Finite model theory

Problems 11

Tuesday 27.11.2018

1. Let φ be a first-order formula with vocabulary τ ($\{\min, \max\} \subseteq \tau$) and $\vec{x}$ and $\vec{y}$ tuples of variables not appearing free in φ. Show that

$$[\mathrm{TC}_{\vec{x},\vec{y}}\,\varphi\,]\vec{\mathrm{min}}\vec{\mathrm{max}}$$

is equivalent to φ.

2. Let $\varphi(\vec{x}) \in \mathrm{FO}(pos\mathrm{TC})$ be a formula. Construct a formula $count_{\varphi}(\vec{x}) \in \mathrm{FO}(pos\mathrm{TC})$ such that for all ordered finite models $\mathfrak{A}$ and $\vec{a}$:

$$\mathfrak{A} \models count_{\varphi}[\vec{a}/\vec{x}] \Leftrightarrow \vec{a} = |\{\vec{b} \mid \mathfrak{A} \models \varphi[\vec{b}/\vec{x}]\}|,$$

where $\vec{a} = (a_0, \ldots, a_{k-1})$ is taken to encode the number

$$a_0 n^{k-1} + a_1 n^{k-2} + \cdots + a_{k-1},$$

and where $\mathrm{Dom}(\mathfrak{A}) = \{0, ..., n-1\}$.

3. Let $\varphi = \varphi_1 \wedge \varphi_2$, where φ_i is of the form

$$[\mathrm{TC}_{\vec{x},\vec{y}}\,\psi_i\,]\vec{\mathrm{min}}\vec{\mathrm{max}}$$

and ψ_i is quantifier-free for $i = 1, 2$. Show that then φ is also equivalent to a formula of the form

$$[\mathrm{TC}_{\vec{x}',\vec{y}'}\,\psi\,]\vec{\mathrm{min}}\vec{\mathrm{max}},$$

where ψ is quantifier-free.

4. Show that existential second-order logic Σ^1_1 is closed under $\vee$ and $\forall x$ in the following sense: if $\varphi, \psi \in \Sigma^1_1$, then there is $\theta \in \Sigma^1_1$ such that for all $\mathfrak{A}$:

$$\mathfrak{A} \models \varphi \vee \psi \Leftrightarrow \mathfrak{A} \models \theta$$

and

$$\mathfrak{A} \models \forall x \varphi \Leftrightarrow \mathfrak{A} \models \theta.$$

5. Show that for ordered finite models: $\mathrm{NPTIME} = co-\mathrm{NPTIME}$ iff $\Sigma^1_1 \equiv \mathrm{SO}$. Here, for each vocabulary τ, $co-\mathrm{NPTIME}$ contains the complements of NPTIME-classes:

$$\{K \subseteq O[\tau] \mid (O[\tau] \setminus K) \in \mathrm{NPTIME}\}.$$