Wiener process \( W(t), t \geq 0 \)

Defined by its increment
\[ W(t) - W(s), \ t > s. \]

- Increments over non-overlapping intervals of equal length are i.i.d. with mean zero and finite variance.
- \( W(t) - W(s) \sim W(0, \sigma^2) \) where
  \( \sigma^2 \) is proportional to \( t - s \).
- Standard Wiener process has by definition \( W(0) = 0 \) and
  \[ W(t) - W(s) \sim W(0, t - s) \]

\[ \text{Solution: For given } \Delta t > 0, \]
we write \( \Delta W(t) = W(t + \Delta t) - W(t) \)

Hence \( \Delta W(t) \sim W(0, \Delta t) \).
Euler integration of ODE

Consider the ODE

\[ \frac{dx}{dt} = f(x, t), \quad x(0) = x_0. \]

This is in the limit \( \Delta t \to 0 \) of

\[ \frac{x(t+\Delta t) - x(t)}{\Delta t} = f(x(t), t), \]

which can also be written as

\[ x(t+\Delta t) = x(t) + f(x(t), t) \Delta t, \quad x(0) = x_0. \]

System (3) can be used to numerically solve (1), and it is called the Euler method of integrating an ODE.

If \( f \) is continuous, we can also define solutions of (1) as the limit of solutions of (3) for \( \Delta t \to 0 \).
Likewise, we define solutions of
\[
\frac{dx}{dt} = f(x,t) + g(x,t) \frac{dW(t)}{dt} \tag{Ito}
\]
(\text{where } W(t) \text{ is the standard Wiener process}) as the limit \(\Delta t \to 0\) of the Euler integration

\[
x(t+\Delta t) = x(t) + f(x(t),t) \Delta t + g(x(t),t) \Delta W(t)
\]
or equivalently

\[
\Delta x = f(x,t) \Delta t + g(x,t) \Delta W
\]

where \(\Delta x(t) = x(t+\Delta t) - x(t)\).

The stochastic diff. equ. \(\circ\) is often written as

\[
dx = f(x,t) dt + g(x,t) dW
\]

The above interpretation of solutions of \(\circ\) as a limit of Euler integration \(\circ\) is called the Ito interpretation.
The Ornstein–Uhlenbeck (OU) process

Consider the linear SDE

\[ dx = -ax \, dt + \sigma \, dW \]

with initial condition \( x(0) = x_0 \).

Formal integration gives the solution

\[ x(t) = x_0 e^{-at} + b e^{-at} \int_0^t e^{az} \, dW_z \]

The integral is interpreted as the limit \( n \to \infty \) of the Euler integration

\[ \sum_{i=0}^{n-1} e^{az_i} (W(z_i+1) - W(z_i)) \]

where \( z_i = t \Delta z_n \) and \( \Delta z_n = \frac{t}{n} \).

The Wiener increments \( W(z_{i+1}) - W(z_i) \) are i.i.d. and \( W(0, \Delta z_n) \) distributed. Hence,

\[ \sum_{i=0}^{n-1} e^{az_i} (W(z_i+1) - W(z_i)) \sim W(0, \sum_{i=0}^{n-1} e^{az_i} \Delta z_n) \]
Taking the limit \( n \to \infty \), we then get
\[
\int_0^t e^{az} \frac{dw}{dz} \, dz \sim N(0, \int_0^t e^{2az} \, dz) \\
\sim N(0, \frac{1}{2a}(e^{2at} - 1))
\]
Substituting this result into (9), we get
\[
x(t) \sim N(x_0 e^{-at}, \frac{b^2}{2a}(1-e^{-2at}))
\]
The process defined by the SDE (8) is called the Ornstein-Uhlenbeck (OU) process.
From (13) we find
\[
\lim_{t \to \infty} x(t) \sim N(0, \frac{b^2}{2a})
\]
i.e., the distribution of \( x(t) \) in the limit becomes stationary, i.e., independent of time.
Stationary distributions are for SDEs, what equilibria are for ODEs.

The OU process is often used as an approximation of the solution of a non-linear SDE with small variance.

Consider the non-linear SDE:

\[ dx = f(x)dt + g(x)dW \]

with \( f(\bar{x}) = 0, \ f'(\bar{x}) < 0, \ g(\bar{x}) \neq 0 \) for some constant \( \bar{x} \).

Taylor expansion of (15) in \( x \) near \( \bar{x} \) and keeping only the dominant terms gives the linear SDE

\[ d\xi = f'(\bar{x})\xi dt + g(\bar{x})dW \]

where \( \xi = x - \bar{x} \).
Equation (16) defines an OU-process with stationary solution
\[ \xi(t) \sim N(0, \frac{3(x)^2}{2|f'(x)|}) \]
and hence
\[ \xi(t) \sim N(x, \frac{3(x)^2}{2|f'(x)|}) \]

As an approximation of solutions of (15) it is only good if \( q(x)^2 \) is small.
The Ornstein-Uhlenbeck process defined by
\[ dx = -ax \, dt + \sigma \, dW, \quad a > 0, \]
has stationary distribution
\[ N(0, \frac{b^2}{2a}). \]

Stationary (i.e., time-independent) distribution

What about structure in time?

The autocovariance of a process tells how \( x(t) \) and \( x(s) \) for \( t \neq s \) are correlated to one another. For a stationary process only the time difference \( \tau := t - s \) matters.

Some [additional notes]...
**Definition**

Let \( \{X_t\} \) be a stationary process with mean
\[
\bar{X} := \mathbb{E}\{X(t)\}
\]
(which is independent of time), and define the autocovariance function \( C : \mathbb{R} \rightarrow \mathbb{R} \) as
\[
C(\tau) := \mathbb{E}\{(X(t+\tau) - \bar{X})(X(t) - \bar{X})\}.
\]

**Example**

Take the stationary OU process of \( \text{\ref{19}} \). How to calculate its autocovariance function?

There are several ways. A robust method is championed in the lecture notes Section 6. Here we introduce a method that tends to be more direct but it assumes we already know the variance of the stationary

\[
\text{var}(X) = \mathbb{E}\{(X - \bar{X})^2\}.
\]
distribution, which in the case of the OU process we do.

From (19) for fixed \( t > 0 \) we get

\[
\frac{d x(t+z)}{dx(t)} = -a x(t+z) dt + b dW(t+z)
\]

\[
\rightarrow \text{(note that } z \text{ is the time variable)}
\]

Multiply left and right with \( x(t) \), which gives

\[
\frac{d x(t) x(t+z)}{dx(t)} = \frac{a x(t) x(t+z) dt}{x(t)} + b x(t) dW(t+z)
\]

Taking expectations on both sides, we get

\[
d C(z) = -a C(z) dz
\]

which in an ODE with solution

\[
C(z) = e^{-az} C(0)
\]
From the definition (21) of $C(2)$ we see that $C(0)$ is the variance of the process, which for the stationary Ornstein process we know:

\[ C(0) = \frac{b^2}{2a}. \]

Substitution of (26) into (25) given:

\[ C(2) = e^{-a^2} \frac{b^2}{2a}. \]
Remark.

The solution of a SDE is a stochastic process.
Such a process is characterized by its distribution and its time-structure.

We will focus on the mean and variance and autocovariance function of the process.

To calculate these, however, we usually will need its calculus.

That is the topic of next lecture.