Pop. density $x \in \mathbb{R}$, stock density $\theta \in \mathbb{R}$

$$\frac{dx}{dt} = f(x, \theta)$$

$$d\theta = g(\theta) dt + hw(\theta) dw$$ (indep. of $x$)

Suppose there exist numbers $\bar{x} \in \mathbb{R}$ and $\bar{\theta} \in \mathbb{R}$ such that

$$f(\bar{x}, \bar{\theta}) = 0, \exists x f(x, \bar{\theta}) < 0$$

$$g(\bar{\theta}) = 0, \; g'(\bar{\theta}) < 0$$

$$h(\bar{\theta}) \neq 0.$$ 

In other words, $(\bar{x}, \bar{\theta})$ is a stable equilibrium of the deterministic system.

$$\frac{dx}{dt} = f(x, \theta)$$

$$\frac{d\theta}{dt} = g(\theta)$$

Linearization of 1 near $(\bar{x}, \bar{\theta})$ gives

$$\frac{dx}{dt} = f_x(\bar{x}, \bar{\theta}) x dt + f_{x\theta}(\bar{x}, \bar{\theta}) \theta dt$$

$$\frac{d\theta}{dt} = g'(\bar{\theta}) \theta dt + h(\bar{\theta}) dw$$
where $\xi := x - \bar{x}$ and $\eta := \theta - \bar{\theta}$.

In matrix notation we have

$$\begin{equation} d(\eta) = A(\eta) \, dt + B d(W) \end{equation}$$

where

$$A = \begin{pmatrix} \partial x f(x, \bar{x}) & \partial \theta f(x, \bar{\theta}) \\ 0 & g'(\bar{\theta}) \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ \text{d}(\bar{\theta}) \end{pmatrix}$$

This is an example of the general multi-dimensional Ornstein-Uhlenbeck process

$$dx = Ax \, dt + B \, dW$$

where $x \in \mathbb{R}^m$, $\bar{W} \in \mathbb{R}^n$ (vector of indep. Wiener processes), $A \in \mathbb{R}^{m \times m}$ a hyperbolically stable matrix (i.e., all eigenval. have strictly neg. real part) and $B \in \mathbb{R}^{m \times n}$ in a non-zero matrix.
General solution is

\[ x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A}BdW(s) \]

Hence

\[ x(t) \sim N(\mu(t), \Psi(t)) \]

with mean

\[ \mu(t) = e^{tA}x(0) \]

and covariance matrix

\[ \Psi(t) = \int_0^t e^{(t-s)A}BB^T e^{(t-s)A^T}ds \]

(explain the multi-variate normal distr.)

Since \( A \) is assumed to be hyperbolically stable, there exists a limiting stationary distr.

\[ \lim_{t \to \infty} x(t) \sim N(0, \Psi_\infty) \]

where

\[ \Psi_\infty = \int_0^\infty e^{tA}BB^T e^{tA^T}dt \]

(explain ).
While given an explicit expression of the covar. matrix of the stationary distribution, it is not clear how to evaluate the integral.

Here is an alternative method of computing \( \Phi_\infty \) as the solution of a linear algebraic equation.

Using Itô's lemma we have from \( \Phi \)

\[
dxx^T = Axx^T dt + xx^T A^T dt + BB^T dt \\
+ BdWx^T + x dW^T B^T
\]

Taking expectations on both sides

\[
A \mathbb{E}\{xx^T\} = A \mathbb{E}\{xx^T\} dt + \mathbb{E}\{xx^T\} A^T dt + BB^T dt
\]

which is an ODE. At the stationary distribution \( \mathbb{E}(xx^T) \) does not change, i.e., \( A \mathbb{E}\{xx^T\} = 0 \). Moreover, since \( \mathbb{E}\{x\} = 0 \) (see \( \Phi \)), we have \( \Phi_\infty = \mathbb{E}\{xx^T\} \).
Hence, $\Phi_0$ is a solution of

$$0 = A \Phi_0 + \Phi_0 A^T + BB^T$$

subject to the symmetry cond.

$$\Phi_0 = \Phi_0^T$$

It can be shown that if $d\det A \neq 0$ and $B \neq 0$, then 17 with 18 then a unique solution.

Example:

Take $A$ and $B$ from 17.

Then

$$\Phi_0 = \begin{pmatrix}
\frac{-a_{12}^2 b_{22}^2}{2 a_{11} a_{22} (a_{11} + a_{22})} & \frac{a_{12} b_{22}^2}{2 a_{22} (a_{11} + a_{22})} \\
\frac{a_{12} b_{22}^2}{2 a_{22} (a_{11} + a_{22})} & \frac{-b_{22}^2}{2 a_{22}}
\end{pmatrix}$$

(discuss!).
Example (interference comp).

\[ \begin{align*}
\frac{dx}{dt} &= r(x) x \left(1 - \frac{x}{k(x)} \right) dt \\
\dot{\theta} &= -a \theta dt + b dW \quad (a > 0, b > 0)
\end{align*} \]

where
\[ r(x) = \beta_0 e^x - \delta \]
\[ k(x) = \beta_0 e^x - \delta \]

(we note that \( \beta = \beta_0 e^\theta \sim \log W \))

(Interference comp.)

Then \( \theta = 0 \) and \( x = k(\theta) \)

satisfy conditions \( \circ \).

The one-approx. mean \( (x, \theta) \) given

\[ \lim_{\theta \to \infty} (x) \sim N \left[ (\theta), K_\theta \right] \]

Here we plot how \( \overline{x} \), \( \text{var} x \)

and \( \text{covar} (x, \theta) \) depend on \( \beta_0 \).
Example (site-comp)

Same as prev. example, but with

\[ \nu(\theta) = \rho_0 e^\theta - \delta \]
\[ k(\theta) = 1 - \frac{\delta}{\rho_0} e^\theta \]

Then given

mean

\[
\begin{align*}
0 & \quad \rho_0 \\
\end{align*}
\]

var

\[
\begin{align*}
0 & \quad \rho_0 \\
\end{align*}
\]

covar

\[
\begin{align*}
0 & \quad \rho_0 \\
\end{align*}
\]
Example 1 (predator-prey)

\[ R \rightarrow \text{prey} \]
\[ X \rightarrow \text{predator} \]

\[ R + X \xrightarrow{c/3} X \] (prey captured)

\[ R + X \xrightarrow{\beta} X + X \] (birth)

\[ X \xrightarrow{d} \varnothing \] (death)

Pop. level model:

\[
\frac{dR}{dt} = \frac{2}{3} R (1 - \frac{R}{M}) - \frac{R}{3} RX \quad \text{(fert)}
\]

\[
\frac{dX}{dt} = \beta RX - dX \quad \text{(growth)}
\]

Fertil dynamics in \( t = 3 \text{tme} \):

\[
\frac{dR}{d\lambda} = \lambda R \left(1 - \frac{R}{M}\right) - \beta RX
\]

\( R \rightarrow R(x) = \begin{cases} 
\mu (1 - \frac{c}{m}) & \text{if } 0 \leq x \leq \frac{2}{\mu} \\
0 & \text{if } x > \frac{2}{\mu}
\end{cases} 
\] (over exploitation)
Slow dynamics in time:

\[ \frac{dx}{dt} = \begin{cases} \frac{\beta \mu_0}{\beta \mu_0 - \delta} \times (1 - \frac{\beta_0 u_0}{\beta_0 (\beta_0 - \delta) \mu_0}) & \text{if } 0 \leq x \leq \frac{\beta_0}{\beta} \\ -\delta x & \text{if } x > \frac{\beta_0}{\beta} \end{cases} \]

Take \( \mu = 1 \), \( \beta = 1 \), then we get

\[ \frac{dx}{dt} = \begin{cases} \nu x (1 - \frac{x}{\kappa}) & \text{if } 0 \leq x \leq \beta^* \\ -\delta x & \text{if } x > \beta^* \end{cases} \]

with

\[ \nu = \beta - \delta \]
\[ \kappa = \frac{\beta_0 - \delta}{\beta^2} \]

In this model with \( \beta = \beta_0 e^\theta \), we get

mean: \[ \frac{\beta_0}{\beta} \]
\[ \beta_0 \]
\[ \beta_0 \]

var: \[ \frac{\beta_0}{\beta} \]
\[ \beta_0 \]
\[ \beta_0 \]

covar: \[ \frac{\beta_0}{\beta} \]
\[ \beta_0 \]
\[ \beta_0 \]
Interference competition:

Site competition:

Resource competition:
Interference competition:

![Graph showing interference competition with dashed line indicating deterministic equilibrium and grey dots for simulation results.]

Site competition:

![Graphs showing site competition with different values of $\beta_0$.]

Resource competition:

![Graphs showing resource competition with different values of $\beta_0$.]
Autocovar in the multi-dim OU process.

Back to the multi-dim OU process

\[ dx = Ax \, dt + BdW \]

with nonstationary drift.

\[ \ln x(t) \sim N(0, F_a) \]

To calculate the autocovar

\[ C(2) : = \mathbb{E}\{ (x(t+2) \times (t)^T \} \quad \text{(multi \ ln)} \]

of the nonstationary process, we re-write (30) for fixed \( t \) and variable \( z \):

\[ dx(t+2) = A x(t+2) \, dz + BdW(t+2) \]

Right - multiplication with \( x(t)^T \) on both sides and taking expectations, we get:

\[ dC(2) = AC(2) \, dz \]
There is an ODE with solution

\[ C(t) = e^{tA} C(0) \]

where, of course,

\[ C(0) = f \]

in the covariance matrix of the stationary distribution.

Since \( A \) is assumed to be hyperbolically stable, it follows that \( C(t) \to 0 \) (null matrix) as \( t \to \infty \).

But not necessarily monotonically.

If \( A \) has complex eigenvalues, the autocovariance function decays through damped oscillations. The stochastic orbit then will show "phase forgetting quasi-cycles".

We will come back to this.