Geometric measure theory part II

Exercise set 12

Exercise 1 Let $X = \mathbb{R}/\mathbb{Z}$, let $T_\alpha(x) = x + \alpha \mod 1$ and let μ be the Lebesgue measure restricted to $[0,1)$. Prove that (X, T, μ) is uniquely ergodic if and only if α is irrational. (Hint: show that if ν is invariant to T_α, then it is invariant to T_β for all $\beta \in [0,1)$. Then recall that Lebesgue is the unique translation invariant probability measure on $[0,1)$)

Exercise 2 Prove the unique ergodic theorem. That is, show that if (X, T, μ) is a uniquely ergodic system and $f \in L^1(\mu)$ continuous at μ almost all $x \in X$. Then

$$\frac{1}{N}\sum_{k=0}^{N-1} f(T^k x) \to \int_X f \, d\mu$$

uniformly for all $x \in X$. (Hint: you can first prove the statement for continuous f, the rest is pretty standard measure theory.)