Geometric measure theory part II

Exercise set 9

Exercise 1 (4 points) Let (X, μ, T) be a measure preserving system. Then the following are equivalent:

(1) (X, μ, T) is ergodic.
(2) The only Borel sets with $\mu(T^{-1}B \Delta B) = 0$ are those with $\mu(B) = 0$ or $\mu(B) = 1$.
(3) For every Borel set B with $\mu(B) > 0$, it holds that $\mu\left(\bigcup_{n\in\mathbb{N}} T^{-n}(B)\right) = 1$.
(4) For all Borel sets A and B with $\mu(A) > 0$ and $\mu(B) > 0$, there exists $n \in \mathbb{N}$ so that $\mu\left(T^{-n}A \cap B\right) > 0$.

(Hint: I guess it is most straight forward to prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.)

Exercise 2 (2 points) Let (X, μ, T) be a measure preserving system. Then the following are equivalent:

(1) (X, μ, T) is ergodic.
(2) If $f(x) = f(Tx)$ for all x (and f is measurable), then there exists c, so that $f(x) = c$ for μ almost all x.
(3) If $f(x) = f(Tx)$ for μ almost all x (and f is measurable), then there exists c, so that $f(x) = c$ for μ almost all x.

The fact that (1) is equivalent to (2) was proved in the lectures, so it is for example enough to show that (1) implies (3), since (3) implies (2) is trivial.

Exercise 3 (2 points) Let (X, T, μ) be ergodic system with T being continuous and $\mu(U) > 0$ for all open sets $U \subset X$. Show that μ almost all points of X have dense orbits. In other words, show that

$$\mu\Bigg(\{x \in X : (T^n x)_{n\in\mathbb{N}} \text{ is dense subset of } X\}\Bigg) = 1.$$

Exercise 4 (2 points) Prove Poincré's recurrence theorem, which states the following: Let (X, T, μ) be a measure preserving system and let $E \subset X$ with $\mu(E) > 0$. Then μ almost all points of E return to E infinitely often under iteration of T. (a.k.a. there exists $F \subset E$ with $\mu(F) = \mu(E)$, so that for every x in F and $N \in \mathbb{N}$, there exists $n \geq N$ so that $T^n x \in F$.)