Fast-flow decomposition in multi-type birth-death processes

**Example.**

\[ \rightarrow \text{Site competition mechanism of the logistic equation in section 1-2 of the printed lecture notes} \]

- \( \Box \): site owner/occupied site
- \( \checkmark \): indiv. without site
- \( e \): empty site

\[ \begin{align*}
\Box & \xrightarrow{f} \Box + \checkmark & \text{birth} \\
\checkmark + \checkmark & \xrightarrow{\gamma} \Box & \text{colonization} \\
\Box & \xrightarrow{\delta} e & \text{death} \\
\checkmark & \xrightarrow{k} \emptyset & \text{death}
\end{align*} \]
\[ n : \text{number of } \mathbb{R} \]
\[ u : \quad \mathbb{Y} \]
\[ K : \quad \mathbb{E} \]
\[ a : \text{system size} \]

<table>
<thead>
<tr>
<th>event</th>
<th>( \Delta m, \Delta n )</th>
<th>pop. rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>fast:</td>
<td>birth</td>
<td>0, 1</td>
</tr>
<tr>
<td>slow:</td>
<td>cell</td>
<td>1, 1, 1</td>
</tr>
<tr>
<td>slow:</td>
<td>death ( \mathbb{X} )</td>
<td>-1, 0</td>
</tr>
<tr>
<td>fast:</td>
<td>death ( \mathbb{X} )</td>
<td>0, 1, -1</td>
</tr>
</tbody>
</table>

Suppose that birth and death \( \mathbb{X} \) are fast processes relative to the other two. Note that these two fast processes do not affect \( n \).

So, we can consider the birth-death process for \( n \) on the fast time scale assuming that \( n \) stays constant.
Table for rate processes for constant $m$.

<table>
<thead>
<tr>
<th>process</th>
<th>$an$</th>
<th>pop. rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>birth</td>
<td>1</td>
<td>$\beta m$</td>
</tr>
<tr>
<td>death</td>
<td>-1</td>
<td>$\alpha n$</td>
</tr>
</tbody>
</table>

Hence we get a birth-death process with

\[ B_n = \beta m \quad \text{(indep. of } n) \]
\[ D_n = \alpha n \]

Note that $n=0$ is not "absorbing", i.e., the pop. can recover from being extinct, because the birth rate $\beta m$ is constant.
\[
\begin{align*}
\frac{dP_0}{dt} &= -\beta m P_0 + \alpha P_1 \\
\frac{dP_n}{dt} &= \beta m P_{n-1} - (\beta m + \alpha n) P_n + \alpha cn P_{n-1}
\end{align*}
\]

This system actually contains a true stationary contribution.

Define the mob flux:
\[J_n = \beta m P_{n-1} - \alpha n P_n\]

Then
\[
\frac{dP_n}{dt} = -J_n
\]

so that at equilibrium we have
\[J_n = 0\] for \(n = 1, 2, 3, \ldots\)
Hence, at equilibrium,

\[ P_n = \frac{\beta m}{\alpha n} P_{n-1} \]

which we can solve recursively for each \( P_n \geq 0 \), which gives

\[ P_n = \frac{1}{n!} \left( \frac{\beta m}{\alpha} \right)^n P_0 \]

Since \( \sum_{n=0}^{\infty} P_n = 1 \) by definition, we can solve for \( P_0 \), i.e.,

\[ P_0 = e^{-\frac{\beta m}{\alpha}} \]

and so

\[ P_n = \frac{1}{n!} \left( \frac{\beta m}{\alpha} \right)^n e^{-\frac{\beta m}{\alpha}} \]

which is the Poisson distribution with mean

\[ \mathbb{E}[n] = \frac{\beta m}{\alpha} \]
Now let's turn to the slow dynamics of \( m \).

\[ n \sim \text{Poisson} \left( \frac{\beta m}{\alpha} \right) \]

| process | \( \Delta m \) | pop. rate | \[ E \left[ \gamma \frac{k-m}{\gamma} \right] = \gamma \frac{k-m}{\gamma} \frac{\beta m}{\alpha} \]
<table>
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</tr>
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<tbody>
<tr>
<td>coll.</td>
<td>1</td>
<td>( \Delta m )</td>
</tr>
<tr>
<td>death</td>
<td>-1</td>
<td>( \Delta m )</td>
</tr>
</tbody>
</table>

For the slow process we thus find

\[
B_m = \frac{\beta \gamma}{\alpha \gamma} m(k-m) \\
D_m = \Delta m
\]
Linear birth-death process

If \( K \gg m \) and \( \frac{\lambda}{\sigma} \) is not very small, \( \mu \).

If \( K \gg m \), then \( K - m \approx K \)
and we get a linear process for \( m \) with

\[
B_m = \frac{\beta m}{2} \cdot m = \frac{\beta \lambda K}{2} m
\]

\[
D_m = 5 m
\]

we have seen in the notes of "SPM 26-03-2019.pdf" (and before) that the probability of eventual extinction is

\[
E_m = \begin{cases} 
1 & \text{if } \frac{\beta \lambda K}{2} m \leq \delta \\
\left( \frac{a_0}{\beta \lambda K} \right) \frac{m}{m} & \text{if } \frac{\beta \lambda K}{2} m > \delta
\end{cases}
\]

(sure extinction)

(invasion with

\( \text{prob. prob.} \))

that is, if we start with

\( m \) individuals in the \( \times \)-state
and the number of individuals in the \( \circ \)-state in Poisson \(( \frac{a_0}{\beta \lambda K} m)\)

Discuss:

Note that the invasion condition \( \frac{\beta \lambda K}{2} \geq \delta \) is different from that of the deterministic model in "SPM 2019-04-15.pdf" page 8, and different from that of the full two-type model in "SPM-04-15.pdf" page 6.
Non-linear birth-death process for semi-large $\varepsilon$.

Translate $\star$ on page 6 into

$\varepsilon_i = \varepsilon i$, $x_i = \varepsilon m$, $b_i(x_i) = \varepsilon B_m$, and $d_i(x_i) = \varepsilon D_m$ and $k = \varepsilon K$:

\[ b(x) = \frac{\beta y}{\alpha} x (k-x) \]
\[ d(x) = \delta x . \]

Lead to the FPE approx.

\[ \partial_t \rho = - \partial_x (b\rho - d\rho) \rho + \frac{\varepsilon}{2} \partial_{xx} (b(x)\rho + d(x)\rho) \]

and corresponding SDE

\[ dx = (b(x)-d(x))dt + \sqrt{\varepsilon(b(x)+d(x))^2} \, dw \]

Note:
The aim of these notes was to illustrate how timescale separation works in birth-death processes.