Geometric measure theory part II

Exercise set 10

Exercise 1 Prove the Borel–Cantelli lemma, which states that if E_n is a sequence of sets with $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then

$$\mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n\right) = 0.$$

Exercise 2 Let (X, T, μ) be a measure preserving system and let $f \in L^1(\mu)$. Show that $\lim_{n\to\infty} \frac{1}{n}|f \circ T^n(x)| = 0$ for μ almost all $x \in X$. (Hint: Exercise 1 might help.)

Exercise 3 Do Exercise 4.8 from the lecture notes.

Exercise 4 Prove the following "invariant" version of the ergodic theorem: Let (X, T, μ) be a measure preserving system and let $f \in L^1(\mu)$. Then there exists a function $\tilde{f} \in L^1(\mu)$, so that

$$\frac{1}{N} \sum_{k=0}^{N-1} f(T^k x) \to \tilde{f}(x)$$

μ almost all $x \in X$. Furthermore, $\tilde{f} \circ T(x) = \tilde{f}(x)$ for μ almost all $x \in X$, and if $B = T^{-1}B$, then $\int_B f \, d\mu = \int_B \tilde{f} \, d\mu$.

Exercise 5 Assume the following maximal lemma: Suppose that $\int f \, d\mu > 0$, then

$$\mu(\{x \in X : f(x) + \cdots + f(T^{n-1}x) > 0 \text{ for all } n \in \mathbb{N}\}) > 0.$$

Prove the ergodic theorem using this Maximal lemma. (Hint: assume that $\int f \, d\mu = 0$ and study the function $f + \varepsilon$.)