INTEGRAL EQUATIONS

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1. Introduction

These lecture notes are based on the course Integral Equations as lectured in Spring 2017 and Spring 2019 in the Department of Mathematics and Statistics at the University of Helsinki.

Integral equations and research focused on them has played a central role in the applications of mathematics to physics, biology and engineering. In addition integral equations have affected the development of modern functional analysis in a significant way, as the following examples well illustrate.

Example 1.0.1. Consider the ordinary differential equation

(1.0.1) \[ y' = f(x, y) \]

By integrating both sides over the interval \([x_0, x]\) we get

(1.0.2) \[ y(x) = y(x_0) + \int_{x_0}^{x} f(s, y(s)) \, ds. \]

Note that this equation is generally nonlinear with respect to the unknown function \(y\). So, for simplicity suppose that

(1.0.3) \[ f(s, y(s)) = k(s)y(s) \]

for some function \(k\). In this case (1.0.2) reduces to a linear integral equation

(1.0.4) \[ y(x) = y(x_0) + \int_{x_0}^{x} k(s)y(s) \, ds. \]

This is a simple example of a Volterra equation of the second kind corresponding to the initial value problem

(1.0.5) \[ y'(x) = k(x)y(x), \quad y(x_0) = y_0, \]

which is easy to solve via separation of variables — easier than (1.0.4) actually!

Date: January 17, 2019.
The general form of a Volterra equation of the second kind is
\[(1.0.6) \quad y(x) = f(x) + \int_{x_0}^{x} k(s, x)y(s)ds\]
for a suitable function \(k\).

**Example 1.0.2.** If \(\rho\) is the charge density in \(\mathbb{R}^3\), then the (static) electric potential \(u\) satisfies a partial differential equation known as Poisson’s equation
\[(1.0.7) \quad \Delta u = -\frac{\rho}{\varepsilon_0}, \quad \varepsilon_0 \text{ is the vacuum permitivity.}\]

Here the *Laplace operator* \(\Delta\) is defined by
\[\Delta u(x) = \sum_{i=1}^{3} \frac{\partial^2 u(x)}{\partial x_i^2}.\]

Let us suppose that \(\rho = 0\) in the complement of a bounded set, and that \(u(x) \to 0\) as \(|x| \to 0\) at least as rapidly as \(1/|x|\). Then the solution to the Poisson’s equation is given by
\[(1.0.8) \quad u(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{4\pi\varepsilon_0|x-y|} dy,\]
which is also known as Poisson’s formula. This will be proven in the course *Partial Differential Equations I*. Note that this gives an explicit formula for \(u\) in terms of the charge density.

Let us investigate the same problem but in a bounded domain \(D \subset \mathbb{R}^3\). In addition we require that the boundary \(\partial D\) is grounded, i.e.,
\[(1.0.9) \quad \begin{cases} u|_{\partial D} = 0 \\ \Delta u = -\frac{\rho}{\varepsilon_0} \text{ in } D, \quad \text{supp}(\rho) \subset D. \end{cases}\]

Unfortunately, explicit formulas for the solution of this problem exist only when \(D\) is sufficiently symmetric, for example a sphere or a cube. So what can be done in the general situation?

Let \(u_0\) be the whole space solution given by the Poisson’s formula,
\[(1.0.10) \quad u_0(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{4\pi\varepsilon_0|x-y|} dy,\]
and search for \(u\) in the form
\[(1.0.11) \quad u = u_0 + v.\]

Now
\[(1.0.12) \quad -\frac{\rho}{\varepsilon_0} = \Delta u = \Delta u_0 + \Delta v \Rightarrow \Delta v = 0,\]
and for the boundary values we get
\[(1.0.13) \quad 0 = u|_{\partial D} = u_0|_{\partial D} + v|_{\partial D} \Rightarrow v|_{\partial D} = -u_0|_{\partial D}.\]
So denoting
\[f = -u_0|_{\partial D}\]
so we arrive at a boundary value problem known as the Dirichlet problem:

\begin{align}
\Delta v &= 0 \text{ in } D, \\
v|_{\partial D} &= f. 
\end{align}

This problem can be solved using integral equations in various ways.

First, write \( v \) as a single layer potential,

\begin{equation}
(1.0.16) \quad v(x) = \int_{\partial D} \frac{\varphi(y)}{4\pi|x-y|} dS(y)
\end{equation}

where \( \varphi \) is an (unknown) charge distribution on the boundary \( \partial D \). Using (1.0.15) one can prove that for \( \varphi \) one gets a Fredholm equation of the first kind

\begin{equation}
(1.0.17) \quad \int_{\partial D} \frac{1}{|x-y|} \varphi(y) dS(y) = 4\pi f(x), \quad x \in \partial D.
\end{equation}

This is an integral equation for functions defined on the boundary \( \partial D \), and after having solved \( \varphi \) from this one recovers \( v \) from (1.0.16), and then \( u = u_0 + v \). However, one can’t usually solve (1.0.17) explicitly. The best one can do is to prove that a unique and a sufficiently regular solution exists, or solve it numerically.

We may also write \( v \) using an unknown dipole distribution \( \psi(y), y \in \partial D \), in the form

\begin{equation}
(1.0.18) \quad v(x) = \int_{\partial D} \psi(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} dS(y),
\end{equation}

where \( \nu \) is the outer unit normal vector of \( \partial D \). This leads us to the double layer potential integral equation

\begin{equation}
(1.0.19) \quad \frac{1}{2} \psi(x) + \frac{1}{4\pi} \int_{\partial D} \psi(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} dS(y) = f(x), \quad x \in \partial D.
\end{equation}

which is a Fredholm equation of the second kind.

Both forms have their benefits, and if they are uniquely solvable, then \( v \) is uniquely defined. However, this is not necessarily the case, and hence we want to understand the structure of (1.0.17) and (1.0.19) in more detail.

2. **Volterra Equations**

Volterra equations have been named after Italian mathematician Vito Volterra (1860–1940). He investigated these equations in the first years of the 20th century.

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\[1\] In 1931, Volterra was one of the 12 university professors who refused to take an oath of loyalty to Benito Mussolini—the other 1238 did.
2.1. Unique Solvability of Equations of the Second Kind. Consider interval $[a, b] \subset \mathbb{R}$. Let $K \in C([a, b] \times [a, b])$, $f \in C([a, b])$, where both can be complex-valued. We wish to find a function $\varphi \in C([a, b])$ such that

\[(2.1.1)\quad \varphi(s) - \lambda \int_a^s K(s, t)\varphi(t) \, dt = f(s), \quad a \leq s \leq b.\]

Here $\lambda \in \mathbb{C}$ is a parameter. The equation (2.1.1) is known as the Volterra equation of the second kind.

To find a solution, we make an Ansatz, and try to find $\varphi$ as a sum of a series

\[(2.1.2)\quad \varphi(s) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(s) \quad \text{for some functions } \varphi_n.\]

Suppose that (2.1.2) converges absolutely on the interval $[a, b]$ when $|\lambda| \leq \lambda_0$ for some $\lambda_0 > 0$. By substituting (2.1.2) to (2.1.1) one gets by grouping together terms having an equal power of $\lambda$,

\[
\sum_{n=0}^{\infty} \lambda^n \varphi_n(s) - \lambda \int_a^s K(s, t) \sum_{n=0}^{\infty} \lambda^n \varphi_n(t) \, dt = f(s) \\
\Leftrightarrow \varphi_0(s) + \lambda \left( \varphi_1(s) - \int_a^s K(s, t) \varphi_0(t) \, dt \right) + \lambda^2 \left( \varphi_2(s) + \int_a^s K(s, t) \varphi_1(t) \, dt \right) + \cdots + \lambda^n \left( \varphi_n(s) + \int_a^s K(s, t) \varphi_{n-1}(t) \, dt \right) + \cdots = f(s).
\]

This equation is satisfied for all $\lambda$, $|\lambda| \leq \lambda_0$ if and only if

\[
\begin{align*}
\varphi_0(s) &= f(s) \\
\varphi_1(s) &= \int_a^s K(s, t) \varphi_0(t) \, dt \\
& \vdots \\
\varphi_n(s) &= \int_a^s K(s, t) \varphi_{n-1}(t) \, dt \\
& \vdots
\end{align*}
\]

The formulas in (2.1.4) yield an iterative algorithm for finding each $\varphi_n$ when $f$ and $K$ are given. We show that in fact (2.1.4) and (2.1.2) define a solution of (2.1.1) for all $\lambda \in \mathbb{C}$.

**Theorem 2.1.1.** Assume that $f \in C([a, b])$ and $K \in C([a, b] \times [a, b])$, and define $\varphi_n$ as in (2.1.4). Then $\varphi(s) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(s)$ is absolutely convergent for all $s \in [a, b]$, and defines a continuous function $\varphi \in C([a, b])$ for all $\lambda \in \mathbb{C}$ which is a solution of (2.1.1).

**Proof.** Let

\[M = \sup_{s \in [a, b]} |f(s)|, \quad N = \sup_{(s, t) \in [a, b] \times [a, b]} |K(s, t)|.\]

Then for all $s \in [a, b]$ we have

\[|\varphi_0(s)| = |f(s)| \leq M.\]

Using this we can estimate

\[|\varphi_1(s)| \leq \int_a^s |K(s, t)||\varphi_0(t)| \, dt \leq MN(s - a),\]
and
\[ |\varphi_2(s)| \leq \int_a^s |K(s,t)||\varphi_1(t)| \, dt \leq N \int_a^s MN(t-a) \, dt = \frac{MN^2(s-a)^2}{2}. \]
By induction on \( n \) we get the estimate
\[ |\varphi_n(s)| \leq \frac{MN^n(s-a)^n}{n!}. \]
To see this note first that clearly holds for \( n = 0 \). Suppose then that
\[ |\varphi_{n-1}(s)| \leq \frac{MN^{n-1}(s-a)^{n-1}}{(n-1)!}. \]
Then
\[ |\varphi_n(s)| \leq \int_a^s |K(s,t)||\varphi_{n-1}(s)| \, dt \leq \frac{MN^n}{(n-1)!} \int_a^s (t-a)^{n-1} \, ds = \frac{MN^n(s-a)^n}{n!} \]
proving (2.1.5). Hence
\[ \lambda^n|\varphi_n(s)| \leq \frac{M(\lambda N(s-a))^n}{n!}, \]
and since
\[ \sum_{n=0}^{\infty} \frac{(\lambda N(s-a))^n}{n!} = e^{\lambda N(s-a)} \]
converges, by the comparison test we see that
\[ \varphi(s) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(s) \]
converges absolutely and uniformly and defines a continuous function on the interval \([a,b]\). The fact that this function is a solution to (2.1.1) follows from the derivation of (2.1.4). \( \square \)

**Remark 2.1.1.** Note that if instead of (2.1.1) one had considered an equation
\[ \varphi(s) - \lambda \int_a^b K(s,t)\varphi(t) \, dt = f(s), \quad a \leq s \leq b, \]
one would still recover with the same ansatz a useful iterative scheme. Namely,
\[ \varphi_0 = f, \quad \varphi_n(s) = \int_a^b K(s,t) \varphi_{n-1}(t) \, dt, \quad n = 1, 2, \ldots \]
However, proving the convergence is then much more delicate. Of course for all \( s \in [a,b] \) it holds
\[ |\varphi_0(s)| \leq M, \]
but already for the next term the trivial estimate only gives
\[ |\varphi_1(s)| \leq \int_a^b |K(s,t)||\varphi_0(t)| \, dt \leq MN(b-a), \]
and by induction one gets
\[ |\varphi_n(s)| \leq \int_a^b |K(s,t)||\varphi_{n-1}(t)| \, dt \leq M(N(b-a))^n. \]
Hence by a comparison to a geometric series one can prove the series (2.1.2) converges to a continuous solution if

$$|\lambda|N(b - a) < 1.$$  

Hence one either needs a small $|\lambda|$, or for a given fixed $\lambda$ either a small kernel (i.e $N$ small), or a short interval. Equation (2.1.8) is an example of a Fredholm–equation of second kind, and getting rid of these smallness assumptions is one of the central themes of the course.

Theorem 2.1.1 shows the existence of a solution, and gives an explicit iterative algorithm for computing the solution. Next we prove uniqueness.

**Theorem 2.1.2.** The equation (2.1.1) has at most one solution $\varphi \in C([a, b])$.

**Proof.** Let $\varphi_1, \varphi_2 \in C([a, b])$ be solutions of the equation (2.1.1). Let $\psi = \varphi_1 - \varphi_2$. Then for all $s \in [a, b]$ we have

$$\varphi_1(s) - \lambda \int_a^s K(s, t) \varphi_1(t) \, dt = f(s) = \varphi_2(s) - \lambda \int_a^s K(s, t) \varphi_2(t) \, dt,$$

i.e the difference $\psi$ satisfies a homogenous Volterra–equation

$$\psi(s) - \lambda \int_a^s K(s, t) \psi(t) \, dt = 0.$$

It suffices to show that $\psi = 0$, i.e., $\varphi_1 = \varphi_2$. Let

(2.1.9) \hspace{1cm} L = \sup_{s \in [a, b]} |\psi(s)|, \hspace{0.5cm} N = \sup_{(s, t) \in [a, b] \times [a, b]} |K(s, t)|.

Hence

$$|\psi(s)| \leq \lambda \int_a^s |K(s, t)||\psi(t)| \, dt \leq LN\lambda(s - a).$$

By iterating this estimate (this is known as the Bootstrap Argument) we have

$$|\psi(s)| \leq \lambda \int_a^s |K(s, t)| |\psi(t)| \, dt \leq LN^2 \lambda^2(s - a)^2$$

and via induction for all $n \in \mathbb{N},$

$$|\psi(s)| \leq \frac{LN^n \lambda^n(s - a)^n}{n!}.$$

which holds for all $s \in [a, b]$. When $n \to \infty$, the right hand side of the above estimate tends to 0, whereas the left hand side is independent of $n$, hence we must have $\psi(s) = 0$ for all $s \in [a, b]$.

\square

**Example 2.1.1.** Consider the Volterra equation

(2.1.10) \hspace{1cm} \varphi(s) = s^2 + \int_0^s (s - t)\varphi(t) \, dt, \hspace{0.5cm} s \in [0, 1].

We wish to see how the iteration used in Theorem (2.1.1) converges to the solution of this equation. First one needs the exact solution. This is easiest to do by reducing the equation (2.1.10) to an initial value problem for an ordinary differential equation. Note
that if $\varphi$ is a continuous solution of (2.1.1), then the equation implies that it is also
differentiable, hence one may differentiate (2.1.10) and one gets

(2.1.11) \[ \varphi'(s) = 2s + [(s - t)\varphi(t)]_{t=s} + \int_0^s \frac{\partial(s - t)}{\partial s} \varphi(t) \, dt = 2s + \int_0^s \varphi(t) \, dt. \]

Differentiating this once more one finally gets a simple ordinary differential equation

(2.1.12) \[ \varphi''(s) = 2 + \varphi(s). \]

In order that this is uniquely solvable, one also needs suitable initial condition. These
follow immediately from the integral equation itself. First of all, evaluating (2.1.10) at
$s = 0$ one gets

(2.1.13) \[ \varphi(0) = 0, \]

and from the once differentiated equation (2.1.11) one obtains

(2.1.14) \[ \varphi'(0) = 2 + \varphi(0) = 2. \]

It is easy to see, that the unique solution of the initial value problem (2.1.12) – (2.1.14) is

(2.1.15) \[ \varphi(s) = e^s + e^{-s} - 2. \]

On the other hand, for the first terms of the iteration corresponding to (2.1.10) (note that
now $\lambda = 1$), we get

\[ \varphi_0(s) = s^2, \]

so

\[ \varphi_1(s) = \int_0^s (s - t)\varphi_0(t) \, dt = \int_0^s (s - t)t^2 \, dt = \frac{s^4}{3} - \frac{s^4}{4} = \frac{s^4}{12}, \]

\[ \varphi_2(s) = \int_0^s (s - t)\varphi_1(t) \, dt = \int_0^s (s - t)\frac{t^4}{12} \, dt = \frac{s^6}{60} - \frac{s^6}{72} = \frac{s^6}{360} \]

and

\[ \varphi_3(s) = \int_0^s (s - t)\varphi_2(t) \, dt = \int_0^s (s - t)\frac{t^6}{360} \, dt = \frac{s^8}{2520} - \frac{s^8}{2880} = \frac{s^8}{20160}. \]

Thus up to first four terms of the iteration we have

(2.1.16) \[ \varphi(s) = s^2 + \frac{s^4}{12} + \frac{s^6}{360} + \frac{s^8}{20160} + \cdots. \]

On the other hand, using the power series expansion of the exponential function, one has

\[ e^s + e^{-s} - 2 = \sum_{n=0}^{\infty} \frac{(1 + (-1)^n)s^n}{n!} - 2 = \sum_{k=1}^{\infty} \frac{2s^{2k}}{(2k)!} = s^2 + \frac{s^4}{12} + \frac{s^6}{360} + \frac{s^8}{20160} + \cdots, \]

which is exactly (2.1.16). Thus the iterative solution scheme produces the beginning of
the Taylor–series of the solution. Note however, that this is not always true (Think why!).
2.2. Connection to Ordinary Linear Differential Equations. There is a close connection between ordinary differential equations and certain Volterra–equations. To understand this, consider an ordinary differential equation

\[ y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x), \quad y^{(k)} = \frac{d^k y}{dx^k}. \]  

Here \( y \in C^n(I) \), \( p_k \in C(I) \), and \( I = (-a, a) \), \( a > 0 \) is an open interval. We want to reduce this to a Volterra–equation of the second kind.

The following notation will be useful: given \( \varphi \in C(I) \), and \( k \in \{1, 2, \ldots\} \), denote by \( I_k(\varphi) \) the integral function of order \( k \) of \( \varphi \) which vanishes at origin up to order \( k-1 \), i.e.

\[ \frac{d^k I_k}{dx^k} = \varphi, \quad \frac{d^l I_k(0)}{dx^l} = 0, \quad l = 1, \ldots, k-1. \]

Note that this determines \( I_k(\varphi) \) uniquely.

Let \( z(x) = y^{(n)}(x) \). Then if \( C_k = y^{n-k}(0) \) one has

\[ y^{n-1}(x) = I_1(z)(x) + C_1, \]
\[ y^{n-2}(x) = I_2(z)(x) + C_1x + C_2, \]
\[ \vdots \]
\[ y^{(k)}(x) = I_{n-k}(z)(x) + C_1 \frac{x^{n-k-1}}{(n-k-1)!} + \cdots + C_{n-k} \]
\[ \vdots \]
\[ y(x) = I_n(z)(x) + C_1 \frac{x^{(n-1)}}{(n-1)!} + \cdots + C_n. \]

Inserting these into (2.2.1) yields

\[ z + p_1(x)I_1(z)(x) + \cdots + p_n(x)I_n(z)(x) + \sum_{k=1}^{n} C_k f_k(x) = f(x) \]

where

\[ f_k(x) = p_k(x) + xp_{k+1}(x) + \cdots + \frac{x^{n-k}}{(n-k)!}p_n(x). \]

The next lemma gives an explicit and a very useful expression for \( I_k(\varphi) \) directly in terms of \( \varphi \):

**Lemma 2.2.1.** For all continuous \( \varphi \) and \( k \in \{1, 2, \ldots\} \) one has

\[ I_k(\varphi)(x) = \int_0^x \frac{(x-s)^{k-1}}{(k-1)!} \varphi(s) \, ds. \]

**References**