HOMEWORK 4

(1) (20pts) Let $A = \mathbb{Z}[i]$.
  (a) The principle ideal (2) of $A$ generated by 2 is not a prime ideal. Hint: Find $x, y \notin (2)$, but $xy \in (2)$.
  (b) The principle ideal (3) of $A$ is a prime ideal. Hint: If $xy \in (3)$, consider the norm of $xy$, use divisibility of 3 to conclude that one of $x, y$ has to be in (3). You need to consider what happens when you have 3 divides $a^2 + b^2$ with $a, b \in \mathbb{Z}$.
  (c) The principle ideal (5) of $A$ is not a prime ideal.
  (d) The principle ideal (7) of $A$ is a prime ideal.

(2) (15pts) Let $A = \mathbb{Z}[\sqrt{-5}]$. Show that:
  (a) The ideal $(3, 1 + \sqrt{-5})$ is a prime ideal.
  (b) The ideal $(3, 1 - \sqrt{-5})$ is a prime ideal.
  (c) $(3, 1 + \sqrt{-5}) \cdot (3, 1 - \sqrt{-5}) = (3)$.

(3) (10pts) $A$: Dedekind domain. $\alpha, \beta, \gamma \subset A$ ideals. Suppose $p_1, p_2, p_3 \subset A$ are prime ideals. Suppose

$$\alpha = p_1 p_3^2, \beta = p_1^2 p_2^2 p_3^2, \gamma = p_1^5 p_3$$

the decomposition of ideals. What is the decomposition for $\alpha + \beta + \gamma$?

(4) (15pts) [This exercise seems a bit tricky, try your best!]
  Let $f(T) = T^3 + T - 1$. Recall the fact: any polynomial in $\mathbb{R}[T]$ of odd degree has at least one real root (you can prove it by intermediate value theorem in calculus).
  (a) Show $f(T)$ has *only one* real root. (HINT: can use either calculus method or algebra method).
  (b) Show $f(T)$ has no roots in $\mathbb{Q}$.
  (c) Show that $f(T)$ is irreducible in $\mathbb{Q}[T]$. (HINT: try to make use of last item; also, Eisenstein criterion seems useless here...)
  Remark: We will make use of this exercise in our later lecture.

Addendum (NOT HW).

- You may wonder when is $(p)$ a prime ideal in $\mathbb{Z}[i]$, and when it is not. If you try some more computations with $p = 11, 13, 17$ etc, you will quickly find some pattern. There is indeed a general criterion, using something called Legendre symbol.