ON THE PROBABILITY OF INVASION IN A MULTI-TYPE BRANCHING PROCESS WITH A SINGLE BIRTH STATE

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Consider a multi-type branching process with states 0, . . . , n, and where 0 corresponds to the unique birth state, and let \( b_j \) denote the birth rate and \( d_j \) the death rate in state \( j \), and let \( t_{ij} \) be the transition rate from state \( j \) to state \( i \). For the conservation of probability mass we necessarily have

\[
(1) \quad t_{jj} = -\sum_{i \neq j} t_{ij} \quad \forall j.
\]

Let further \( p_j[l] \) denote the probability that an individual presently in state \( j \) will produce \( l \) offspring during the rest of its stay in the same state \( j \), and let \( q_j(k) \) denote the probability that an individual presently in state \( j \) will produce \( k \) offspring during the rest of its life in the present state and all other states it will visit thereafter. Then

\[
(2) \quad p_j[l] = \left( \frac{b_j}{b_j + d_j - t_{jj}} \right)^l \frac{d_j - t_{jj}}{b_j + d_j - t_{jj}}
\]

(i.e., the probability that there are \( l \) birth-events followed by a single non-birth event which terminates the stay in state \( j \) either by a death event or a transition to another state), and

\[
(3) \quad q_j[k] = p_j[k] \frac{d_j}{d_j - t_{jj}} + \sum_{l=0}^{k} p_j[l] \sum_{i \neq j} \left( q_i[k - l] \frac{t_{ij}}{d_j - t_{jj}} \right)
\]

(i.e., the probability of producing \( k \) offspring in state \( j \) followed by a death event plus the probability of producing \( l \) offspring in state \( j \) and \( k - l \) offspring during the rest of the individual's life after a transition to another state).

Let \( f_j(z) \) and \( g_j(z) \) denote the probability generating functions of the distributions \( \{p_j[l]\}_{l \geq 0} \) and \( \{q_j[k]\}_{k \geq 0} \). Then

\[
(4) \quad f_j(z) = \frac{d_j - t_{jj}}{(1 - z)b_j + d_j - t_{jj}}
\]

and after some pretty straightforward calculations, also involving equation (4),

\[
(5) \quad g_j(z)((1 - z)b_j + d_j) = d_j + \sum_{\forall i} g_i(z)t_{ij}.
\]
Differentiation of equation (5) gives

\[ R_j d_j - b_j = \sum_{\forall i} R_i t_{ij} \]

where we used that \( g_j(1) = 1 \) and \( g_j'(1) = E_j\{k\} = R_j \), which is the reproduction ratio of state \( j \). Note that in particular \( R_0 \) is the well-known basic reproduction ratio. Define

\[ R := \begin{pmatrix} R_0 & \ldots & R_n \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}, \quad B := \begin{pmatrix} b_0 & \ldots & b_n \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \]

\[ D := \begin{pmatrix} d_0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & d_n \end{pmatrix}, \quad T := \begin{pmatrix} t_{00} & \ldots & t_{0n} \\ \vdots & \ddots & \vdots \\ t_{n0} & \ldots & t_{nn} \end{pmatrix} \]

Since there is only one birth state, \( R \) is equal to the so-called next generation matrix. Equation (6) can be written in matrix notation as

\[ R(D - T) = B \]

or equivalently

\[ R = B(D - T)^{-1} \]

which is possible because \( D - T \) is strictly diagonally dominant and thus can be inverted.

Next, let \( z_j \) denote the probability of the eventual extinction of the branching process starting in state \( j \). Then, substitution of \( z = z_0 \) in equation (5) gives

\[ z_j((1 - z_0)b_j + d_j) = d_j + \sum_{\forall i} z_i t_{ij} \]

where we used that \( g_j(z_0) = z_j \) for all \( j \). Let \( \pi_j = 1 - z_j \) denote the probability of invasion starting from state \( j \), then from equation (10) and equation (1) we get that

\[ \pi_j(\pi_0 b_j + d_j) = \pi_0 b_j + \sum_{\forall i} \pi_i t_{ij} \]

Define

\[ \Pi := \begin{pmatrix} \pi_0 & \ldots & \pi_n \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \]

then equation (11) can be rewritten as

\[ \Pi(\pi_0 B + D - T) = \pi_0 B. \]
Right-multiplication with \((D - T)^{-1}\), using equation (9), subsequently gives
\[
\Pi(\pi_0 R + I) = \pi_0 R
\]
or equivalently,
\[
\Pi = \pi_0 R(\pi_0 R + I)^{-1}
\]
where \(I\) is the identity matrix. We can do this because \(\pi_0 R + I\) is the product of two non-singular matrices, namely \(\pi_0 B + D - T\), which is strictly diagonally dominant, and \((D - T)^{-1}\). Hence \(\pi_0 R + I\) is non-singular itself and can be inverted. Formal expansion of the right hand side of equation (15) gives
\[
\Pi = \pi_0 R \sum_{i=0}^{\infty} (-1)^i \pi_0^i R^i
\]
which converges whenever all eigenvalues of \(\pi_0 R\) lie inside the unit circle in the complex plane, i.e., whenever \(\pi_0 R_0 < 1\). Writing out equation (18) for the upper leftmost element (i.e., the only element that matters, really), we get
\[
\pi_0 = \frac{\pi_0 R_0}{1 + \pi_0 R_0}
\]
i.e., \(\pi_0 = 0\) or
\[
\pi_0 = \frac{R_0 - 1}{R_0}
\]
whenever the latter is positive, i.e., whenever \(R_0 > 1\). If \(R_0 \leq 0\), then \(\pi_0 = 0\) is the only solution. Thus, in conclusion, we have shown that
\[
\pi_0 = \begin{cases} 
0 & \text{if } R_0 \leq 0 \\
\frac{R_0 - 1}{R_0} & \text{if } R_0 > 1.
\end{cases}
\]
I would like to emphasize that this result is exact.