Finite model theory
Problems 10
Tuesday 20.11.2018

1. Let $\Sigma = \{0, 1\}$. The problem HALT

$$\text{HALT} = \{w_M \mid M \text{ a Turing machine that halts with empty input } \lambda\ \}$$

is known to be undecidable (not decidable by any Turing machine M), where w_M is a word encoding M is some fixed way. Sketch the idea of the proof that the set SAT(FO[τ])

$$\text{SAT}(\text{FO}[\tau]) = \{\varphi \in \text{FO}[\tau] \mid \mathfrak{A} \models \varphi \text{ for some finite model } \mathfrak{A}\},$$

is also undecidable for a suitable vocabulary τ. Hint: Construct a reduction from HALT to SAT by simulating the behavior of M with formulas.

2. Let τ be a finite relational vocabulary, and K a class of finite τ models. Let $< \notin \tau$ and let $K_<$ be the following class of $\tau \cup \{<\}$ models:

$$K_< = \{\langle \mathfrak{A}, < \rangle \mid \mathfrak{A} \in K \text{ and } < \text{ is an ordering of } \text{Dom}(\mathfrak{A})\}.$$

Let $\mathcal{L}$ be a logic and C a complexity class. We say that $\mathcal{L}$ *strongly captures* C if for all τ and classes K of finite τ-models:

$$K_< \in C \Leftrightarrow K = \text{Mod}(\varphi),\ \text{for some } \varphi \in \mathcal{L}[\tau].$$

Show that IFP does not strongly capture PTIME.

3. Show that Σ^1_1 strongly captures NPTIME.

Let $\varphi \in \text{FO}[\tau]$, where $\tau = \tau_1 \cup \{<\}$ and τ_1 is finite. Sentence φ is *order-invariant* if for all finite τ_1-models $\mathfrak{A}$ and all linear orderings $<, <'$ of Dom($\mathfrak{A}$):

$$\langle \mathfrak{A}, < \rangle \models \varphi \Leftrightarrow \langle \mathfrak{A}, <' \rangle \models \varphi.$$

4. Give an example of a first-order sentence φ which is not order-invariant.

5. Show that the set of order-invariant sentences of FO is undecidable. Hint: You may assume (or show using Exercise 1) that the set of sentences of FO that are valid (true in all finite models) is not decidable.