SIXTH EXERCISES FOR GMT

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In the first two exercises, a *Kakeya set* refers to a set which contains a **full line** in every direction, and not just a line segment. The proof of Theorem 7.2 from the lectures shows that there exist Kakeya sets of vanishing Lebesgue measure in $\mathbb{R}^2$.

Exercise 1 (1 point)**.** Construct a set $E \subset \mathbb{R}^2$ with positive Lebesgue measure such that $\pi_e(E)$ has empty interior for all $e \in S^1$. This shows that **Exercise 1** from the previous set is sharp. *Hint*: Kakeya sets.

Exercise 2 (2 points)**.** Use Marstrand's projection theorem to show that planar Kakeya sets have Hausdorff dimension 2. Don't worry about measurability questions here!

Hint: For a line $L = \{(x, y) : y = ax + b\}$, consider the dual point $p(L) = (a, b)$. So, the Kakeya set $K \subset \mathbb{R}^2$ gives rise to a dual set $p(K) \subset \mathbb{R}^2$. Start by arguing that $\mathcal{H}^1(p(K)) > 0$. Then, use Marstrand's projection theorem to show that $\dim[K \cap (\{t\} \times \mathbb{R})] = 1$ for almost every $t \in \mathbb{R}$. Finally, recall Lemma 5.38 from the lecture notes.

In the next exercise, a Kakeya set contains a unit line segment in every direction.

Exercise 3 (3 points)**.** Consider the proof from the lecture notes that planar Kakeya sets have box dimension 2 (i.e. Section 7.1). Run the same argument in $\mathbb{R}^d$. Which lower bound do you get for the (lower) box dimension of Kakeya sets in $\mathbb{R}^d$? You can take for granted that $\mathcal{H}^{d-1}(S^{d-1} \cap B(e, r)) \sim r^{d-1}$ for $0 < r < 1$ and $e \in S^{d-1}$.

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