Example (SIS)

\[ S + I \xrightarrow{\beta} S + I \quad \text{(transmission)} \]
\[ I \xrightarrow{\gamma} S \quad \text{(recovery)} \]

Deterministic model:
(i.e., infinite system size)

\[ \frac{dS}{dt} = -\beta SI + \gamma I \]
\[ \frac{dI}{dt} = \beta SI - \gamma I \]

Total pop. density \( N = S + I \) is constant.

\[ \Rightarrow \] Sufficient to keep track of \( I \) alone.

Stable equilibria:

\[ I = \begin{cases} 
0 & \text{if } N \leq \frac{\gamma}{\beta} \\
N - \frac{\gamma}{\beta} & \text{if } N > \frac{\gamma}{\beta}
\end{cases} \]  

\( \text{(sub-critical)} \)  

\( \text{(supercritical)} \)
Finite system size $R$:

<table>
<thead>
<tr>
<th>Process</th>
<th>$\Delta i$</th>
<th>Per capita rate</th>
<th>Pop. rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>trans.</td>
<td>$+1$</td>
<td>$\beta \frac{n-i}{i}$</td>
<td>$\beta \frac{n-i}{i}$</td>
</tr>
<tr>
<td>recov.</td>
<td>$-1$</td>
<td>$\gamma$</td>
<td>$\gamma$</td>
</tr>
</tbody>
</table>

Note: $i$ and $n$ are now numbers of individuals.

The per capita rates are easiest to see if we write the reactions from an "$I$-centered" point of view:

1. $\frac{\beta n-i}{i} \rightarrow 2I$ ("birth")
2. $I \rightarrow \emptyset$ ("death")

Hence

\[ B_i = \beta \frac{n-i}{i} \]
\[ D_i = \gamma i \]
The conditional probability distribution (i.e., conditional on non-extinction) is given by the vector equation

\[
\frac{d\vec{p}^c}{dt} = (A + D, \vec{p}^c, I) \vec{p}^c
\]

where \( \vec{p}^c = (p_1^c, \ldots, p_n^c) \) and

\[
A = \begin{pmatrix}
-B_1 - D_1 & D_2 & 0 & \cdots \\
B_1 & -B_2 - D_2 & D_3 & \cdots \\
0 & B_2 & -B_3 - D_3 & \cdots \\
0 & 0 & \cdots & B_n \\
\end{pmatrix}
\]

in the tri-diagonal matrix given in section 8.6 of the printed lecture notes, and

\( I \in \mathbb{R}^{n \times n} \) is the identity matrix.

The equilibrium of (5) satisfies

\[
0 = (A + D, \vec{p}^c, I) \vec{p}^c
\]

and is called the quasi-stationary distribution.
To actually calculate the normalized dominant eigenvector of $A$, we realize that this eigenv. in the sense of the cone edge of the matrix $B = A + \mu I$ where $\mu \approx \max |A_{ij}|$ (maxing out of $A$).

Then $B\mathbf{e}_1$ is obtained (approx.) by iterating the following procedure:

$$\mathbf{p}_{next} = \frac{B\mathbf{p}_{old}}{\|B\mathbf{p}_{old}\|_1},$$

where $\|\cdot\|_1$ in the sum norm.
The point of this:

The time till extinction rapidly increases with system size (keeping pop. density \( n/\lambda \) constant), while the variance of stationary distribution decreases very slowly.
\[ \text{Log (Expected extinction time)} \]

as a function of system size. (dotted line)

Not that \( \log \mathcal{E}[T] \) eventually becomes linear in \( \Omega \), i.e., \( \mathcal{E}[T] \) increase exponentially with \( \Omega \).

This defines the "realm" of the semi-large system size approximation we will look at next:

Semi-large system size is large enough that we can ignore extinction, but still small enough that the demographic stochasticity (read "variance") cannot be ignored.
Semi-large system size.

Fokker-Planck approx.

\[ \partial_t p = -\partial_x (b - d(p) p) + \frac{\epsilon}{2} \partial_{xx} (b + d(p) p) \]

where:

\[ \epsilon = \frac{1}{\sigma^2} \quad \text{(inverse system size)} \]

\[ x = \frac{\xi}{\sigma} = \xi \]

\[ p(x, t) = \Omega \mathcal{P}(t) \]

\[ h(x) = \epsilon B; \]

\[ d(x) = \epsilon D; \]

\[ (\text{See section 9.1 of the printed lecture notes}) \]

Writing:

\[ \mu(x) := h(x) - d(x) \]

\[ \sigma^2(x) := h(x) + d(x) \]

We will approximate the stationary distribution of (9).
Fokker-Plank approximation.

10. \[ \frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\mu p) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}(\sigma^2 p) \]

where

\[ \mu = b - d \quad \text{and} \quad \sigma^2 = b + d \]

Stationary drift \( \bar{p} \) satisfies

11. \[ 0 = -\frac{\partial}{\partial x}(\mu \bar{p}) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}(\sigma^2 \bar{p}) \]

Hence

12. \[ C_0 = -\mu \bar{p}(x) + \frac{\sigma^2}{2} \frac{\partial}{\partial x}(\sigma^2 \bar{p}(x)) \]

Case of reflecting boundary at \( x_{\text{max}} \) implies that \( C_0 = 0 \).

Solving for \( \bar{p}(x) \) we get

13. \[ \bar{p}(x) = \frac{C_1}{\sigma^2(x)} \exp \left( \frac{1}{\sigma^2(x)} \left( \frac{\mu}{\sigma^2(x)} \right) x \right) \]

where \( C_1 \) is a normalization constant such that

14. \[ \int_{x_{\text{min}}}^{x_{\text{max}}} \bar{p}(x) \, dx = 1 \]
\( \Omega = 10; \ \mathbb{E}\{T\} = 4; \)

\( \Omega = 20; \ \mathbb{E}\{T\} = 14; \)

\( \Omega = 30; \ \mathbb{E}\{T\} = 53; \)

\( \Omega = 100; \ \mathbb{E}\{T\} = 1165.139; \)

\[
\text{Expected extinction time in units of mean recovery time } \gamma. \\
\]

- Note the gross overestimation of the extinction time in the exact model on page 4.
Ornstein-Uhlenbeck approx.

The Fokker-Planck equation has corresponding SDE

\[ \text{(6) } dx = (b(x) - \mu(x)) dt + \sqrt{\sigma^2(x)} dW \]

on with \[ \text{(11) } \]

\[ \text{(7) } \dot{x} = \mu(x) dt + \sqrt{\sigma^2(x)} dW \]

The deterministic equilibrium \[ \bar{x} = x_{\text{max}} - \frac{\gamma}{\beta} \]

is the solution of \( \mu(x) = 0 \) and since \( \mu'(\bar{x}) < 0 \), it is stable.

Linearization of (7) around \( \bar{x} \) gives the OU process

\[ \text{(8) } d(x - \bar{x}) = \mu'(\bar{x})(x - \bar{x}) dt + \sqrt{\sigma^2(\bar{x})} dW \]

which has stationary solution
\[
x \sim \mathcal{N}(\bar{x}, \frac{3\sigma^2(x)}{2|\mu'(\bar{x})|})
\]

but normalized over the interval \([L, X_{\text{max}}]\)

\[
\Omega = 10; \quad \delta(T) = 8;
\]

\[
\Omega = 20; \quad \delta(T) = 111;
\]

\[
\Omega = 30; \quad \delta(T) = 1262;
\]

\[
\Omega = 100; \quad \delta(T) = 15158098762;
\]

quasi-stationary distribution according to the \(O(1)\)-approx.

Note that the extinction times are very much overestimated, even more so than the ext times based on the \(E(1)\)-approx.
Even though the extinction times of FP and OU approx are (notoriously) bad, the quasi-stationary distributions are quite ok.

\[ \text{Note} \]
- From page 5 we found that the expected time till extinction grows asymptotically exponential with system size.
- From the OU approx on page 10 we see that the variance is inversely proportional to system size, and hence decreases only slowly.