From SDE to PDE

The Ito SDE

1.\[ dx = f(x)dt + g(x)dw \]

and the PDE

2.\[ \partial_t p(x,t) = -\partial_x (f(x)p(x,t)) + \frac{1}{2} \partial_{xx} (g(x)^2 p(x,t)) \]

(called the Fokker-Planck equation)

describe the same stochastic process \( \{X(t)\}_{t\geq 0} \) but focus on different aspects.

While the SDE\(^1\) describes in the first place stochastic orbits, the PDE\(^2\) describes how the probability density \( p(x,t) \) of \( X \) at time \( t \) changes with time.

\[ \begin{array}{c}
\text{prob. dens. at } t=t_0. \\
\text{stochastic orbit (or sample path)}
\end{array} \]
So far we have focussed on the SDE, but the PDE representation is particularly useful for calculating a stationary distribution.

A stationary distribution is a time-independent solution of the PDE and hence satisfies

$$ 0 = \frac{1}{2} \frac{df}{dx} (f(x)p(x)) + \frac{1}{2} \frac{d^2}{dx^2} (g(x)^2 p(x)) $$

It immediately follows that

$$ f(x)p(x) + \frac{1}{2} \frac{df}{dx} (g(x)^2 p(x)) = k $$

for some constant $k$, the value of which depends on the particular boundary conditions.

We consider the case where $p(\pm \infty) = 0$ and $p'(\pm \infty) = 0$. If $f(x)$, $g(x)$ and $g'(x)$ are bounded then necessarily $k = 0$.

(Boundedness of $f$, $g$, $g'$ is a bit unusual. Actually we only need that the left hand side of (4) vanishes at $x = \pm \infty$. )
with \( k = 0 \), we rewrite (4) as

\[
\frac{f(x)}{q^2(x)} (q(x)^2 p(x)) + \frac{1}{2} \frac{df}{dx} (q(x)^2 p(x)) = 0
\]

and solve for \( q(x)^2 p(x) \), which gives

\[
q(x)^2 p(x) = C e^{-2 \int_{-\infty}^{x} \frac{f(y)}{q(y)^2} dy}
\]

and so

\[
p(x) = C \frac{1}{q(x)^2} e^{-2 \int_{-\infty}^{x} \frac{f(y)}{q(y)^2} dy}
\]

where \( C \) is a normalization constant to ensure that

\[
\int_{-\infty}^{\infty} p(x) \, dx = 1
\]

(\( p \) is a probability density, after all).

The above gives the basic idea of calculating stationary distribution for the PDE (3)

With different boundary conditions, results differ from (7).
How does the PDE follow from the SDE?

Using Ito's lemma, we have that

\[ d\ln(x) = \ln(x) (f(x) \, dt + g(x) \, dW) + \frac{1}{2} \ln''(x) g(x)^2 \, dt \]

Taking expectations on both sides, we get

\[ dE(\ln(x)) = E(\ln'(x) f(x)) \, dt + \frac{1}{2} E(\ln''(x) g(x)^2) \, dt \]

Note that this is now an ordinary differential equation.

Let \( p(x, t) \) be the probability density of \( X \) at time \( t \).

Rewriting the expectations in explicitly using the density \( p(x, t) \), we get
\[ \begin{align*}
\partial_t \int_{-\infty}^{+\infty} \phi(x,t) \, dx &= \\
&= \int_{-\infty}^{+\infty} \phi'(x) \phi''(x) \, dx + \\
&\quad + \frac{1}{2} \int_{-\infty}^{+\infty} \phi''(x) g(x^2 p(x,t)) \, dx
\end{align*} \]

Integration by parts gives

\[ \begin{align*}
\partial_t \int_{-\infty}^{+\infty} \phi(x,t) \, dx &= \\
&= -\int_{-\infty}^{+\infty} \phi(x) \phi'(x) \, dx + \\
&\quad + \frac{1}{2} \int_{-\infty}^{+\infty} \phi''(x) (g(x^2 p(x,t))) \, dx
\end{align*} \]

where we used that \( p(x,t) \) and \( \phi(x,t) \) converge to zero sufficiently fast as \( x \to \pm \infty \).

Next take a sequence of twice continuously differentiable functions \( \{ \phi_n \} : \mathbb{R} \to [0,1] \) such that \( \phi_n \to \phi_0 \) uniformly as \( n \to \infty \), where
\[ h_{\infty}(x) = \begin{cases} 1 & \text{for } x \leq x_0 \\ 0 & \text{for } x > x_0 \end{cases} \]

Substituting \( h_\infty \) in (12) by \( h_\infty \) from (13) and letting \( n \to \infty \), we get

\[ \partial_t \int_{x_0}^{x_0} p(x,t) \, dx = - \int_{x_0}^{x_0} \partial_x (f(x)p(x,t)) \, dx + \frac{1}{2} \int_{x_0}^{x_0} \partial_{xx} \left( g(x)^2 p(x,t) \right) \, dx \]

Finally, differentiation with respect to \( x_0 \) gives the PDE (12), which is what we were looking for.

(Examples are given as exercises)

(Example set given on exercise)

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