Geometric measure theory part II

Exercise set 11

Exercise 1 Let X be compact and μ a probability measure on X. Let $1 \leq p \leq \infty$ and let f_n be sequence in $L^p(\mu)$ with $\|f_n - f\|_p \to 0$ as $n \to \infty$. Show that there exists a subsequence $n(k)$ so that $f_{n(k)}(x) \to f(x)$ μ almost everywhere.

Exercise 2 Exercise 6.2 from the lecture notes (Show that $k_i/N \to c_i^t$, for almost all $\mathtt{i}$ as $N \to \infty$).

Exercise 3 Let μ be a Bernoulli measure on $\Sigma = \{1, \ldots, \kappa\}^{\mathbb{N}}$ and for $k \in \{1, \ldots, \kappa\}$ define $E_k = \{\mathtt{i} \in \Sigma : \sigma^n \mathtt{i} \notin [k] \text{ for all } n\}$. Prove that $\mu(E) = 0$.

Exercise 4 In the proof of Theorem 6.4 in the lecture notes, why is it enough to consider the case $\dim_{\mathrm{H}} E < 1$? Also, explain why we can assume that $f_i(B(0,1)) \subset B(0,1)$ for all i and $f_i(U(0,1)) \cap f_j(U(0,1)) = \emptyset$ for all $i \neq j$.

Exercise 5 Prove the following theorem: Let $\Phi = \{f_i\}_{i=1}^{\kappa}$ be a self-similar IFS in $\mathbb{R}^2$ having attractor E and let each f_i be of the form $f_i(x) = c_i R_{\theta_i} O_i x + d_i$, where $0 < c_i < 1$, R_{θ_i} are rotations by angle θ_i, and O_i is either the identity or reflection with respect to x-axis (In deed, every self similar IFS can be expressed in this form). Then for every $\varepsilon > 0$ there is a IFS $\Phi' = \{g_i\}_{i=1}^{M}$, with $g_i(x) = cR_\theta x + t_i$ for some c, θ, and translations t_i, so that its attractor E' satisfies $E' \subset E$ and $\dim_{\mathrm{H}} E' \geq \dim_{\mathrm{H}} E - \varepsilon$.

More over if the group generated by $\{\theta_i\}_{i=1}^{\kappa}$ is dense in $[0, \pi]$, then θ can be chosen so that $\theta/\pi \in \mathbb{R} \setminus \mathbb{Q}$.

Hint: first get rid of the reflections, then proceed pretty much as in the proof of Theorem 6.1. no need to write all the details again, but be precise about how to choose θ.