Geometric measure theory part II

Exercise set 7

Exercise 1 When calculating the Hausdorff dimension of the Cantor set $C_{1/3}$, we constructed the measure μ by giving $\frac{1}{2}$ mass to the interval $[0, \frac{1}{3},]$ and to $[\frac{2}{3}, 1]$ and then continuing with the same pattern inside these intervals. Show that this construction really gives a radon measure μ on $C_{1/3}$. (Hint: Recall Caratheodory's construction)

Exercise 2 let $\{f_i\}_{i=1}^{\kappa}$ be an IFS in $\mathbb{R}^d$. According to Banach fixed point theorem (, because $\mathbb{R}^d$ is a complete metric space), each f_i has a unique fixed point a_i. Let $L^* = \max_i L_i$ and $r^* = \max_{i,j} |a_i - a_j|$, and set $R = r^*(1 - L^*)^{-1}$.

Show that $\bigcup_{i=1}^{\kappa} f_i(B(a_j, R)) \subset B(a_j, R)$ for any of the fixed points a_j.

Exercise 3 Let $f_1, f_2, f_3, f_4 \colon \mathbb{R}^d \to \mathbb{R}^d$, with

$$f_1(x) = \frac{1}{3}x,\ f_2(x) = \frac{1}{3}R_{\pi/6}x + (1/3, 0),\ f_3(x) = -\frac{1}{3}R_{-\pi/6}x + (2/3, 0),$$
$$f_4(x) = \frac{1}{3}x + (2/3, 0),$$

where R_α denotes the rotation by angle α. The limit set is called Von Koch snowflake.

Give an IFS of two mappings that has Von Koch snow flake as an invariant set. Hint: Drawing the picture might help.

Exercise 4 Consider an IFS in $\mathbb{R}^d$, with invariant set E. Show that $\sup_{x \in \mathbb{R}^d, r>0} |Z(x, r)| < \infty$ if and only if $\sup_{x \in E, r>0} |Z(x, r)| < \infty$.

Exercise 5 Consider a self-similar IFS in $\mathbb{R}^d$, with invariant set E, satisfying the strong separation condition. Show that $\sup_{x \in E, r>0} |Z(x, r)| < \infty$.