i$, 1 and i. The images of these three points are

$$f(-i) = 0, \qquad f(i) = \infty, \qquad f(1) = i.$$

Möbius transformations map circlines onto circlines. Since $\infty \in f\Big(\partial\mathbb{D}(0,1)\Big)$, the image of $\partial\mathbb{D}(0,1)$ is a line. Since $i \in f\Big(\partial\mathbb{D}(0,1)\Big)$ and $0 \in f\Big(\partial\mathbb{D}(0,1)\Big)$, the image of the boundary is the imaginary axis.

16.3 The image of a disc and the image of a half-plane

Every straight line L cuts $\overline{\mathbb{C}}$ into two disjoint half-planes; we denote them by $\mathbb{H}_1$ and $\mathbb{H}_2$.

Since Möbius transformations are bijective, continuous mappings $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, and the inverse of a Möbius transformation is also a Möbius transformation, and Möbius transformations preserve circlines, we obtain the following remark:

Remark 16.3.1. Let $z_0, \omega_0 \in \mathbb{C}$, $r, R > 0$. The image of $\mathbb{D}(z_0, r)$ or the complement of a closed disc, that is, $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}(z_0, r)}$, or a half-plane, is either a disc $\mathbb{D}(\omega_0, R)$ or $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}(\omega_0, R)}$ or a half-plane.

Example 16.3.2. Let $f(z) = \frac{z+i}{iz+1}$. Find the image of the unit disc.

Solution. First we find $f\Big(\partial\mathbb{D}(0,1)\Big)$: We pick three points from $\partial\mathbb{D}(0,1)$ and then their image points.

$$f(1) = 1, \qquad f(-1) = -1, \qquad f(i) = \infty.$$

Since one of the image points is ∞, we know that the image is a line. Since -1 and 1 are in the line, the image is the real line with ∞ point.

Because of remark 16.3.1, either $f\Big(\mathbb{D}(0,1)\Big) = \mathbb{H}_+$ or $f\Big(\mathbb{D}(0,1)\Big) = \mathbb{H}_-$. Since $f(0) = i \in \mathbb{H}_+$, we know by 16.3.1 that $f\Big(\mathbb{D}(0,1)\Big) = \mathbb{H}_+$.

Remark 16.3.3.

1. When $f(z) = z + b$, $b \in \mathbb{C}$, f is a translation, then for $w_0 = z_0 + b$ and $w_1 = z_1 + b$ is the equation $w_0 - w_1 = z_0 - z_1$. The distance between two input points is invariant under a translation.
2. When $f(z) = az + b$, $a, b \in \mathbb{C}$, f is a rigid motion, then the distance between two input points is not invariant under f. But if $w_j = az_j + b$, when $j = 0, 1, 2$, then

$$\frac{w_0 - w_1}{w_0 - w_2} = \frac{z_0 - z_1}{z_0 - z_1}.$$

   The ratio of the distances of these input points is invariant under on rigid motion.
3. What about Möbius maps?

16.4 Cross-ratios and the triplet representation of a Möbius transformation

Theorem 16.4.1. *Suppose that each $\{z_1, z_2, z_3\}$ and $\{\omega_1, \omega_2, \omega_3\}$ is an ordered triplet of distinct points in $\overline{\mathbb{C}}$. Then, there exists a unique Möbius transformation f such that $f(z_k) = \omega_k$, $k = 1, 2, 3$ and this $f(z) = \omega$ is given by the following:*

$$\frac{(\omega - \omega_1)(\omega_2 - \omega_3)}{(\omega - \omega_2)(\omega_1 - \omega_3)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)}$$

Proof. The map

$$g : z \mapsto \frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)}$$

takes z_1, z_2, z_3 to 0, ∞, 1, respectively, and the map

$$h : \omega \mapsto \frac{(\omega - \omega_1)(\omega_2 - \omega_3)}{(\omega - \omega_2)(\omega_1 - \omega_3)}$$

takes $\omega_1, \omega_2, \omega_3$ to 0, ∞, 1, respectively. The composition map $f := h^{-1} \circ g$ is a Möbius transformation which takes z_k to ω_k for $k = 1, 2, 3$ and

$$\frac{(\omega - \omega_1)(\omega_2 - \omega_3)}{(\omega - \omega_2)(\omega_1 - \omega_3)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)}.$$

The last step to prove uniqueness is left to the reader. □

Definition 16.4.2. The cross ratio of four complex numbers z_0, z_1, z_2, z_3, denoted by $[z_0, z_1, z_2, z_3]$ is the number

$$[z_0, z_1, z_2, z_3] = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_2)(z_1 - z_3)}.$$

If one of the points is ∞, then $[z_0, z_1, z_2, z_3]$ is defined as a limit.

Remark. Suppose $z_2 = \infty$. Take a_j so that $\lim_{j\to\infty} a_j = \infty$, and then

$$\lim_{j\to\infty} \frac{(z_0 - z_1)(a_j - z_3)}{(z_0 - a_j)(z_1 - z_3)} = \lim_{j\to\infty} \frac{z_0 - z_1}{z_1 - z_3} \frac{1 - \overbrace{\frac{z_3}{a_j}}^{\to 0}}{\underbrace{\frac{z_0}{a_j}}_{\to 0} - 1} = \frac{z_0 - z_1}{z_3 - z_1}.$$

Remark. The importance of this cross ratio is, that it is preserved by Möbius transformations, that is

$$[z_0, z_1, z_2, z_3] = \left[(z_0), f(z_1), f(z_2), f(z_3) \right]$$

for all Möbius transformation functions f.

The claim is trivial, if f is a translation $f(z) = z + c$, $c \in \mathbb{C}$. Similarly, if f is a dilation $f(z) = kz$, since all k-factors will cancel. Similarly, if f is a rotation $f(z) = \exp(i\alpha)\, z$.

With inversions note that $f(z_1) - f(z_2) = -\frac{z_1 - z_2}{z_1 z_2}$. Then

$$\frac{f(a_1) - f(a_2)}{f(a_1) - f(a_3)} \cdot \frac{f(a_3) - f(a_4)}{f(a_2) - f(a_4)} = \frac{a_1 - a_2}{a_1 - a_3} \cdot \frac{a_3 - a_4}{a_2 - a_4}.$$

Thus inversion does not affect the cross ratio either.

Since every Möbius transformation is a combination of transformations, dilations, rotations and inversions, we see that none of the Möbius transformations affect the cross ration and as a corollary of Theorem 16.4.1, we have the following theorem:

Theorem 16.4.3. *The cross ratio is invariant under Möbius transformations.*

Remark 16.4.4. One consequence of this formula is that Mobius transformations cannot map any four given points to any other four given points, unless their cross ratios are equal.

Example 16.4.5. There exists no Mobius transformation that maps $0, 1, 2, 3$ to $0, 1, 2, 10$.

Remark 16.4.6. However, one can use this cross ratio formula to find a Mobius transformation that maps three given points z_1, z_2, z_3 to any other three given points w_1, w_2, w_3, w, that is

$$\begin{aligned} f\left(z_1\right) &= w_1, \\ f\left(z_2\right) &= w_2, \\ f\left(z_2\right) &= w_3, \\ f(z) &= w, \end{aligned}$$

The identity should be valid $\left[z_1, z_2, z_3, z\right] = \left[w_1, w_2, w_3, w\right]$ and one can use this identity to solve for $f(z)$ in terms of z.

Example 16.4.7. Let the points $-1, 0, 1$ be given and we like to map them to $-1, i, 1$, respectively. By solving the equation

$$\begin{aligned} \left[-1, 0, 1, z\right] &= \left[-1, i, 1, w\right] \\ \frac{\left(-1-0\right)}{\left(-1-1\right)} \frac{\left(1-z\right)}{\left(0-z\right)} &= \frac{\left(1-i\right)\left(1-f\left(z\right)\right)}{\left(-1-1\right)\left(i-f\left(z\right)\right)} \\ \frac{z-1}{2z} &= \frac{\left(1+i\right)\left(f\left(z\right)\right)}{2\left(f\left(z\right)-i\right)} \\ f\left(z\right) &= \frac{z+i}{iz+1} \end{aligned}$$

we found the mapping f.

Example 16.4.8. Find the mapping f such that

$$\begin{aligned} f\left(-1\right) &= -1, \\ f\left(0\right) &= i, \\ f\left(1\right) &= 1. \end{aligned}$$

Solution. By letting $z_1 = -1$, $z_2 = 0$, $z_3 = 1$, $w_1 = -1$, $w_2 = i$, $w_3 = 1$ and $w = f(z)$ we can use Möbius transformation. Thus the mapping is

$$\begin{aligned} \frac{\left(z-z_1\right)\left(z_2-z_3\right)}{\left(z-z_2\right)\left(z_1-z_3\right)} &= \frac{\left(w-w_1\right)\left(w_2-w_3\right)}{\left(w-w_2\right)\left(w_1-w_3\right)} \\ \frac{\left(z+1\right)\left(0-1\right)}{\left(z-0\right)\left(-1-1\right)} &= \frac{\left(w+1\right)\left(i-1\right)}{\left(w-i\right)\left(-1-1\right)} \\ \frac{z+1}{z}\left(-1\right) &= \frac{w+1}{w-1}\left(i-1\right) \\ w + zwi &= i + z \\ w\left(1+iz\right) &= i + z \\ w &= \frac{z+i}{zi+1}. \end{aligned}$$

Example 16.4.9.

1. Find the Möbius transformation f such that:

$$f(0) = 0, \qquad f(1) = 1, \qquad f(-1) = \infty.$$

2. Find the image of $\mathbb{D}\big(0, 1\big)$ under this mapping.

Solution.

1. By using definition 16.4.2 and Theorem 16.4.1, we get

$$\begin{aligned}
[z, 0, 1, -1] &\Leftrightarrow [w, 0, 1, \infty] \\
\frac{z-0}{z-1} \cdot \frac{1-(-1)}{0-(-1)} &= \frac{\omega - 0}{\omega - 1} \\
2\frac{z}{z-1} &= \frac{\omega}{\omega - 1} \\
2z\omega - 2z &= \omega z - \omega \\
z\omega - 2z &= -\omega \\
z\omega + \omega &= 2z \\
\omega &= \frac{2z}{z+1}.
\end{aligned}$$

2. Note that $f(i) = 1 + i$. Since $-1 \mapsto \infty, 1 \mapsto 1, i \mapsto 1 + i$, then $f\Big(\partial\mathbb{D}\big(0, 1\big)\Big) = \{z : \operatorname{Re} z = 1\}$. Since $f(0) = 0$, $f\Big(\mathbb{D}\big(0, 1\big)\Big) = \{z : \operatorname{Re} z < 1\}$.

Remark 16.4.10. If a half-plane $\mathbb{H}$ is given and we need to find a Möbius transformation which maps $\mathbb{H}$ onto some disc $\mathbb{D}$, then we pick three points z_1, z_2 and z_3 from $\partial\mathbb{H}$ and three points ω_1, ω_2 and ω_3 from $\partial\mathbb{D}$ to find f such that $f(z_k) = \omega_k, k = 1, 2, 3$. Then we get a Möbius transformation such that $f(\mathbb{H}) = \mathbb{D}$ or $f(\mathbb{H}) = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. If $f(\mathbb{H}) = \mathbb{D}$ we are done. If $f(\mathbb{H}) = \overline{\mathbb{C}} \setminus \mathbb{D}$, we use a rotation g which gives $g(\mathbb{H}) = \overline{\mathbb{C}} \setminus \overline{\mathbb{H}}$. Then, we combine them to get $f \circ g$ and obtain $f(g(\mathbb{H})) = f\Big(\overline{\mathbb{C}} \setminus \mathbb{H}\Big) = \mathbb{D}$.

16.5 Möbius transformations preserve angles

Möbius transformation preserves the angles of objects, although they do not, in general, preserve the shape of objects.

Theorem 16.5.1. *Let γ_1 and γ_2 be two smooth C^1-curves that intersect at $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$, suppose that $\gamma_k'(t_0) \neq 0$ so that the curves aren't stationary at the intersection, and suppose that f is a Möbius transformation. Then the angle α at which the smooth curves intersect each other at t_0 is simply the difference in the arguments of their tangents at time t_0:*

$$\alpha = \operatorname{Arg}\Big(\gamma_2'(t_0)\Big) - \operatorname{Arg}\Big(\gamma_1'(t_0)\Big).$$

Proof. The transformed curves $f \circ \gamma_k$, $k = 1, 2$ are still smooth (at least when ignoring the possible pole of the Möbius transformation), and they intersect each other at $f(z_0) = f(\gamma_1(t_0)) = f(\gamma_2(t_0))$. Their tangents at time t_0 are given by the chain rule as

$$(f \circ \gamma_k)'(t_0) = f'(\gamma_k(t_0)) \cdot \gamma_k'(t_0) = f'(z_0) \cdot \gamma_k'(t_0).$$

Since f is a Möbius transformation, $f'(z_0) \neq 0$, so we can calculate

$$\begin{aligned}
&\operatorname{Arg}\Big(f'(z_0) \cdot \gamma_2'(t_0)\Big) - \operatorname{Arg}\Big(f'(z_0) \cdot \gamma_1'(t_0)\Big) \\
&= \operatorname{Arg}\big(f'(z_0)\big) + \operatorname{Arg}\Big(\gamma_2'(t_0)\Big) - \operatorname{Arg}\big(f'(z_0)\big) - \operatorname{Arg}\Big(\gamma_1'(t_0)\Big)
\end{aligned}$$

$$= \operatorname{Arg}\Big(\gamma_2'(t_0)\Big) - \operatorname{Arg}\Big(\gamma_1'(t_0)\Big)$$
$$= \alpha.$$

□

Thus if two (smooth) curves intersect at an angle α, when we transform them using a Möbius transformation, their images also intersect at an angle α. So angles are not distorted, only distances are.

The assumption that $f'\left(z_0\right) \neq 0$ is important!

Example 16.5.2. The real and imaginary axis meet at right angles, but they map under $f(z) = z^2$ to the positive real axis and negative real axis, respectively, which are 180° apart!

Example 16.5.3. Let $f(z) = \frac{z-1}{z-3}$.

1. What is the image of $\partial\mathbb{D}\big(0, 1\big)$ under f? Now

$$1 \mapsto 0,$$
$$i \mapsto \frac{2}{5} - \frac{i}{5},$$
$$-1 \mapsto \frac{1}{2}.$$

   The image is a circle which goes through the points 0, $\frac{2}{5} - \frac{i}{5}$ and $\frac{1}{2}$.

2. What is the image of the real axis? Now

$$-1 \mapsto 1/2,$$
$$0 \mapsto 1/3,$$
$$1 \mapsto 0.$$

   So the image of the real axis is the real axis.

The unit circle has right angles with the real axis at -1 and 1. Since Möbius transformations preserves angles, the image of the unit circle meets the image of the real axis at right angles at $f\left(-1\right) = 1/2$ and $f\left(1\right) = 0$. Hence the image of the unit circle is a circle with radius $\frac{1}{4}$ and center $\left(\frac{1}{4}, 0\right)$. Since $0 \mapsto \frac{1}{3} \in D\left(\left(\frac{1}{4}, 0\right), \frac{1}{4}\right)$ the unit circle maps onto $D\left(\left(\frac{1}{4}, 0\right), \frac{1}{4}\right)$.

17 Conformal mappings

17.1 On conformal mappings

Definition 17.1.1. Let Ω be open set in $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ analytic in Ω. The function f is conformal at $z \in \Omega$ if $f'(z) \neq 0$. The function f is conformal in Ω if f is conformal at every point in Ω, i.e. $f'(z) \neq 0$ for all $z \in \Omega$. If f is conformal at every $z \in \Omega$ and f is bijective, f is called a conformal mapping.

Remark 17.1.2.

1. The geometric property where angles are preserved under the analytic mappings is conformality.
2. Mappings which preserve angles of objects are conformal.

An analytic mapping $f : \Omega \to \Omega$ is a conformal mapping if $f'(z) \neq 0$ for all $z \in \Omega$ and f is bijective (that is, one-to-one and onto).

Example 17.1.3. Let

$$A = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) > 0\} \quad \text{and}$$
$$B = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\} = IH^+.$$

The mapping $f : A \to IH^+$ where $z \mapsto z^2$ is analytic in A and $f'(z) = 2z \neq 0$ for all $z \in A$. Hence f is conformal.

17.1.4 Conformality at ∞ as analyticity at ∞

Suppose that f is defined in $\mathbb{D}(\infty, r)$ and gets values in $\mathbb{C}$ there.

- Then f is conformal at ∞ if $z \mapsto f\left(\frac{1}{z}\right)$ is conformal at 0.

Suppose that f is defined in an open disc $\mathbb{D}(z_0, r)$ and $f(z_0) = \infty$.

- If $z_0 \in \mathbb{C}$, then f is conformal at z_0, if $z \mapsto \frac{1}{f(z)}$ is conformal at z_0.
- If $z_0 = \infty$, then f is conformal at ∞, if $z \mapsto \frac{1}{f\left(\frac{1}{z}\right)}$ is conformal at 0.

Example 17.1.5. Let $f(z) = \frac{z}{z^2-4}$, f is analytic in $\mathbb{C} \setminus \{-2, 2\}$. We have to set $f(\infty) = 0$ and $f(\pm 2) = \infty$ in order to get a continuous $f : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$. Now

$$f(\pm 2) = \infty : \quad \frac{1}{f(z)} = \frac{z^2-4}{z} = z - \frac{4}{z} \text{ is analytic at } \pm 2 \text{ and conformal at } \pm 2.$$

$$f(\infty) = 0 : \quad g(z) = \begin{cases} f\left(\frac{1}{z}\right) = \frac{1/z}{1/z^2-4} = \frac{z}{1-4z^4}, & 0 < |z| < r, \\ 0, \quad \text{when } z = 0. \end{cases}$$

Then in the latter case $g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{f} = 1 \neq 0$, and g is conformal at 0 and f is conformal at 0.

Example 17.1.6.

1. Find the image of the real axis under the mapping $f : z \mapsto \frac{z+i}{z+1}$.
2. Find the image of the boundary of $\mathbb{D}(0, 1)$ under the mapping $f : z \mapsto \frac{z+i}{z+1}$.

Solution.

1. We pick $1, 0, -1 \in \mathbb{R}$ and calculate that $f(-1) = \infty$, $f(0) = i$, and $f(1) = \frac{1}{2} + \frac{i}{2}$.

$$\{(x, y) : y = -x + 1\} = \{z : \operatorname{Im} z = -\operatorname{Re} z + 1\}.$$

2. Now

$$f(-1) = \infty \quad \text{(i.e. the image is a line)}, \qquad f(-i) = 0, \qquad f(1) = \frac{i+1}{2}$$

$$\partial\mathbb{D}\big(0, 1\big) \perp \underbrace{\{z : \operatorname{Im} z = 0\}}_{\text{The real axis}}$$

The angle at (0, 1) is $\frac{\pi}{2}$. By conformality,

$$f\Big(\partial\mathbb{D}\big(0, 1\big)\Big) \perp f\big(\{z : \operatorname{Im} z = 0\}\big) \quad \text{at} \quad f\big(1\big) = \frac{1}{2} + \frac{i}{2}.$$

Hence $f\Big(\partial\mathbb{D}\big(0, 1\big)\Big) = \{(x, y) : x = y\}$, or we could calculate

$$f\big(-1\big) = \infty, \qquad f(-i) = 0 \quad \text{and} \quad f\big(1\big) = \frac{1+i}{2}.$$

Examples 17.1.7.

1. Möbius transformations are conformal. We refer to 16.5 ($f'(z_0) \neq 0$).
2. $f(z) = z^2$ is not conformal at 0 but is at other points.

Example 17.1.8. Find the Möbius transformation f such that

$$f\big(1\big) = 0,$$
$$f\big(-1\big) = \infty,$$
$$f(i) = i.$$

Now

$$\frac{\big(z-1\big)\big(-1-i\big)}{\Big(z-\big(-1\big)\Big)\big(1-i\big)} = \frac{\big(w-0\big)(\infty - i)}{(w-\infty)\big(0-i\big)} = \frac{\big(w-0\big)}{\big(0-i\big)}\big(-1\big) = \frac{w}{-1}(-i).$$

Thus the transformation is

$$w = \frac{\big(z-1\big)\big(-i+1\big)}{\big(z+1\big)\big(i-1\big)}\big(-1\big) = \frac{z-1}{z+1},$$

and the given conditions apply

$$w\big(1\big) = 0,$$
$$w(i) = \frac{i-1}{i+1} = -\frac{1-i}{1+i} = -\frac{\big(1-i\big)^2}{1+1} = -\frac{1-2i-1}{2} = i,$$
$$w\big(-1\big) = \infty.$$

17.1.9. We refer to 16.5. Any analytic mapping f when $f'(z_0) \neq 0$ preserves angles between smooth curves.

17.2 On conformal mapping problems

Let Ω_1 and Ω_2 be two domains in $\overline{\mathbb{C}}$. The problem or task is to find out, if there exists a bijective conformal map $f : \Omega_1 \to \Omega_2$.

If the mapping exists, then the task is to find it.

Example 17.2.1. Find a bijective conformal map f which maps the sector

$$\{z : 0 < \operatorname{Arg} z < \frac{\pi}{2}, 0 < |z| < 1\}$$

onto $\mathbb{D}(0,1)$. See Figure 17.1.

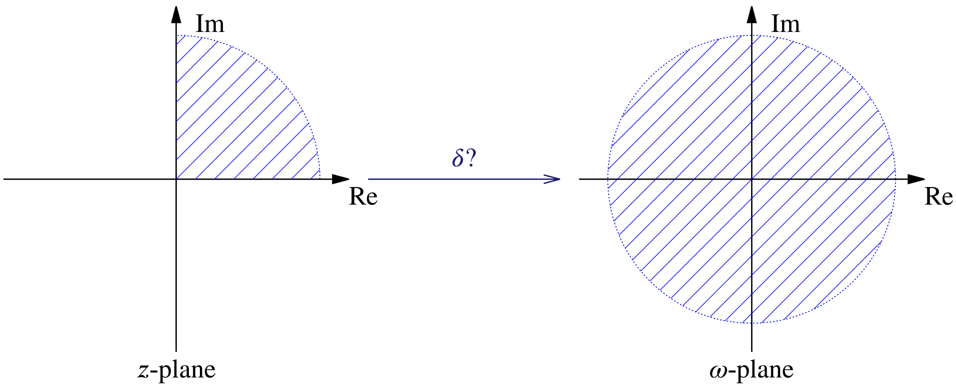


Figure 17.1: How to find a bijective conformal mapping δ which maps the sector $\{z : 0 < \operatorname{Arg} z < \frac{\pi}{2}, 0 < |z| < 1\}$ onto $\mathbb{D}(0,1)$? (Example 17.2.1.)

Solution. The asked mapping δ can constructed by composing suitable bijective conformal mappings $\omega_1, \omega_2, \omega_3$ and ω_4. Now we need to

1. map the real axis to the real axis using the cross ratio formula,
2. the image if the unit circle is the imaginary axis.

Let (as shown in Figure 17.2)

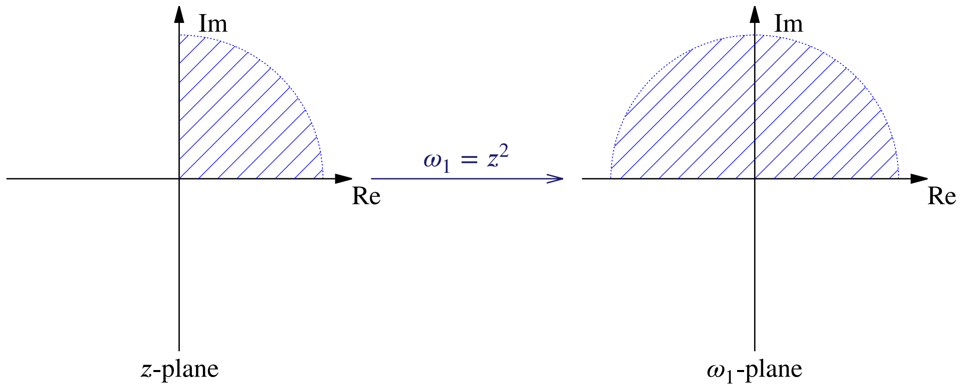
$$\omega_1 = z^2, \qquad \omega_2 = \frac{1+z}{1-z}, \qquad \omega_3 = \omega_1 = z^2, \quad \text{and} \quad \omega_4 = \frac{z-i}{z+i}.$$

Now

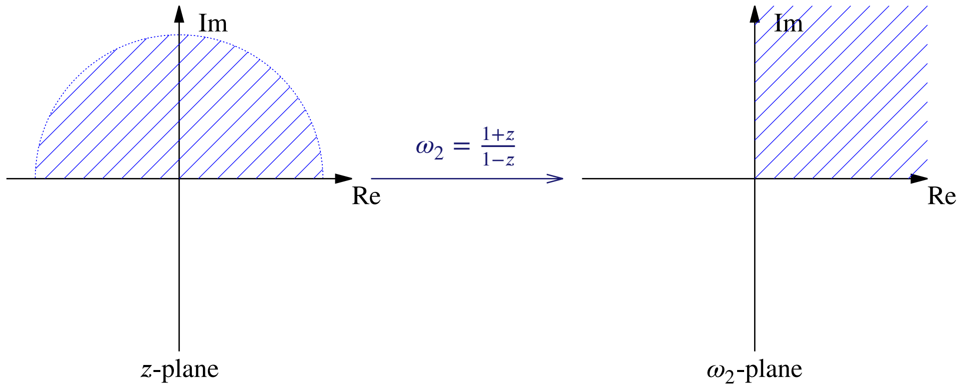
$$\omega_2 \circ \omega_1 = \frac{1+z^2}{1-z^2}, \qquad \omega_3 \circ \omega_2 \circ \omega_1 = \left(\frac{1+z^2}{1-z^2}\right)^2, \qquad \omega_4 \circ \omega_3 \circ \omega_2 \circ \omega_1 = \frac{\left(\frac{1+z^2}{1-z^2}\right)^2 - i}{\left(\frac{1+z^2}{1-z^2}\right)^2 + i};$$

$$\omega = \delta(z) = \omega_4 \circ \omega_3 \circ \omega_2 \circ \omega_1(z) = \frac{(1+z^2)^2 - i(1-z^2)^2}{(1+z^2)^2 + i(1-z^2)^2}$$

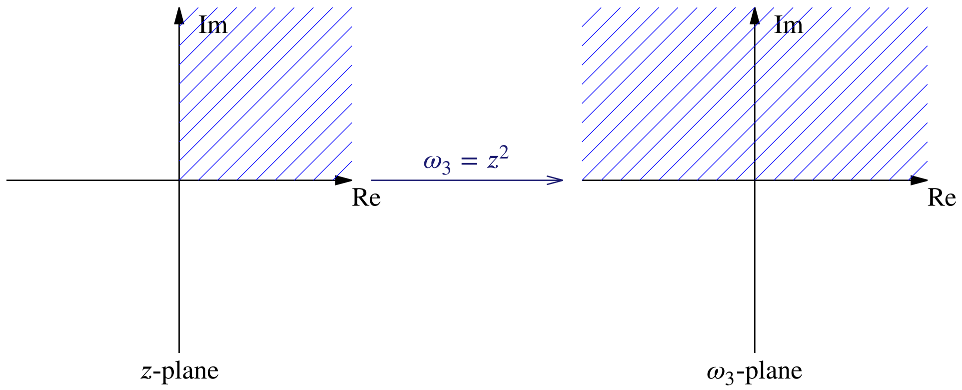
And the asked bijective conformal mapping δ has been constructed.



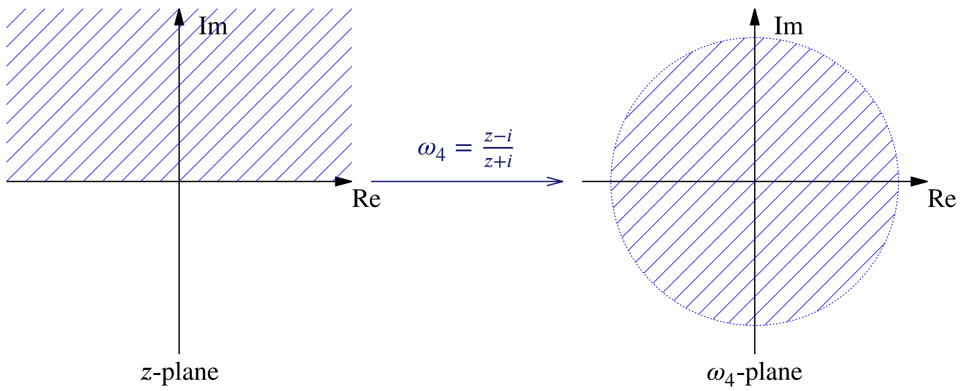
(a) The mapping ω_1, $z \mapsto z^2$ maps the quadrant to a half-circle.



(b) The mapping ω_2, $z \mapsto \frac{1+z}{1-z}$ maps the half-circle to a quarter-plane.



(c) The mapping $\omega_3 = \omega_1$, $z \mapsto z^2$ maps the quarter-plane to a half-plane.



(d) The mapping ω_4, $z \mapsto \frac{z-i}{z+i}$ maps the half-plane to a disc.

Figure 17.2: Construction of the bijective conformal mapping $\delta = \omega_4 \circ \omega_3 \circ \omega_2 \circ \omega_1$ in the solution to the example 17.2.1.

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Alphabetical Index