Finite model theory
Problems 9
Tuesday 13.11.2018

1. Let τ be a finite vodabulary with $\leq$, min, max, and S. Let $\mathfrak{A}$ be a finite ordered τ-model with $\mathrm{Dom}(A) = \{0, \ldots, n-1\}$. Construct FO(DTC)[$\tau$] formulas $\varphi_+(x, y, z)$ and $\varphi_*(x, y, z)$ such that for any $a, b, c \in \mathrm{Dom}(\mathfrak{A})$:

$$\begin{aligned} \mathfrak{A} \models \varphi_+[a/x, b/y, c/z] &\Leftrightarrow a + b = c, \\ \mathfrak{A} \models \varphi.[a/x, b/y, c/z] &\Leftrightarrow a \cdot b = c. \end{aligned}$$

2. Let $\tau = \{\leq, R_1, R_2\}$, where $ar(R_i) = 3$ for $i = 1, 2$. Let $\mathfrak{A}$ be a finite ordered τ-model with $\mathrm{Dom}(A) = \{0, \ldots, n-1\}$ where $\leq$ is interpreted naturally. Construct FO[τ]-sentences φ_1 and φ_2 such that for all $\mathfrak{A}$ as above:

$$\mathfrak{A} \models \varphi_1 \Leftrightarrow R_1^{\mathfrak{A}} = \{(a, b, c) \in \mathrm{Dom}(\mathfrak{A})^3 \mid a + b = c\},$$

and for all $\mathfrak{A}$ such that $\mathfrak{A} \models \varphi_1$:

$$\mathfrak{A} \models \varphi_2 \Leftrightarrow R_2^{\mathfrak{A}} = \{(a, b, c) \in \mathrm{Dom}(\mathfrak{A})^3 \mid a \cdot b = c\}.$$

3. Let τ be as in Exercise 1. Show that there is no FO[τ]-formula $\varphi_+(x, y, z)$ such that for all finite ordered τ-models $\mathfrak{A}$:

$$\mathfrak{A} \models \varphi_+[a/x, b/y, c/z] \Leftrightarrow a + b = c.$$

(Hint: Use our knowledge of the expressive power of FO over orderings and the expressivity obtained using $\varphi_+(x, y, z)$.)
Let $\tau = \{E\}$, and $\mathfrak{B}$ a finite graph. Define CSP($\mathfrak{B}$) as

$$\{\mathfrak{A} \mid \text{there is a homomorphism } h : \mathfrak{A} \to \mathfrak{B}\},$$

where $\mathfrak{A}$ ranges over finite graphs. Show that

4. CSP($\mathfrak{B}$) is the class of 3-colorable graphs assuming $\mathfrak{B}$ is a complete graph with three vertices.

5. CSP($\mathfrak{B}$) can be axiomatized in MSO[τ] for any finite $\mathfrak{B}$. (Hint: Generalize the idea used when defining 3-colorable graphs in MSO.)