Finite model theory
Problems 8
Tuesday 6.11.2018

1. A bipartite graph $\mathbb{G}$ is balanced if there is $U \subseteq \mathrm{Dom}(\mathbb{G})$ such that $|U| = |\mathrm{Dom}(\mathbb{G}) \setminus U|$ and all edges of $\mathbb{G}$ are between elements of U and $\mathrm{Dom}(\mathbb{G}) \setminus U$. Show that there is no sentence $\varphi \in \mathrm{MSO}$ such that for all finite graphs $\mathbb{G}$:

$$\mathbb{G} \models \varphi \Leftrightarrow \mathbb{G} \text{ is balanced.}$$

Hint: Show that if such a sentence exists then the non-regular language

$$L = \{w \in \{a, b\}^+ \mid w \text{ has the same number of } a\text{'s and } b\text{'s}\}$$

would be definable in MSO.

2. Show that $\mathrm{FO(DTC)} \leq \mathrm{FO(TC)}$.

3. Let τ be relational and finite. Show that for any $\varphi \in \mathrm{PFP}[\tau]$ there is an *equivalent* sentence $\varphi^* \in \mathcal{L}^{\omega}_{\infty,\omega}[\tau]$, that is, for all finite τ-models $\mathfrak{A}$:

$$\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{A} \models \varphi^*.$$

4. Let $\tau = \emptyset$. Show that there is no $\varphi \in \mathrm{PFP}[\tau]$ such that for all finite $\mathfrak{A}$:

$$\mathfrak{A} \models \varphi \Leftrightarrow |\mathrm{Dom}(\mathfrak{A})| \text{ is even.}$$

5. Show that on *ordered* finite structures, $\mathrm{SO} \leq \mathrm{PFP}$.