Strong logics

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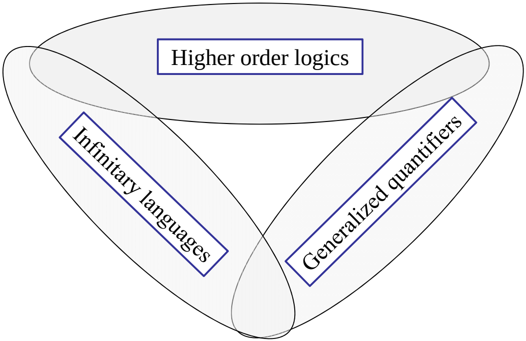
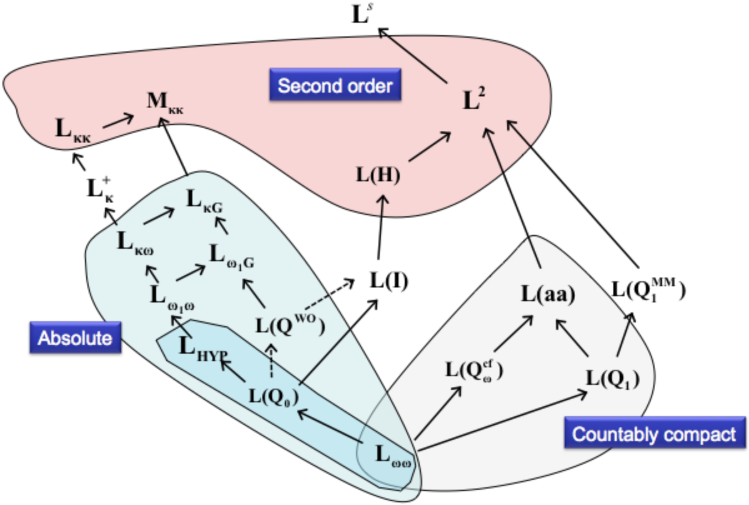
Why strong logics?

- Definability in set theory and definability in strong logics are “symbiotically" connected to each other.
- Large cardinal properties such as weak compactness, strong compactness, supercompactness, etc can be characterized by means of strong infinitary logics.
- Combinatorial principles, are closely related to model-theoretic properties of strong logics.
- Many strong logics cannot be properly understood without large cardinals.
- Strong logics can be used to give interesting inner models for set theory.


Higher order logics
Infinitary languages
Generalized quantifiers



L^s
Second order
L^2
$M_{\kappa\kappa}$
$L_{\kappa\kappa}$
$L(H)$
L_κ^+
$L_{\kappa G}$
$L_{\kappa\omega}$
$L_{\omega_1 G}$
$L_{\omega_1\omega}$
$L(I)$
$L(aa)$
$L(Q_1^{MM})$
$L(Q^{WO})$
Absolute
L_{HYP}
$L(Q_\omega^{cf})$
$L(Q_1)$
$L(Q_0)$
Countably compact
$L_{\omega\omega}$

Simple theory of types

- First order variables $x, y, z, \ldots$ for individuals.
- Second order variables $X, Y, Z, \ldots$ for sets.
- Third order variables $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ for sets of sets.
- Etc...
- First order logic: $x = y$, $R(x_1, \ldots, x_n)$
- Second order logic: $X(y)$
- Third order logic: $X(y), \mathcal{X}(Y)$
- Etc...

Definition (Henkin)

A general model consists of a sequence (D_n), where D_0 is a non-empty set and always $D_{n+1} \subseteq \mathcal{P}(D_n)$ for all n. The general model is standard, if $D_{n+1} = \mathcal{P}(D_n)$ for all n.

Alternatively, can think of a general model as a many-sorted model, or as a first order structure with extra predicates.

Completeness Theorem, Compactness Theorem, Löwenheim-Skolem Theorem

Theorem (Henkin)

A set of sentences of simple theory of types is consistent if and only if it has a (countable) general model.

Basic examples

We give an overview of the main examples of strong logics, such as

- First order logic $L_{\omega\omega}$ (not really “strong"),
- Logic with the generalized quantifier Q_0,
- Infinitary logics $L_{\kappa\lambda}$,
- Logic with the generalized quantifier Q_α,
- Härtig-quantifier I.
- Second order logic L^2.
- Later: cofinality logic, stationary logic

- In many cases can manipulate a general model into standard model.
- $L(Q_1)$ is an example.
- $L_{\kappa\lambda}$: Chain models.
- L^2 very difficult, but a little easier with Boolean valued models.

First order logic

Definition

1. A vocabulary is a (usually finite) set L of relation symbols. Each relation symbol has an arity.
2. **First order logic** has atomic formulas $x = y$ and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$.

Definition

1. An L-structure (occasionally τ-structure) $\mathcal{M}$ is a non-empty set M with an interpretation $R^{\mathcal{M}}$ as an n-ary relation on M whenever $R \in L$ is n-ary.
2. An assignment (into M) is a mapping s of a finite set of variables into M. The empty mapping is denoted $\emptyset$.
3. A modified assignment $s(a/x)$ is the modification of s at x to get the value $a \in M$.
4. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment s into M and a formula φ, is defined by induction on φ.
5. Truth $\mathcal{M} \models \varphi$ for sentences (formulas without free variables) is defined as $\mathcal{M} \models_\emptyset \varphi$.

Definition

1. The relation

$$T \models \varphi$$

of logical consequence means $\mathcal{M} \models T$ implies $\mathcal{M} \models \varphi$ for all $\mathcal{M}$.

2. Validity

$$\models \varphi$$

means $\emptyset \models \varphi$ i.e. $\mathcal{M} \models \varphi$ for all $\mathcal{M}$.

The logic $L(Q_0)$

Definition

1. The logic $L(Q_0)$ has atomic formulas $x = y$ and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$ and Q_0.
2. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment s into M and a formula φ, is defined by induction on φ using:

$$\mathcal{M} \models_s Q_0 x \varphi \iff |\{a \in M : \mathcal{M} \models_{s(a/x)} \varphi\}| \geq \aleph_0.$$

Example

Let φ be $\forall x \neg Q_0 y(y < x)$. Together with some simple axioms θ this characterizes the standard model $\mathcal{N}$ of arithmetic. Hence for all arithmetical ψ

$$\mathcal{N} \models \psi \iff \models (\theta \wedge \varphi) \rightarrow \psi.$$

Theorem (Mostowski)

The validity relation $\models$ of $L(Q_0)$ is non-arithmetical.

Weak second order logic L^2_w

Definition

1. The logic L^2_w has atomic formulas $x = y$, $X(y)$ and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$.
2. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment (of individual and set variables) s into M and a formula φ, is defined by induction on φ using:

$$\mathcal{M} \models_s \exists X\varphi \iff \mathcal{M} \models_{s(A/X)} \varphi \text{ for some finite } A \subseteq M.$$

Note. $L(Q_0)$ can be translated into L^2_w, but not conversely. However, they are *essentially* equivalent (see later).

ω-logic L^ω

Definition

1. The logic L^ω has a distinguished predicate symbol N and distinguished constant symbols $\underline{n}$. Its atomic formulas are $t = t'$, $N(t)$ and $R(t_1, \ldots, t_n)$ if $R \in L$ of arity n and $t, t', t_1, \ldots, t_n$ are terms. The logical operations are $\wedge, \neg, \exists$.
2. It is assumed that N is always interpreted as $\mathbb{N}$ and $\underline{n}$ as n.

Note. L^ω, $L(Q_0)$ and L^2_w are *essentially* equivalent (see later).
Orey proved a Completeness theorem for L^ω based on the rule "From $\varphi(\underline{n})$ for all n, infer $\forall x(N(x) \to \varphi(x))$".

Infinitary logic

$$\exists^{\geq n} x \varphi(x) \quad \Leftrightarrow_{def} \quad \exists x_1 \ldots \exists x_n (\bigwedge_{1 \leq i < i \leq n} \neg x_i = x_j \wedge \varphi(x_i))^1$$

$$Q_0 x \varphi \quad \Longleftrightarrow \quad \bigwedge_{n < \omega} \exists^{\geq n} x \varphi$$

[1] We agree that $\exists^{\geq 0}\varphi$ is always true.

The logic $L_{\omega_1\omega}$

Definition

1. The logic $L_{\omega_1\omega}$ has atomic formulas $x = y$ and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$ and $\bigwedge$.
2. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment s into M and a countable set Φ of formulas[a], is defined by induction on φ using:

$$\mathcal{M} \models_s \bigwedge \Phi \iff \mathcal{M} \models_s \varphi \text{ for all } \varphi \in \Phi.$$

[a] We assume that there is a finite set of variables which contains all the free variables occurring in the formulas of Φ.

Example

Suppose $r \subseteq \mathbb{N}$. The structure $(\mathbb{N}, <, r)$ can be characterized in $L_{\omega_1\omega}$ with some simple axioms plus the sentence:

$$\forall x(\bigvee_{n<\omega} \neg\exists^{\geq n} y(y < x) \wedge (P(x) \leftrightarrow \bigvee_{n\in r} \theta_n(x)))$$

where

$$\theta_n(x) \iff \exists^{\geq n} y(y < x) \wedge \neg\exists^{\geq n+1} y(y < x).$$

Example

Define for countable ordinals α by transfinite induction:

$$\theta_\alpha(x) \iff \bigwedge_{\beta<\alpha} \exists y(y < x \wedge \theta_\beta(y))$$

$$\wedge \forall y(y < x \to \bigvee_{\beta<\alpha} \theta_\beta(y))$$

Lemma

Suppose $\alpha < \omega_1$. A linear order satisfies

$$\forall x \bigvee_{\beta<\alpha} \theta_\beta(x) \wedge \bigwedge_{\beta<\alpha} \exists x \theta_\beta(x)$$

if and only if the order-type of the linear order is α.

Admissible fragments

Definition

Suppose A is a countable transitive model of some nice axioms of (weak) set theory (e.g. Kripke-Platek set theory KP, then A is an admissible set). Let

$$L_A = L_{\omega_1\omega} \cap A.$$

Let HYP be the smallest admissible set. Then L_{HYP} is the smallest admissible fragment. It is the result of restricting infinite conjunctions and disjunctions in $L_{\omega_1\omega}$ to recursive ones.

L_A satisfies the so-called Barwise Compactness Theorem (see later).

The logic $L(Q_1)$

Definition

1. The logic $L(Q_1)$ has atomic formulas $x = y$ and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$ and Q_1.
2. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment s into M and a formula φ, is defined by induction on φ using:

$$\mathcal{M} \models_s Q_1 x\varphi \iff |\{a \in M : \mathcal{M} \models_{s(a/x)} \varphi\}| \geq \aleph_1.$$

Note: Always

$$Q_1 y \exists x \varphi \to \exists x Q_1 y \varphi \vee Q_1 x \exists y \varphi.$$

The logic $L(Q_\alpha)$

Definition

1. The logic $L(Q_\alpha)$ has atomic formulas $x = y$ and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$ and Q_α.
2. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment s into M and a formula φ, is defined by induction on φ using:

$$\mathcal{M} \models_s Q_\alpha x\varphi \iff |\{a \in M : \mathcal{M} \models_{s(a/x)} \varphi\}| \geq \aleph_\alpha.$$

Open problem!

- We can axiomatize “there are more than there are natural numbers".
- It is impossible to axiomatize “there are at least as many as there are natural numbers".
- We can axiomatize “there are more than there are real numbers".
- We do not know whether one can axiomatize “there are as many as there are real numbers" (unless we assume CH).

The Magidor-Malitz logic $L(Q_1^{MM})$

Definition

1. The logic $L(Q_1^{MM})$ has atomic formulas $x = y$ and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$ and Q_1^{MM}.
2. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment s into M and a formula φ, is defined by induction on φ using:

$$\mathcal{M} \models_s Q_1^{MM} xy\varphi \iff \text{there is } X \subseteq M \text{ such that}$$

$$|X| \geq \aleph_1 \text{ and for all } a, b \in X : \mathcal{M} \models_{s(ab/xy)} \varphi.$$

Example

- $Q_1 x\varphi(x) \equiv Q_1^{MM} xy(\varphi(x) \wedge \varphi(y))$.
- An equivalence relation (M, E) has uncountably many classes iff $(M, E) \models Q_1^{MM} xy \neg xEy$.
- A tree T has an uncountable branch iff $(T, <_T) \models Q_1^{MM} xy(x \leq_T y \vee y <_T x)$.
- An Aronszajn tree T is Souslin iff $(T, <_T) \models \neg Q_1^{MM} xy(x \not<_T y \wedge y \not<_T x)$.

Full second order logic L^2

Definition

1. The logic L^2 has atomic formulas $x = y$, $X(y_1, \ldots, y_m)$ where X is a relation variable of arity m, and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$.
2. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment (of individual and relation variables) s into M and a formula φ, is defined by induction on φ using:

$$\mathcal{M} \models_s \exists X\varphi \iff \mathcal{M} \models_{s(A/X)} \varphi \text{ for some } A \subseteq M^m.$$

Full second order logic L^2

Note. $L(Q_0)$, $L(Q_1)$ and $L(Q_1^{MM})$ can be translated into L^2, but not conversely. E.g.

$$Q_1 x\varphi(x) \iff \begin{array}{l} \neg\exists A(\text{“}A \text{ is a linear order"} \\ \wedge\forall x(\varphi(x) \to A(x,x)) \\ \wedge\text{“}A \text{ is of type } \omega\text{"}) \end{array}$$

The logic $L_{\kappa\lambda}$

Definition

1. The logic $L_{\kappa\lambda}$ has atomic formulas $x = y$ and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$ and $\bigwedge$.
2. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment s (of individual variables and sequences of variables of length $< \lambda$) into M and a set Φ of $< \kappa$ formulas[a], is defined by induction on φ using:

$$\mathcal{M} \models_s \bigwedge \Phi \iff \mathcal{M} \models_s \varphi \text{ for all } \varphi \in \Phi.$$

$$\mathcal{M} \models_s \exists \vec{x} \varphi \iff \mathcal{M} \models_{s(\vec{a}/\vec{x})} \varphi$$

for some sequence $\vec{a}$ of elements of M.

[a] We assume that there is set of $< \lambda$ variables which contains all the free variables occurring in the formulas of Φ.

- $L(Q_0)$, L^ω, L^2_w can be all translated into $L_{\omega_1\omega}$
- $L(Q_\alpha)$ can be translated into $L_{\aleph_\alpha\aleph_\alpha}$
- L^2 and $L_{\kappa\lambda}$ are incomparable.

Problem session 1

1. Show that $L(Q_0)$ can be translated into L^2_w. Suggest why the converse might not be true.
2. Show that L^ω and L^2_w can be translated into $L_{\omega_1\omega}$. Suggest why the converse might not be true.
3. Suppose $\alpha < \omega_1$. A linear order satisfies

$$\forall x \bigvee_{\beta<\alpha} \theta_\beta(x) \wedge \bigwedge_{\beta<\alpha} \exists x \theta_\beta(x)$$

   if and only if the order-type of the linear order is α.
4. Prove $\models Q_1 y \exists x \varphi \to \exists x Q_1 y \varphi \vee Q_1 x \exists y \varphi$.
5. Show that $L(Q_0)$, $L(Q_1)$ and $L(Q_1^{MM})$ can be translated into L^2. Suggest why the converse might not be true.
6. Suggest, why neither of $L(Q_0)$ and $L(Q_1)$ can be translated to the other.

Abstract model theory

- As a general concept covering the strong logics above—as well as many others—we introduce the concept of an abstract logic.
- We define what it means for an abstract (i.e. strong) logic to satisfy the LS-theorem, have a Hanf-number, have a compactness property, satisfy interpolation, and have a Δ-extension.
- We discuss the so-called Lindström's Theorem which characterizes first order logic among all abstract logics.

Abstract logic

Let τ be a vocabulary. We use Str(τ) to denote the class of all τ-structures.

Definition An abstract (“strong") logic is a pair $L = (S, T)$, where S is a set and T is a relation between structures and elements of S, such that L is closed under isomorphisms, renaming, free expansions, negation, conjunction, and existential quantification, all defined below.

Definition

- For $\varphi \in S$ we write $\text{Mod}_{L,\tau}(\varphi) = \{\mathfrak{M} \in \text{Str}(\tau) : T(\mathfrak{M}, \varphi)\}$.
- A model class is *L-definable* if it is $\text{Mod}_{L,\tau}(\varphi)$ for some $\psi \in S$.
- An abstract logic L is said to be *closed under negation*, if for all vocabularies τ and all $\varphi \in S$ there is $\neg\varphi \in S$ such that $\text{Mod}_{L,\tau}(\neg\varphi) = \text{Str}(\tau) \setminus \text{Mod}_{L,\tau}(\varphi)$.
- We say L is *closed under conjunction* if for all vocabularies τ and all $\varphi, \psi \in S$ there is $\varphi \wedge \psi \in S$ such that $\text{Mod}_{L,\tau}(\varphi \wedge \psi) = \text{Mod}_{L,\tau}(\varphi) \cap \text{Mod}_{L,\tau}(\psi)$.

Definition

- We say L is *closed under existential quantification*, if for all vocabularies τ, for all constant symbols c in τ and for all $\varphi \in S$, there is $\varphi' \in S$ such that:

$$\mathrm{Mod}_{L,\tau\setminus\{c\}}(\varphi') = \{\mathfrak{M} : (\mathfrak{M}, c^{\mathfrak{M}}) \in \mathrm{Mod}_{L,\tau}(\varphi) \text{ for some } c^{\mathfrak{M}} \in M\}.$$

- We say that L is *closed under renaming* if whenever $\pi : \tau \to \tau'$ is a bijection which respects arity, and we extend π in a canonical way to $\hat{\pi} : \mathrm{Str}(\tau) \to \mathrm{Str}(\tau')$, then for all $\varphi \in S$, there is $\varphi' \in S$ such that $\{\hat{\pi}(\mathfrak{M}) : \mathfrak{M} \in \mathrm{Mod}_{L,\tau}(\varphi)\} = \mathrm{Mod}_{L,\tau'}(\varphi')$.
- We say that L is *closed under free expansions* if whenever $\tau \subseteq \tau'$ and $\varphi \in S$, there is $\varphi' \in S$ such that $\mathrm{Mod}_{L,\tau}(\varphi) = \{\mathfrak{M} \restriction \tau : \mathfrak{M} \in \mathrm{Mod}_{L,\tau'}(\varphi')\}$.
- Finally, we say that L is *closed under isomorphisms*, if whenever $\varphi \in S$, $\mathfrak{M} \in \mathrm{Mod}_{L,\tau}(\varphi)$ and $\mathfrak{M} \cong \mathfrak{N}$, then also $\mathfrak{N} \in \mathrm{Mod}_{L,\tau}(\varphi)$.

First order logic as an abstract logic

Example

We get an abstract logic $L_{\omega\omega} = (S_0, T_0)$ by letting S_0 be the set of all first order sentences and T_0 the usual truth predicate of first order logic:

$$T_0(\mathfrak{M}, \varphi) \iff \mathfrak{M} \models \varphi.$$

For example, the closure under free expansions can be satisfied simply by choosing $\varphi' = \varphi$. Other abstract logics arise from infinitary languages, generalized quantifiers, higher order logic and combinations of such.

Definition

- An abstract logic $L = (S, T)$ is a sublogic of another abstract logic $L' = (S', T')$, in symbols $L \leq L'$, if for all $\varphi \in S$ there is $\varphi' \in S'$ such that for all τ, $\mathrm{Mod}_{L,\tau}(\varphi) = \mathrm{Mod}_{L',\tau}(\varphi')$.
- If $L \leq L'$ and $L' \leq L$, we say that L and L' are equivalent, $L \equiv L'$.

Definition

- An abstract logic $L = (S, T)$ is (κ, λ)-compact if for any $\Sigma \subseteq S$ of cardinality $\leq \kappa$, if $\bigcap\{\mathrm{Mod}_{L,\tau}(\varphi) : \varphi \in \Sigma\} = \emptyset$, then $\bigcap\{\mathrm{Mod}_{L,\tau}(\varphi) : \varphi \in \Sigma_0\} = \emptyset$ for some $\Sigma_0 \subseteq \Sigma$ of cardinality $< \lambda$.
- κ-compact means (κ, ω)-compact.
- Countably compact means $\aleph_0$-compact.
- Compact means κ-compact for all κ.
- κ is weakly compact for L, if L is (κ, κ)-compact.
- λ is strongly compact for L, if L is (κ, λ)-compact for all κ.
- Suppose $X, Y \subseteq \mathcal{P}(S)$. (X, Y) is a compactness property for L if for for any $\Sigma \in X$, if $\bigcap\{\mathrm{Mod}_{L,\tau}(\varphi) : \varphi \in \Sigma\} = \emptyset$, then $\bigcap\{\mathrm{Mod}_{L,\tau}(\varphi) : \varphi \in \Sigma_0\} = \emptyset$ for some $\Sigma_0 \in Y$.

Examples

Example

- $L(Q_0)$ is not countably compact (Exercise).
- $L(Q_1)$ is countable compact (later), not $\aleph_1$-compact.
- An inaccessible κ is weakly compact[a] iff it is weakly compact for $L_{\kappa\omega}$.
- An inaccessible κ is strongly compact[b] iff it is strongly compact for $L_{\kappa\omega}$.
- Define $\mathrm{aa}s\varphi(s)$ iff "$\varphi(s)$ is satisfied by a club of countable subsets s". $L(\mathrm{aa})$ is countable compact (later), not $\aleph_1$-compact.
- Define $Q_\omega^{cof}xy\varphi(x, y)$ iff "$\varphi(x, y)$ is a linear order with cofinality ω". $L(Q_\omega^{cof})$ is compact (later).

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[a] i.e. $\kappa \to (\kappa)^2_2$

[b] i.e. for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

Barwise Compactness Theorem

Example

Suppose A is a countable admissible set. Let X be the set of Σ_1-definable subsets of A. Then (X, A) is a compactness property for the infinitary logic $L_A = L_{\infty\omega} \cap A$. (proof omitted, see Barwise "Admissible Sets and Structures")

Let $H(\kappa)$ denote the set[2] of sets of hereditary cardinality $< \kappa$.
If L is an abstract logic, let $L \cap H(\kappa)$ mean the sublogic consisting of sentences (as well as their vocabulary) that are elements of $H(\kappa)$.

Theorem (Stavi 1977)

Suppose κ is a measurable cardinal[a] and U is a normal[b] ultrafilter on κ. If $L = (S, T)$ is any abstract logic, then the set of $\lambda < \kappa$ that are weakly compact for $L \cap H(\lambda)$ belongs to U.

[a] I.e. there is a κ-complete non-principal ultrafilter on κ.
[b] I.e. every $f : \kappa \to \kappa$ such that $f(\alpha) < \alpha$ for all α is constant on a set in U.

[2] See later for details about $H(\kappa)$

- U gives rise to $i : V \to M$, where $M \cong V^\kappa/U$ is transitive.
- κ is the first ordinal moved by i and $M^\kappa \subseteq M$. Thus i is the identity of $H(\kappa)$.
- κ is weakly compact for $L \cap H(\kappa)$ in M, for if not, then M satisfies "there is a theory $\Sigma \subseteq S \cap H(\kappa)$ of cardinality κ such that if Σ_0 is any subset of Σ of cardinality $< \kappa$, then Σ_0 has a model, but Σ itself does not have a model".
- Since $i(\Sigma) \cap H(\kappa) = \Sigma \cap H(\kappa)$, the cardinal κ is the least cardinal λ in M such that $i(\Sigma) \cap H(\lambda)$ does not have a model.
- Recall: $M \models \varphi(\kappa) \iff \{\alpha < \kappa : \varphi(\alpha)\} \in U$. (Loś Lemma!)
- Hence in V the set of $\lambda < \kappa$ that are the least λ such that $\Sigma \cap H(\lambda)$ does not have a model belongs to U (and so $\neq \emptyset$), a contradiction.
- Hence κ is weakly compact for $L \cap H(\kappa)$ in M.
- Hence by Loś again, the set of $\lambda < \kappa$ that are weakly compact for $L \cap H(\lambda)$ belongs to U.

Corollary

Below a measurable cardinal there are many cardinals that are weakly compact for $L_{\omega_1\omega}$, $L(Q_1)$, L^2, etc.

Definition

- An abstract logic $L = (S, T)$ satisfies the Downward Löwenheim-Skolem Property down to κ, $LS(\kappa)$, if for every τ of cardinality $\leq \kappa$, every non-empty $\mathrm{Mod}_{L,\tau}(\varphi)$, $\varphi \in S$, contains a model of cardinality $\leq \kappa$.
- Downward Löwenheim-Skolem Property means $LS(\aleph_0)$.
- The Löwenheim-number ℓ_L of L is the least cardinal κ such that L satisfies $LS(\kappa)$.
- An abstract logic $L = (S, T)$ satisfies the Downward Löwenheim-Skolem-Tarski Property down to κ $LST(\kappa)$ if for every τ of cardinality $\leq \kappa$ and every $\mathfrak{M} \in \mathrm{Mod}_{L,\tau}(\varphi)$, $\varphi \in S$, there is a model $\mathfrak{N} \subseteq \mathfrak{M}$ of cardinality $\leq \kappa$ such that $\mathfrak{N} \in \mathrm{Mod}_{L,\tau}(\varphi)$.
- The LST-number of L is the least cardinal κ such that L satisfies $LST(\kappa)$.

Definition

- An abstract logic $L = (S, T)$ satisfies the Upward Löwenheim-Skolem Property $ULS(\kappa, \lambda)$ if for every τ of cardinality $\leq \lambda$ and $\varphi \in S$ such that $\mathrm{Mod}_{L,\tau}(\varphi)$ contains a model of cardinality $\geq \lambda$, $\mathrm{Mod}_{L,\tau}(\varphi)$ contains also a model of cardinality $\geq \kappa$.
- The Hanf-number h_L of L is the least cardinal λ such that L satisfies $ULS(\kappa, \lambda)$ for all κ.
- An abstract logic $L = (S, T)$ satisfies the Upward Löwenheim-Skolem-Tarski Property $ULST(\kappa, \lambda)$ if for every τ of cardinality $\leq \lambda$ and every $\mathfrak{M} \in \mathrm{Mod}_{L,\tau}(\varphi)$ of cardinality λ, there is a model $\mathfrak{N} \supseteq \mathfrak{M}$ of cardinality $\geq \kappa$ such that $\mathfrak{N} \in \mathrm{Mod}_{L,\tau}(\varphi)$.
- The ULST-number of L is the least cardinal λ such that L satisfies $ULST(\kappa, \lambda)$ for all κ.

Spectrum problem

Definition

- The spectrum of $\varphi \in L$ is the class $\mathrm{Sp}(\varphi) = \{|M| : M \in Mod_{L,\tau}(\varphi)\}$.
- The spectrum problem of L is the problem of characterizing which classes of cardinals are spectra.
- Finite Spectrum Problem: Are the finite parts of first order spectra closed under complements. (Open Problem related to P=NP).

Lemma (Exercise)

- *The Löwenheim number of L is the supremum of the minima of spectra.*
- *The Hanf number of L is the supremum of the suprema of bounded spectra (or its successor if the supremum is in some bounded spectrum).*

A characterisation of first order logic

Lindström's Theorem *Suppose L is an abstract logic such that* $L_{\omega\omega} \leq L$. *Then the following conditions are equivalent:*

(1) *L has the Countable Compactness Property and the Downward Löwenheim-Skolem Property.*

(2) $L \equiv L_{\omega\omega}$.

- Suppose some $\varphi \in S$ is not first order definable, i.e. $\mathrm{Mod}_{L,\tau}(\varphi)$ is not of the form $\mathrm{Mod}_{L_{\omega\omega},\tau}(\psi)$ for any first order ψ. We assume w.l.o.g. that τ is finite and relational.
- For every n there are only finitely many (logically non-equivalent) first order sentences $\psi_i^n, i = 1, ..., k_n$, of vocabulary τ and of quantifier rank at most n.
- Let us call two τ-structures *n-equivalent* if they satisfy the same ψ_i^n. There are only $\leq 2^{k_n}$ different n-equivalence classes, and each class is first order definable.

- Since φ is not definable in first order logic, we can find for any n τ-structures $\mathfrak{M}_n$ and $\mathfrak{N}_n$ such that:

$$\begin{array}{l} T(\mathfrak{M}_n, \varphi) \\ T(\mathfrak{N}_n, \neg\varphi) \\ \mathfrak{M}_n \text{ and } \mathfrak{N}_n \text{ are } n\text{-equivalent.} \end{array} \quad (1)$$

- *Ehrenfeucht and Fraisse* showed that two models $\mathfrak{M}$ and $\mathfrak{N}$ are n-equivalent if and only if there are relations (" back-and-forth sequences") I_i, $i < n$, such that
  - If $(a_1, ..., a_i) I_i (b_1, ..., b_i)$, then $a_1, ..., a_i \in M$ and $b_1, ..., b_i \in N$
  - $() I_0 ()$
  - If $(a_1, ..., a_i) I_i (b_1, ..., b_i)$ then for all $a_{i+1} \in M$ ($b_{i+1} \in N$) there is $b_{i+1} \in N$ ($a_{i+1} \in M$) such that $(a_1, ..., a_{i+1}) I_{i+1} (b_1, ..., b_{i+1})$.
  - If $(a_1, ..., a_{i-1}) I_i (b_1, ..., b_{i-1})$, then for all atomic formulas $\varphi(v_1, ..., v_{i-1})$ we have $\mathfrak{M} \models \varphi(a_1, ..., a_{i-1})$ if and only if $\mathfrak{N} \models \varphi(b_1, ..., b_{i-1})$.

- If there are such relations $I_i, i < \omega$, then we say that that $\mathfrak{M}$ and $\mathfrak{N}$ are ω-equivalent.
- Note:

$$\text{If } \mathfrak{M} \text{ and } \mathfrak{N} \text{ are countable and } \omega\text{-equivalent, then } \mathfrak{M} \cong \mathfrak{N}. \quad (2)$$

- (1) above can be written, supplemented with a little bit of arithmetic, into a sentence $\psi(n)$ in S, using the above back-and-forth characterization of n-equivalence.
- By the Countable Compactness Property there is a model of $\psi(n)$ in which n is non-standard.
- Due to the coding used, this model yields two other models $\mathfrak{M}$ and $\mathfrak{N}$ such that $T(\mathfrak{M}, \varphi)$, $T(\mathfrak{N}, \neg\varphi)$ and $\mathfrak{M}$ and $\mathfrak{N}$ are ω-equivalent.
- By the Downward Löwenheim-Skolem Property, we may assume $\mathfrak{M}$ and $\mathfrak{N}$ are countable.
- But then they are isomorphic by (2).
- Thus L cannot be closed under isomorphisms, contrary to assumption. This ends the proof. $\square$

Theorem (Lindström)

First order logic is characterized by the property that if a sentence has an infinite model it has models of all infinite cardinalities, i.e. the spectra are trivial.

Corollary

If L is a proper extension of first order logic (a “strong logic"), then sentences of L make distinctions between different infinite cardinalities, i.e. there are non–trivial spectra.

Definition (Härtig-quantifier)

Let $\mathfrak{M} \models_s Ixy\varphi(x)\psi(y) \iff$

$$|\{a \in M : \mathfrak{M} \models_{s(a/x)} \varphi(x)\}| = |\{b \in M : \mathfrak{M} \models_{s(b/x)} \psi(x)\}|,$$

i.e. as many elements satisfy $\varphi(x)$ as satisfy $\psi(y)$.

Example (Exercise)

Let $\ell_I = \ell_{L(I)}$ and $h_I = h_{L(I)}$. Let $E(\kappa)$ be the predicate $2^\kappa \geq \kappa^{++}$.

1. If κ is the least κ such that $E(\kappa)$, then $\kappa < \ell_I$
2. If κ is the least κ such that $\neg E(\lambda)$ for all $\lambda \geq \kappa$, then $\kappa < h_I$.

The “identity crises" of ℓ_I and h_I

Theorem (Magidor-V.)

Let $\ell_I = \ell_{L(I)}$ *and* $h_I = h_{L(I)}$.

1. ℓ_I *can consistently be above the first* *measurable* *cardinal, but also consistently below the first* *weakly inaccessible* *cardinal, relative to the consistency of a supercompact cardinal.*
2. h_I *can consistently be above the first* *supercompact* *cardinal but also consistently below the first* *weakly compact* *cardinal.*
3. *Both* $\ell_I < h_I$ *and* $h_I < \ell_I$ *are consistent.*
4. *Both* $\ell_I < 2^\omega$ *and* $2^\omega < \ell_I$ *are consistent.*

(No proofs given for the moment)

The “identity crisis" of $\mathrm{LST}(I)$

Theorem (Magidor-V.)

Let $\mathrm{LST}(I) = \mathrm{LST}(L(I))$.

1. *If* $\mathrm{LST}(I)$ *exists, then there is a weakly inaccessible cardinal and* $\mathrm{LST}(I)$ *is at least the least weakly inaccessible cardinal.*
2. *It is consistent relative to the consistency of a supercompact cardinal that* $\mathrm{LST}(I)$ *is the first weakly inaccessible, and also consistent that it is the first supercompact.*
3. *If* $\kappa = \mathrm{LST}(I)$ *exists, then* $\square_{\lambda,\lambda}$ *fails at every singular* $\lambda \geq \kappa$ *of cofinality* ω*, and SCH holds above* κ.
4. *Each of* $\mathrm{LST}(I) < 2^\omega$, $\mathrm{LST}(I) = 2^\omega$ *and* $2^\omega < \mathrm{LST}(I)$ *are consistent, relative to the consistency of a supercompact.*

(Only proof of (1) given for the moment)

Proof of (1): Let $\mathfrak{A} = (H(\kappa^+), \in, Cd, \vec{f})$, where $\kappa = \mathrm{LST}(I)$, Cd is the set of cardinals $\leq \kappa$, and $\vec{f}$ is a sequence of Skolem functions for $H(\kappa^+)$. Let φ be the conjunction of a suitable finite part of the $L(I)$-theory of $\mathfrak{A}$. By the definition of $\mathrm{LST}(\kappa)$ there is a submodel $\mathfrak{B} = (B, \in \cap B^2, Cd \cap B, \vec{f} \restriction B)$ of $\mathfrak{A}$ of power $\leq \kappa$ satisfying φ. Because $\mathfrak{A}$ has built in Skolem functions, $\mathfrak{B} \prec \mathfrak{A}$. Let M be the transitive collapse of $\mathfrak{B}$ and $i : M \to B$ the associated isomorphism. Now $i : M \to H(\kappa^+)$ is an elementary embedding. Let λ be the largest cardinal in M. As φ is true in $\mathfrak{A}$, every cardinal of M is a real cardinal. Clearly $i(\lambda) = \kappa > \lambda$. Let γ be the first ordinal moved by i. It is easy to see that γ is a limit cardinal. Suppose $f \in M$ is a cofinal δ-sequence in γ for some $\delta < \gamma$. Now $i(f)$ is a cofinal δ-sequence in $i(\gamma)$ whence $i(f)(\beta) > \gamma$ for some $\beta < \delta$. But $i(f)(\beta) = i(f(\beta)) = f(\beta) < \gamma$. Thus γ is weakly inaccessible in M, and therefore, $i(\gamma) \leq \kappa$ is weakly inaccessible in V.

A closure operation on strong logics

Definition

- The **projection** $K \restriction \tau$ of a model class K is the class of reducts $\mathfrak{M} \restriction \tau$, where $\mathfrak{M} \in K$.
- The logic $\Sigma(L)$ consists of all projections of $\mathrm{Mod}_{L,\tau}(\varphi)$ where $\varphi \in L$ and τ is any vocabulary (possibly many-sorted).
- The **Δ-extension** $\Delta(L)$ of L is the logic consisting of model classes K so that $K \in \Sigma(L)$ and $-K \in \Sigma(L)$.
- Note: $\Delta(L)$ is an abstract logic.

Lemma (Exercise)

Δ is a closure property on abstract logics.

Theorem

1. *L is* (κ, λ)*-compact if and only if* $\Delta(L)$ *is. (Exercise)*
2. *L satisfies* $LS(\kappa)$ *if and only if* $\Delta(L)$ *does.*
3. $\Delta(L)$ *satisfies the Souslin-Kleene interpolation property: If* K *and* $-K$ *are both projections, then* K *is definable.*
4. $\Delta(L^2)$ *contains the simple theory of types. (Later.)*

Examples - a preview

Example

1. $\Delta(\mathcal{L}_{\omega\omega}) = \mathcal{L}_{\omega\omega}$ (By Craig Interpolation.)
2. $\Delta(\mathcal{L}_{\omega_1\omega}) = \mathcal{L}_{\omega_1\omega}$ (By Craig Interpolation. Proof in a moment.)
3. $\Delta(L_A) = L_A$ for countable "admissible" A. (By Craig Interpolation.)
4. $\Delta(L(Q_0)) = \Delta(L^2_w) = L_{HYP}$, where HYP is the smallest admissible set. (Later!)
5. Conjecture: $\Delta(L(Q_1))$ cannot be obtained from first order logic by adding finitely many generalised quantifiers.
6. $\Delta(L(I))$ can be consistently $= \Delta(L^2)$ but also $\neq \Delta(L^2)$. (Later!)
7. $\Delta(L^2) \neq L^2$. (Later!)
8. A model class is definable in $\Delta(L^2)$ if and only if it is Δ_2-definable in set theory. (Later!)

Problem session 2

1. Show that $L(Q_0)$ is not $(\kappa, \aleph_0)$-compact for any $\kappa \geq \aleph_0$.
2. Show that a sublogic of a (κ, λ)-compact abstract logic is (κ, λ)-compact.
3. Show that a sublogic of an abstract logic with the Downward Löwenheim-Skolem Property down to κ has the Downward Löwenheim-Skolem Property down to κ.
4. Show that the Löwenheim number of L is the supremum of the minima of spectra, and that the Hanf number of L is the supremum of the suprema of bounded spectra (or its successor if the supremum is in some bounded spectrum).
5. Suppose κ is weakly compact for $L_{\kappa\omega}$. Show that κ is regular.
6. Show that $L(Q_1)$ satisfies the Downward Löwenheim-Skolem-Tarski Property down to $\aleph_1$.

Problem session 3

1. Show that if the extension L of first order logic is (κ, λ^+)-compact, then L satisfies $ULST(\kappa, \lambda)$.
2. Show that $\Delta(\Delta(L)) \equiv \Delta(L)$.
3. Prove: L is (κ, λ)-compact if and only if $\Delta(L)$ is.
4. Let $\ell_I = \ell_{L(I)}$ and $h_I = h_{L(I)}$. Let $E(\kappa)$ be the predicate $2^\kappa \geq \kappa^{++}$. Show that if κ is the least $\kappa \geq \omega$ such that $E(\kappa)$, then $\kappa < \ell_I$, and if κ is the least κ such that $\neg E(\lambda)$ for all $\lambda \geq \kappa$, then $\kappa < h_I$.
5. Show that ℓ_I is *at least* the least κ with $\kappa = \aleph_\kappa$.

Model theoretic preliminaries

We review some fundamental constructions in model theory such as consistency properties and ultraproducts with an eye on generalizing them to strong logics.

Model constructions

- For $L_{\omega\omega}$ we have the Compactness Theorem, but it is often too rough. For $L_{\omega_1\omega}$ we do not have compactness.
- More flexible: Consistency Properties.
- Elaboration of Beth tableaux.
- A bit like forcing.
- First an auxiliary tool: Hintikka sets.

Notation: Pushing negation inside

$$\begin{array}{lcl} \varphi\neg & = & \neg\varphi \text{ if } \varphi \text{ atomic} \\ (\neg\varphi)\neg & = & \varphi \\ (\bigwedge_n \varphi_n)\neg & = & \bigvee_n \neg\varphi_n \\ (\bigvee_n \varphi_n)\neg & = & \bigwedge_n \neg\varphi_n \\ (\forall x\varphi)\neg & = & \exists x\neg\varphi \\ (\exists x\varphi)\neg & = & \forall x\neg\varphi \end{array}$$

Lemma

$\vdash \neg\varphi \leftrightarrow \varphi\neg$

Definition

L countable, C a countable set of new constant symbols and $L' = L \cup C$. A *Hintikka set* is a set H of L'-sentence of $L_{\omega_1\omega}$ (or $L_{\omega\omega}$), which satisfies:

1. $t = t \in H$ for every constant L'-term t.
2. If $\varphi(t) \in H$, $\varphi(t)$ atomic, and $t = t' \in H$, then $\varphi(t') \in H$.
3. If $\neg\varphi \in H$, then $\varphi\neg \in H$.
4. If $\bigvee_n \varphi_n \in H$, then $\varphi_n \in H$ for some n. (In the case of $L_{\omega\omega}$ consider only finite disjunctions.)
5. If $\bigwedge_n \varphi_n \in H$, then $\varphi_n \in H$ for all n. (In the case of $L_{\omega\omega}$ consider only finite conjunctions.)
6. If $\exists x\varphi(x) \in H$, then $\varphi(c) \in H$ for some $c \in C$
7. If $\forall x\varphi(x) \in H$, then $\varphi(c) \in H$ for all $c \in C$
8. For every constant L'-term t there is $c \in C$, such that $t = c \in H$.
9. There is no atomic sentence φ such that $\varphi \in H$ and $\neg\varphi \in H$.

The Hintikka set H is a Hintikka set *for* a sentence φ of $L_{\omega_1\omega}$ if $\varphi \in H$.

A basic lemma

Lemma

If there is a Hintikka set for $\varphi \in L_{\omega_1\omega}$, then φ has a model.

...and vice versa!

Proof.

- Define on C: $c \sim c'$ if $c = c' \in H$.
- $M = \{[c] : c \in C\}$.
- $c^{\mathfrak{M}} = [c]$.
- We let $\mathfrak{M} \models R(t_1, \ldots, t_n)$ if and only if $R(t_1, \ldots, t_n) \in H$.
- By induction on $\varphi(x_1, \ldots, x_n)$: if $d_1 \ldots, d_n \in C$ then:
  1. If $\varphi(d_1, \ldots, d_n) \in H$, then $\mathfrak{M} \models \varphi(d_1, \ldots, d_n)$.
  2. If $\neg\varphi(d_1, \ldots, d_n) \in H$, then $\mathfrak{M} \not\models \varphi(d_1, \ldots, d_n)$.

  In particular, $\mathfrak{M} \models \varphi$ for the φ we started with, since $\varphi \in H$.

□

Whence Hintikka sets?

- How to find useful Hintikka sets?
- Rough method: Add inductively Henkin axioms $\exists x \varphi(x) \to \varphi(c)$. Take maximal consistent extension. This is a Hintikka set for $L_{\omega\omega}$.
- A more refined tool is: consistency property.
- A consistency property is a set Δ of (usually) finite sets S which are consistent and the consistency property has information about how to extend S to a Hintikka set, which will then give a model for S.

Definition

Let L be a countable vocabulary, C a countable set of new constant symbols and $L' = L \cup C$. A *consistency property* is any set Δ of countable sets S of L'-formulas of $L_{\omega_1\omega}$ (or $L_{\omega\omega}$), which satisfies the conditions:

1. If $S \in \Delta$, then $S \cup \{t = t\} \in \Delta$ for every constant L'-term t.
2. If $\varphi(t) \in S \in \Delta$, $\varphi(t)$ atomic, and $t = t' \in S$, then $S \cup \{\varphi(t')\} \in \Delta$.
3. If $\neg\varphi \in S \in \Delta$, then $S \cup \{\varphi\neg\} \in \Delta$.
4. If $\bigvee_n \varphi_n \in S \in \Delta$, then $S \cup \{\varphi_n\} \in \Delta$ for some n.
5. If $\bigwedge_n \varphi_n \in S \in \Delta$, then $S \cup \{\varphi_n\} \in \Delta$ for all n.
6. If $\exists x\varphi(x) \in S \in \Delta$, then $S \cup \{\varphi(c)\} \in \Delta$ for some $c \in C$.
7. If $\forall x\varphi(x) \in S \in \Delta$, then $S \cup \{\varphi(c)\} \in \Delta$ for all $c \in C$.
8. For every constant L'-term t there is $c \in C$ such that $S \cup \{t = c\} \in \Delta$.
9. There is no atomic formula φ such that $\varphi \in S$ and $\neg\varphi \in S$.

The consistency property Δ is a consistency property *for* a set T of infinitary L- sentences if for all $S \in \Delta$ and all $\varphi \in T$ we have $S \cup \{\varphi\} \in \Delta$.

Fragments - to obtain a countability condition.

Definition

Let L be a vocabulary. An L-*fragment* of $L_{\omega_1\omega}$ is any set $\mathcal{F}$ of formulas of $L_{\omega_1\omega}$ in the vocabulary L such that

(1) $\mathcal{F}$ is closed substitutions of terms.

(2) $\mathcal{F}$ contains the atomic L-formulas.

(3) $\neg\varphi \in \mathcal{F}$ if and only if $\varphi \in \mathcal{F}$.

(4) $\wedge\Phi \in \mathcal{F}$ if $\Phi \subseteq \mathcal{F}$ is finite.

(5) $\vee\Phi \in \mathcal{F}$ if $\Phi \subseteq \mathcal{F}$, is finite.

(6) $\wedge\Phi \in \mathcal{F}$ if and only if $\vee\Phi \in \mathcal{F}$, and then $\Phi \subseteq \mathcal{F}$.

(7) $\forall x\varphi \in \mathcal{F}$ if and only if $\varphi \in \mathcal{F}$.

(8) $\exists x\varphi \in \mathcal{F}$ if and only if $\varphi \in \mathcal{F}$.

Note that a fragment is necessarily closed under subformulas.

Existence of fragments

Lemma

Suppose $\varphi \in L_{\omega_1\omega}$ with a countable vocabulary L. Then there is a countable fragment $\mathcal{F} \subseteq L_{\omega_1\omega}$ such that $\varphi \in \mathcal{F}$.

Existence of Hintikka sets

Lemma

Let T be a countable set of L-sentences of $L_{\omega_1\omega}$ and $\mathcal{F}$ a countable fragment such that $T \subseteq \mathcal{F}$. Suppose Δ is a consistency property for T. Then for any $S \in \Delta$ there is a Hintikka set H for T such that $S \subseteq H$.

Proof.

Enumerate all formulas of $\mathcal{F}$ in an ω-sequence. Then construct H as the union of an increasing ω-sequence of elements of Δ dovetailing to make sure all requirements of Hintikka sets are met. □

Note: This is like constructing a generic set over a countable model in forcing, meeting countably many dense sets.

Proof:

- Let *Trm* the set of all constant L'-terms.
- Let $T = \{\varphi_n : n \in \mathbb{N}\}, C = \{c_n : n \in \mathbb{N}\}, Trm = \{t_n : n \in \mathbb{N}\}$.
- Let $\{\psi_n : n \in \mathbb{N}\}$ be a list of all elements of $\mathcal{F}$.
- We define H as the union of an increasing sequence $S_0, S_1, \ldots$, where $S_0 = S$.
- Suppose we have already defined S_n and want to now define S_{n+1}.
- $S_{n+1} = S_n$ unless:

1. If $n = 3^i$, then S_{n+1} is $S_n \cup \{\varphi_i\} \in \Delta$.
2. If $n = 2 \cdot 3^i$, then S_{n+1} is $S_n \cup \{t_i = t_i\} \in \Delta$.
3. If $n = 4 \cdot 3^i \cdot 5^j \cdot 7^k$, $\psi_i = (t = t') \in S_n$, and $\psi_j = \varphi(t) \in S_n$ with $\varphi(t)$ atomic, then S_{n+1} is $S_n \cup \{\varphi(t')\} \in \Delta$.
4. If $n = 8 \cdot 3^i \cdot 5^j$ and $\psi_i = \neg\psi \in S_n$, then S_{n+1} is $S_n \cup \{\psi\neg\}$.
5. If $n = 16 \cdot 3^i \cdot 5^j$ and $\psi_i = \bigvee_m \psi_m \in S_n$, then S_{n+1} is $S_n \cup \{\psi_m\}$ for some m, whichever is in Δ.
6. If $n = 32 \cdot 3^i \cdot 5^j \cdot 7^k$, $j \in \{0, 1\}$ and $\psi_i = \bigwedge_m \psi_m \in S_n$, then S_{n+1} is $S_n \cup \{\psi_j\}(\in \Delta)$.
7. If $n = 64 \cdot 3^i \cdot 5^j$ and $\psi_i = \exists x\varphi \in S_n$, then S_{n+1} is $S_n \cup \{\varphi(c)\}$ for such $c \in C$ that $S_{n+1} \in \Delta$.
8. If $n = 128 \cdot 3^i \cdot 5^j \cdot 7^k$ and $\psi_i = \forall x\varphi \in S_n$, then S_{n+1} is $S_n \cup \{\varphi(c_j)\}(\in \Delta)$.
9. If $n = 256 \cdot 3^i$, then S_{n+1} is $S_n \cup \{t_i = c\}$ for such $c \in C$ that $S_{n+1} \in \Delta$.

Clearly $\bigcup_n S_n$ is a Hintikka set for T.

Consistency property from "consistency"

Let $S \vdash \varphi$ be the "canonical" concept of proof for $L_{\omega_1\omega}$ i.e. from $\bigwedge_n \varphi_n$ derive each φ_n and from $\{\varphi_n : n < \omega\}$ derive $\bigwedge_n \varphi_n$, and similarly for disjunction.

Lemma

The set of finite sets S of formulas of $L_{\omega_1\omega}$ such that only finitely many constants from C occur in S, and $S \nvdash \perp$ is a consistency property.

Proof: (Sketch)

- Suppose $\bigwedge_n \varphi_n \in S \in \Delta$ but $S \cup \{\varphi_n\} \vdash \bot$ for some n. Then $S \vdash \bot$, contradiction.
- Suppose $\exists x \varphi(x) \in S \in \Delta$ but $S \cup \{\varphi(c)\} \vdash \bot$ for all $c \in C$. Then $S \vdash \bot$, because we can choose c so that it does not occur in S. contradiction.

The Completeness Theorem for $L_{\omega\omega}$ or $L_{\omega_1\omega}$

Theorem

TFAE for ($\varphi \in L_{\omega\omega}$ or) $\varphi \in L_{\omega_1\omega}$:

1. $\models \varphi$ *i.e.* φ *is true in all models.*
2. $\vdash \varphi$ *i.e.* φ *has a proof.*

Proof.

If φ has a proof, then clearly $\models \varphi$. If φ does not have a proof, then $\neg\varphi \nvdash \bot$, hence $\neg\varphi$ has a model and $\not\models \varphi$. □

Completeness Theorem for $L(Q_0)$, L_2^w, ω-logic and the simple theory of types (and many others) can be proved in the same way. $L(Q_1)$ and other logics based on uncountable models call for a different method.

- $L_{\omega_1\omega}$ is obviously not countably compact but Barwise Compactness (see before) can be proved using consistency properties (see Barwise: Admissible Sets and Structures, Springer 1975).
- Consistency properties can be used to prove Craig Interpolation for $L_{\omega_1\omega}$.
- Another application: $L_{\omega_1\omega}$ cannot express well-foundedness of a binary relation.

Craig Interpolation Theorem for $L_{\omega_1\omega}$ (or $L_{\omega\omega}$)

Theorem

We assume that L_1 and L_2 are relational. Suppose $\models \varphi \to \psi$, where φ is an L_1-sentence and ψ is an L_2-sentence of $L_{\omega_1\omega}$. Then there is an $L_1 \cap L_2$-sentence θ of $L_{\omega_1\omega}$ such that

1. $\models \varphi \to \theta$
2. $\models \theta \to \psi$

Proof:

- Let us assume that the claim of the theorem is false and derive a contradiction. Since $\models \varphi \to \psi$, the set $\{\varphi, \neg\psi\}$ has no models. We construct a consistency property for $\{\varphi, \neg\psi\}$.
- Let $L = L_1 \cap L_2$.
- Given a set S of sentences, let S_1 consists of all $L_1 \cup C$-sentences in S with only finitely many constant from C, and let S_2 consists of all $L_2 \cup C$-sentences in S with only finitely many constant from C.

Definition

Let us say that θ **separates** S' and S'' if

1. $S' \models \theta$,
2. $S'' \models \neg\theta$,

Definition

Let Δ consist of all finite sets S of sentences of $L_{\omega_1\omega}$ such that $S = S_1 \cup S_2$ and :

($\star$) There is no $L \cup C$-sentence that separates S_1 and S_2.

This is a consistency property. Hence $\{\varphi, \neg\psi\}$ has a model, a contradiction. QED

We prove one case of the proof:

Case 6. Consider $S \in \Delta$ and $\exists x\varphi(x) \in S_1$. Let $c_0 \in C$ be such that c_0 does not occur in S. We claim that the sets $S_1 \cup \{\varphi(c_0)\}$ and S_2 satisfy $(\star)$. Otherwise there is some $\theta(c_0, \ldots, c_{m-1})$ that separates $S_1 \cup \{\varphi(c_0)\}$ and S_2. Let[3] $\theta'(c_1, \ldots, c_{m-1}) = \exists x\theta(x, c_1, \ldots, c_{m-1})$. We show that $\theta'(c_1, \ldots, c_{m-1})$ separates S_1 and S_2, a contradiction. Checking this:

- $S_1 \models \theta'(c_1, \ldots, c_{m-1})$:

$$S_1 \cup \{\varphi(c_0)\} \models \theta(c_0, \ldots, c_{m-1}) \text{ by assumption}$$

$$S_1 \models \varphi(c_0) \to \theta(c_0, \ldots, c_{m-1})$$

$$S_1 \models \forall x(\varphi(x) \to \theta(x, c_1, \ldots, c_{m-1}))$$

$$S_1 \models \exists x\varphi(x) \to \exists x\theta(x, c_1, \ldots, c_{m-1})$$

$$S_1 \models \exists x\theta(x, c_1, \ldots, c_{m-1}) \text{ as } S_1 \models \exists x\varphi(x)$$

$$S_1 \models \theta'(c_1, \ldots, c_{m-1})$$

[3] If c_0 does not occur in θ, then we take $\theta' = \theta$.

Case 6. (Contd.)

- $S_2 \models \neg\theta'(c_0, \ldots, c_{m-1})$:

$$S_2 \models \neg\theta(c_0, \ldots, c_{m-1})$$

$$S_2 \models \forall x \neg\theta(x, c_1, \ldots, c_{m-1})$$

$$S_2 \models \neg\exists x \theta(x, c_1, \ldots, c_{m-1})$$

$$S_2 \models \neg\theta'(c_1, \ldots, c_{m-1})$$

Case 6. (Contd.) Consider $S \in \Delta$ and $\exists x\varphi(x) \in S_2$. Let $c_0 \in C$ be such that c_0 does not occur in S. We claim that the sets S_1 and $S_2 \cup \{\varphi(c_0)\}$ satisfy $(\star)$. Otherwise there is some $\theta(c_0, \ldots, c_{m-1})$ that separates S_1 and $S_2 \cup \{\varphi(c_0)\}$. Let[4] $\theta'(c_1, \ldots, c_{m-1}) = \forall x\theta(x, c_1, \ldots, c_{m-1})$. We show that $\theta'(c_1, \ldots, c_{m-1})$ separates S_1 and S_2, a contradiction. Checking this:

- $S_1 \models \theta'(c_1, \ldots, c_{m-1})$:

$$S_1 \models \theta(c_0, \ldots, c_{m-1})$$

$$S_1 \models \forall x\theta(x, c_1, \ldots, c_{m-1})$$

$$S_1 \models \theta'(c_1, \ldots, c_{m-1})$$

[4] If c_0 does not occur in θ, we choose $\theta' = \theta$.

- $S_2 \models \neg\theta'(c_1, \ldots, c_{m-1})$:

$$S_2 \cup \{\varphi(c_0)\} \models \neg\theta(c_0, \ldots, c_{m-1})$$

$$S_2 \models \varphi(c_0) \to \neg\theta(c_0, \ldots, c_{m-1})$$

$$S_2 \models \forall x(\varphi(x) \to \neg\theta(x, c_1, \ldots, c_{m-1}))$$

$$S_2 \models \exists x\varphi(x) \to \exists x\neg\theta(x, c_1, \ldots, c_{m-1})$$

$$S_2 \models \exists x\neg\theta(x, c_1, \ldots, c_{m-1})$$

$$S_2 \models \neg\theta'(c_1, \ldots, c_{m-1})$$

- $\mathrm{LST}(L_{\omega_1\omega}) = \omega$ (Easy, using Skolem-functions.)
- $h_{L_{\omega_1\omega}} = \beth_{\omega_1}$ (Proof omitted for the moment, uses Erdös-Rado Theorem)
- $\Delta(L_{\omega_1\omega}) = L_{\omega_1\omega}$ (Follows from Craing Interpolation for $L_{\omega_1\omega}$.)

Definition

Suppose M_i, $i \in I$, are sets and F is a filter on I. Let

$$f \sim g \iff \{i \in I : f(i) = g(i)\} \in F \tag{3}$$

for $f, g \in \prod_i M_i$. The set

$$\prod_i M_i/F = \{[f] : f \in \prod_i M_i\}$$

is called a *reduced product* of the sets M_i, $i \in I$. If F is an ultrafilter, it is called an *ultraproduct* of the sets M_i, $i \in I$.

Let L be a vocabulary, F a filter on I, and $\mathfrak{M}_i$, $i \in I$ a collection of L-structures. We can define a new L-structure $\mathfrak{M}$ with

$$M = \prod_i M_i / F \tag{4}$$

as universe as follows. For $R \in L$, $\mathfrak{a}(R) = m$, we define

$$([f_1], \ldots, [f_m]) \in R^{\mathfrak{M}} \iff \{i \in I : (f_1(i), \ldots, f_m(i)) \in R^{\mathfrak{M}_i}\} \in F. \tag{5}$$

Note that (5) is independent of the choice of $f_1, \ldots, f_m$. For $c \in L$ we define

$$c^{\mathfrak{M}} = [f], \text{ where } f(i) = c^{\mathfrak{M}_i}. \tag{6}$$

Definition

The *ultraproduct* $\prod_i \mathfrak{M}_i/F$ of the L-structures $\mathfrak{M}_i (i \in I)$ with respect to the ultrafilter F is the L-structure $\mathfrak{M}$ defined by (4), (5), and (6) above.

If s is an assignment into the set $\prod_i M_i/F$ then $s(x_i) = [f_i]$ for some function f_i. We then denote by s_n the induced assignment $s_n(x_i) = f_i(n)$ into M_n.

Lemma (Loś Lemma)

If F is an ultrafilter and φ is a first-order formula, then

$$\prod_i \mathfrak{M}_i/F \models_s \varphi \iff \{i \in I : \mathfrak{M}_i \models_{s_i} \varphi\} \in F.$$

Among the many corollaries of the Łoś Lemma the simplest is: For all $\mathfrak{M}$ and all ultrafilters F:

$$\prod_i \mathfrak{M}/F \equiv \mathfrak{M}.$$

We can also use the Łoś Lemma to give a new proof of the Compactness Theorem in any vocabulary, countable or uncountable:

Theorem (Compactness Theorem)

Suppose L is a vocabulary (of any cardinality) and T is a set of first-order L-sentences. If every finite subset of T has a model, then T itself has a model.

Proof.

Suppose T is a set of first-order sentences. Let $\mathcal{A}$ be the set of finite subsets of T. The assumption is that each $S \in \mathcal{A}$ has a model $\mathfrak{M}_S$. Let $\widehat{\varphi} = \{S \in \mathcal{A} \ : \ \varphi \in S\}$. Let F be a non-principal ultrafilter on $\mathcal{A}$ extending the set

$$\{\widehat{\varphi} \ : \ \varphi \in T\}.$$

Let $\mathfrak{M} = \prod_{S \in \mathcal{A}} \mathfrak{M}_S / F$. For $\varphi \in T$:

$$\{S \in \mathcal{A} : \mathfrak{M}_S \models \varphi\} \supseteq \widehat{\varphi} \in F.$$

Hence $\mathfrak{M} \models \varphi$ for all $\varphi \in T$. □

Ultraproducts and the appropriate Loś Lemma are behind the proof of

- κ is weakly compact iff it is weakly compact for $L_{\kappa\omega}$.
- κ is strongly compact iff it is strongly compact for $L_{\kappa\omega}$.

We now use ultraproducts to prove the Countable Compactness Theorem for the logic $L(Q_\alpha)$, $\aleph_\alpha = (2^{\aleph_0})^+$, that is, the extension of first-order logic by the quantifier

"there are more ... than there are reals".

Lemma

Suppose If F is an ultrafilter on I and $A \in F$.

1. $F \restriction A = \{X \cap A : X \in F\}$ *is an ultrafilter on A.*
2. $|\prod_{n\in\mathbb{N}} M_n/F| = |\prod_{n\in A} M_n/(F \restriction A)|$.

Proof.

(2): If $[f] \in \prod_n M_n/F$, let

$$\pi([f]) = [f \restriction A].$$

It is easy to see that π is well-defined and a bijection $\prod M_n/F \to \prod_n M_n/F \restriction A$. □

Thus the size of the ultraproduct is unaltered by restriction to a subset of the indices as long as the subset itself was large, i.e. an element of the ultrafilter.

Lemma (Łoś Lemma)

Suppose F is an u.f. on $\mathbb{N}$. If $\varphi \in L(Q_\alpha)$, $\aleph_\alpha = (2^{\aleph_0})^+$, then

$$\prod_n \mathfrak{M}_n/F \models_s \varphi \iff \{n \in \mathbb{N} : \mathfrak{M}_n \models_{s_n} \varphi\} \in F.$$

Proof.

We can concentrate on the case $\varphi = Q_\alpha x_m \psi$. Let

$$X = \{[f] \in \prod M_n/F : \prod_n \mathfrak{M}_n/F \models_{s([f]/x_m)} \psi\}$$

and $X_n = \{a \in M_n : \mathfrak{M}_n \models_{s_n(a/x_m)} \psi\}$. Then by the induction hypothesis $X = \prod_n X_n/F$.

Let $A = \{n \in \mathbb{N} : \mathfrak{M}_n \models_{s_n} \varphi\}$. We want to prove $|X| > 2^\omega \iff A \in F$.

$\Leftarrow$: Note that $n \in A \iff |X_n| > 2^\omega$. So if $A \in F$, then

$$|X| = |\prod_{n \in A} X_n/(F \restriction A)| > 2^\omega.$$

$\Rightarrow$: If $A \notin F$, then $|X| = |\prod_{n \notin A} X_n/(F \restriction -A)| \leq (2^\omega)^\omega = 2^\omega$.

□

Theorem (Fuhrken)

$L(Q_\alpha)$, $\aleph_\alpha = (2^{\aleph_0})^+$, satisfies Countable Compactness.

Proof.

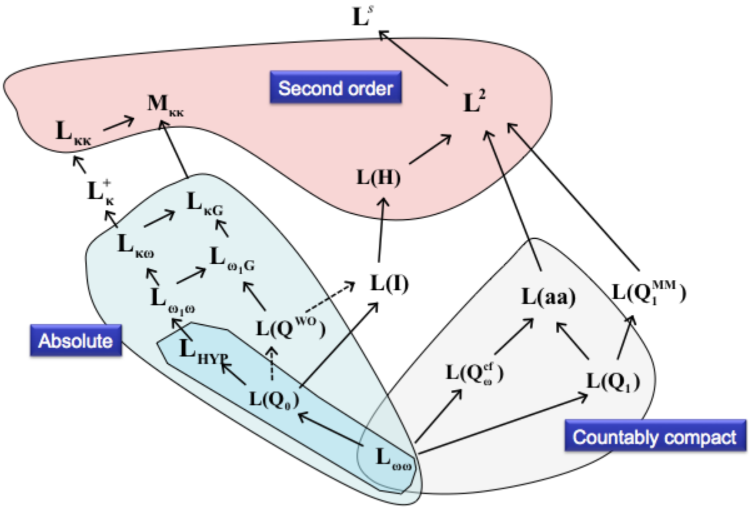
Suppose $T = \{\varphi_0, \varphi_1, \ldots\}$ is a set of sentences of $L(Q_\alpha)$ such that each $\{\varphi_0, \ldots, \varphi_n\}$ has a model $\mathfrak{M}_n$. Suppose F is a non-principal u.f. on $\mathbb{N}$. Let $\mathfrak{M} = \prod_n \mathfrak{M}_n/F$. Since F does not contain finite sets,

$$\{n \in \mathbb{N} : \mathfrak{M}_n \models \varphi_i\} \supseteq \{i, i+1, i+2, \ldots\} \in F.$$

Hence $\mathfrak{M} \models \varphi_i$ for all $i \in \mathbb{N}$. □

Theorem (Shelah)

If $\aleph_\alpha^\omega = \aleph_\alpha$, then $L(Q_{\alpha+1})$ satisfies Countable Compactness.


L^s
Second order
L²
M_κκ
L_κκ
L(H)
L_κ^+
L_κG
L_κω
L_ω1G
L(I)
L(aa)
L(Q_1^MM)
L_ω1ω
L(Q^WO)
Absolute
L_HYP
L(Q_ω^cf)
L(Q_1)
L(Q_0)
Countably compact
L_ωω

Problem session 4

1. Show that h_I is *at least* the least κ with $\kappa = \beth_\kappa$. (Hint: Write a sentence of $L(I)$ which, for each $\alpha < \kappa$, has a model of cardinality $\beth_\alpha$, but which has no models of cardinality $\geq \kappa$.)
2. Prove case 5 (conjunction) of the proof that the set Δ consisting of all finite sets S of sentences of $\mathcal{L}_{\omega_1\omega}$ such that $S = S_1 \cup S_2$ and there is no $L \cup C$-sentence that separates S_1 and S_2, is a consistency property.
3. See next slide.
4. Consider two equivalence relations E_1 and E_2 on ω_1, both with only uncountable classes, such that E_1 has uncountably many classes but E_2 only countably many. Prove that they are $L(Q_1)$-equivalent by using the EF-game of Exercise 3.
5. Use the previous exercise to show that $L(Q_1)$ does not satisfy the Craig Interpolation Theorem. (Hint: Construct an implication so that an interpolating sentence would separate E_1 and E_2 above.)

Problem session 4 (Contd.)

Problem 3: The EF-game for $L(Q_1)$ is defined as follows: We are given two models $\mathcal{A}$ and $\mathcal{B}$ of a fixed finite relational vocabulary. The players play elements, one at a time. The elements played in $\mathcal{A}$ are denoted $a_0, \ldots, a_{n-1}$, and elements played in $\mathcal{B}$ respectively $b_0, \ldots, b_{n-1}$. If Player I at any moment plays a_i, then Player II plays b_i. If Player I plays b_i, then Player II plays a_i. Player II wins as long as always $a_i \mapsto b_i$ $(i = 0, \ldots, n-1)$ is a partial isomorphism (i.e. the a_is satisfy the same atomic formulas as the b_is). Thus $\mathcal{A} \equiv \mathcal{B}$ if and only if for all n Player II has a winning strategy in the n-move game. Now the Q_1-move: Player I plays (instead of a single element) an uncountable subset X of one of the models, say $\mathcal{A}$. Player II responds by playing an uncountable subset Y from the other model, in this case $\mathcal{B}$. Then Player I picks an element $b_i \in Y$. Finally Player II picks an element $a_i \in X$. After this the sets X and Y are forgotten and the game continues. Show by induction on the quantifier-rank m of $\varphi(x_0, \ldots, x_{n-1}) \in L(Q_1)$ that if Player II has a winning strategy for m moves in the position $(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})$, then $\mathcal{A} \models \varphi(a_0, \ldots, a_{n-1}) \iff \mathcal{B} \models \varphi(b_0, \ldots, b_{n-1})$.

Generalized quantifiers

Generalized quantifiers are a corner stone of the area of strong logics. We review the basic properties of the most important generalized quantifiers:

- Q_0: LS-theorem, incompactness, Hanf-number.
- Q_1: LS-theorem, countable compactness, Hanf-number.
- Magidor-Malitz quantifier Q_1^{MM}: LS-theorem, countable compactness assuming $\diamondsuit$, consistency of the failure of countable compactness.
- Cofinality quantifier Q_ω^{cf}: LS-theorem, compactness.
- Stationary logic $L(\mathtt{aa})$: LS-theorem, axiomatization, countable compactness.
- Härtig-quantifier, Henkin-quantifier.

Why completeness theorems?

- A completeness theorem shows that the question, whether a sentence of the logic has a model, is absolute, Δ_1.
- Suppose we construct a structure using CH or $\Diamond$ or large cardinals. If we can express the properties of the structure in a logic which has a Completeness Theorem, we can conclude that the set theoretic assumptions are not needed.

Definition

1. Axioms of first order logic
2. $\neg Qx(x = y \vee x = z)$
3. $\forall x(\varphi \to \psi) \to (Qx\varphi(x) \to Qx\psi(x))$
4. $Qx\varphi(x) \to Qy\varphi(y)$
5. $\mathcal{Q}x\exists y\varphi(x,y) \to (\exists y\mathcal{Q}x\varphi(x,y) \vee \mathcal{Q}y\exists x\varphi(x,y))$
6. Rules: Modus Ponens and Generalization.

Example

$Qx(\varphi \vee \psi) \to (Qx\varphi \vee Qx\psi)$

Proof.

Let $\theta \equiv ((y = y_1 \wedge \varphi) \vee (y = y_2 \wedge \psi))$.

(1) $\vdash \forall x((\varphi \vee \psi) \to \exists y\theta)$

(2) $\vdash Qx(\varphi \vee \psi) \to Qx\exists y\theta$

(3) $\vdash Qx\exists y\theta \to (\exists yQx\theta \vee Qy\exists x\theta)$

(4) $\vdash \theta \to (y = y_1 \vee y = y_2)$

(5) $\vdash (\exists x\theta) \to (y = y_1 \vee y = y_2)$

(6) $\vdash (Qy\exists x\theta) \to Qy(y = y_1 \vee y = y_2)$

(7) $\vdash \neg Qy\exists x\theta$

(8) $\vdash Qx(\varphi \vee \psi) \to \exists yQx\theta$, by (2,3,7)

(9) $\vdash \neg y = y_1 \to (Qx\theta \to Qx\psi)$

(10) $\vdash (\neg y = y_1 \vee \neg y = y_1) \to (Qx\theta \to (Qx\varphi \vee Qx\psi))$

(11) $\vdash (\neg y = y_1 \vee \neg y = y_1) \to (\exists yQx\theta \to (Qx\varphi \vee Qx\psi))$

(12) $\vdash Qx(\varphi \vee \psi) \to (Qx\varphi \vee Qx\psi)$, by (8,11) □

Idea

- We construct a model for a consistent theory T as the union $\mathcal{M}$ of an elementary chain of countable models $(\mathcal{M}_\alpha)$, $\alpha < \omega_1$.
- If T thinks $Q_1 x\varphi(x)$, we make sure for all α $\mathcal{M} \models \varphi(a)$ for some $a \in M_{\alpha+1} \setminus M_\alpha$.
- If T thinks $\neg Q_1 x\varphi(x)$, we make sure there is α such that for all $a \in M$, if $\mathcal{M} \models \varphi(a)$ then $a \in M_\alpha$.

Weak model

Definition

A weak model is a pair $(\mathcal{M}, q)$, where $\mathcal{M}$ is an ordinary model and $q \subseteq \mathcal{P}(M)$. Then we define

$$(\mathcal{M}, q) \models Qx\varphi(x) \Leftrightarrow_{def} \{a \in M : (\mathcal{M}, q) \models \varphi(a)\} \in q.$$

Lemma (Weak Completeness Theorem)

If L is a countable vocabulary and T is a consistent set of sentences of $L(Q_1)$, then T has a countable weak model.

Proof

- Adopt a set C of countably many new constants.
- Extend T by Henkin axioms $\exists x\varphi(x) \to \varphi(c)$ to a consistent T_H
- Extend T_H to a consistent T', which decides every φ.
- Let $c \sim c'$ if $c = c' \in T'$. This is an equivalence relation. Let $[c]$ be the equivalence class of c.
- Let $\{[c] : c \in C\}$ be the domain of a model $\mathcal{M}$, where $R^{\mathcal{M}}([c_1], \ldots, [c_n])$ is defined to hold iff $R(c_1, \ldots, c_n) \in T'$.
- Let $q = \{\{[c] : \varphi(c) \in T'\} : Qx\varphi(x) \in T'\}$.
- By induction, $(\mathcal{M}, q) \models \varphi \iff \varphi \in T'$.
- Hence $(\mathcal{M}, q) \models T$.

Lemma (Weak Omitting Types Theorem)

Suppose L is a countable vocabulary, T is a consistent set of sentences of $L(Q_1)$, and τ_n is a non-principal[a] *type of T for each n. Then T has a countable weak model which omits each τ_n.*

[a] I.e. if $\exists x\varphi$ is consistent with T, then there is $\psi \in \tau_n$ such that $\exists x(\varphi \wedge \neg\psi)$ is.

Proof

- Adopt a set C of countably many new constants.
- Extend T step by step by Henkin axioms $\exists x\varphi(x) \to \varphi(c)$ to a consistent T_H.
- At the same time make sure that for each τ_n and for each $c \in C$ there is some $\varphi(x) \in \tau_n$ such that $\neg\varphi(c) \in T_H$.
- Then continue as before.

Definition

An elementary chain of weak models is a sequence $(\mathcal{M}_\alpha, q_\alpha)$, $\alpha < \gamma$, such that $(\mathcal{M}_\alpha, q_\alpha) \prec (\mathcal{M}_\beta, q_\beta)$ for all $\alpha < \beta < \gamma$. The union

$$\bigcup_{\alpha<\gamma} (\mathcal{M}_\alpha, q_\alpha)$$

of the chain is $(\mathcal{M}, q)$, where $\mathcal{M} = \bigcup_{\alpha<\gamma} \mathcal{M}_\alpha$ and

$$q = \{X \subseteq M : \text{there is } \delta < \gamma \text{ such that } X \cap M_\beta \in q_\beta \text{ for all } \delta < \beta < \gamma\}.$$

Lemma (Chain Lemma)

If $(\mathcal{M}, q) = \bigcup_\alpha (\mathcal{M}_\alpha, q_\alpha)$, then $(\mathcal{M}_\alpha, q_\alpha) \prec (\mathcal{M}, q)$ for all α.

Main Lemma

Lemma (The Main Lemma)

Suppose $(\mathcal{M}, q)$ *is a countable weak model of the axioms and* $\varphi(x)$ *is a formula in the vocabulary*[a] *of* $\mathcal{M}^*$ *such that*

$$(\mathcal{M}^*, q) \models Qx\varphi(x).$$

There is a countable elementary extension $(\mathcal{N}, r)$ *of* $(\mathcal{M}, q)$ *such that* $(\mathcal{N}^*, r) \models \varphi(a)$ *for some* $a \in N \setminus M$*, and moreover, if* $(\mathcal{M}^*, q) \models \neg Qx\psi(x)$*, where* $\psi(x)$ *is any formula, then* $(\mathcal{N}^*, r) \models \psi(a)$ *for no* $a \in N \setminus M$

[a] $\mathcal{M}^*$ is the result of adding in $\mathcal{M}$ a name for each element of M.

- Take a new constant c.
- Let T consist of the $L(Q)$-diagram[5] of $(\mathcal{M}^*, q)$ plus $\varphi(c)$ plus $\neg\psi(c)$ for all $\psi(x)$ such that $(\mathcal{M}^*, q) \models \neg Qx\psi(x)$.
- A nice criterion: A sentence $\theta(c)$ is consistent with T if and only if $(\mathcal{M}^*, q) \models Qx(\theta(x) \wedge \varphi(x))$. The proof of this uses Axiom (5), or rather the Example on slide 107. Hence T is consistent.
- Let $\psi_n(x)$ list all formulas such that $(\mathcal{M}^*, q) \models \neg Qx\psi_n(x)$. Let τ_n be the type $\{\psi_n(x)\} \cup \{\neg x = \underline{a} : a \in M\}$.
- The nice criterion above and Axiom (5) can be used to show that these are non-principal types of T.
- Hence by the Omitting Type Theorem there is a countable weak model $(\mathcal{N}, r)$ of T which omits all these. W.l.o.g. $(\mathcal{M}, q) \prec (\mathcal{N}, r)$.
- $(\mathcal{N}, r)$ is the desired countable weak model.

[5] All sentences true in the model.

Exact Extension Lemma

Lemma (Exact Extension Lemma)

Suppose $(\mathcal{M}, q)$ is a countable weak model of the axioms. There is a countable elementary extension $(\mathcal{N}, r)$ of $(\mathcal{M}, q)$ such that for any formula $\varphi(x)$, $(\mathcal{M}^*, q) \models Qx\varphi(x)$ if and only if $(\mathcal{N}^*, r) \models \varphi(a)$ for some $a \in N \setminus M$. Such an extension is called an exact extension.

Proof.

This follows by iterating the Main Lemma. □

Completeness Theorem

Theorem

If L is a countable vocabulary and T is a (finitely) consistent set of sentences of $L(Q_1)$, then T has a model.

Corollary

1. *If $\varphi \in L(Q_1)$, then $\models \varphi \iff \vdash \varphi$.*
2. *$L(Q_1)$ is countably compact.*

Proof.

By the Weak Completeness Theorem T has a countable weak model $(\mathcal{M}_0, q_0)$ in which all the axioms are true. By the Exact Extension Lemma, iterated ω_1 times, we obtain an elementary chain of weak countable models $(\mathcal{M}_\alpha, q_\alpha)$, taking unions at limits, such that

$$(\mathcal{M}^*_\alpha, q_\alpha) \models Qx\varphi(x) \iff$$

$$\text{For some } a \in M_{\alpha+1} \setminus M_\alpha, (\mathcal{M}_{\alpha+1}, q_{\alpha+1}) \models \varphi(a).$$

Let $(\mathcal{N}, r) = \bigcup_\alpha (\mathcal{M}_\alpha, q_\alpha)$. By the Chain Lemma, $(\mathcal{N}, r) \models T$. By induction, for all $b_1, \ldots, b_n \in N$,

$$(\mathcal{N}, r) \models \varphi(b_1, \ldots, b_n) \iff \mathcal{N} \models \varphi(b_1, \ldots, b_n))$$

□

Proposition

$h_{L(Q_1)} = \beth_\omega$

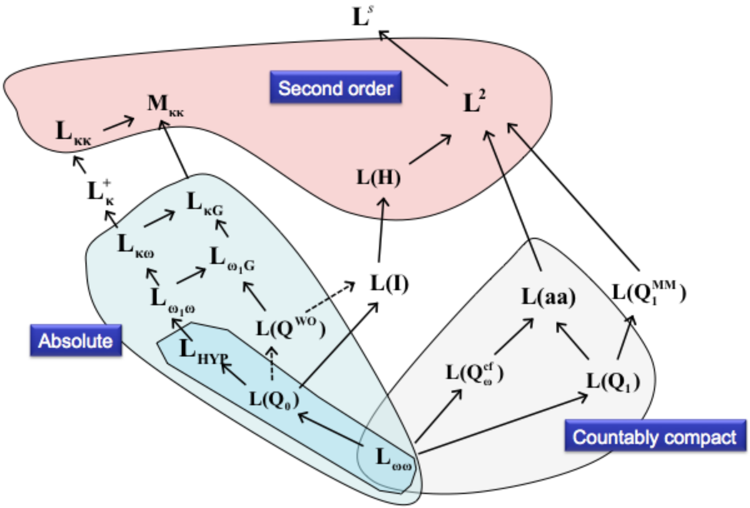
Proof.

For the moment, we only prove $h_{L(Q_1)} \geq \beth_\omega$:

- Let $c_0, \ldots, c_n$ be constant symbols, and R, E binary.
- Let φ say:
  1. $\neg Q_1 x R(c_0, x)$,
  2. $\forall x R(c_n, x)$,
  3. $\forall x \forall y (\forall z (E(z, x) \leftrightarrow E(z, y)) \rightarrow x = y)$,
  4. $\forall y (R(c_{i+1}, y) \rightarrow \forall z (E(z, y) \rightarrow R(c_i, y)))$ $(i = 0, \ldots, n-1)$
- φ has a model of cardinality $\beth_n$ but none bigger.

□

- $\mathrm{LST}(L(Q_1)) = \aleph_1$
- Conjecture: $\Delta(L(Q_1))$ has “no" effective syntax.


L^s
Second order
L^2
$M_{\kappa\kappa}$
$L_{\kappa\kappa}$
L(H)
L^+_κ
$L_{\kappa G}$
$L_{\kappa\omega}$
$L_{\omega_1 G}$
$L_{\omega_1\omega}$
L(I)
$L(Q^{WO})$
L(aa)
$L(Q_1^{MM})$
Absolute
L_{HYP}
$L(Q^{cf}_\omega)$
$L(Q_1)$
$L(Q_0)$
Countably compact
$L_{\omega\omega}$

Cofinality logic

Definition

1. The logic $L(Q^{cof}_\omega)$ has atomic formulas $x = y$ and $R(x_1, \ldots, x_n)$ if $R \in L$ of arity n. The logical operations are $\wedge, \neg, \exists$ and Q^{cof}_ω.
2. The satisfaction relation $\mathcal{M} \models_s \varphi$ for an L-structure $\mathcal{M}$, an assignment s of individual variables into M and set variables into $[M]^\omega$, a formula φ, is defined by induction on φ using:

$$Q^{cof}_\omega xy\varphi(x,y) \to \text{“}\varphi(x,y) \text{ is a linear order of cofinality } \omega\text{"}$$

Theorem

$L(Q^{cof}_\omega)$ satisfies $LST(\aleph_1)$.

Proof.

We are given a model $\mathcal{A}$. Let $\psi_n(x, \vec{z})$ list all $L(Q^{cof}_\omega)$-formulas with the free variables shown. If $\vec{a} \in A$ and $\mathcal{A} \models \exists x \psi_n(x, \vec{a})$, let $g_n(\vec{a})$ be an element b of A such that $\mathcal{A} \models \psi_n(b, \vec{a})$. Let $\varphi_n(x, y, \vec{z})$ list all $L(Q^{cof}_\omega)$-formulas with the free variables shown. If $\vec{a} \in A$ and $\mathcal{A} \models Q^{cof}_\omega xy\varphi_n(x, y, \vec{a})$, let $f_n(\vec{a})$ be a countable cofinal set in $\varphi_n(\cdot, \cdot, \vec{a})^{\mathcal{A}}$. If $\vec{a} \in A$, $Y \subseteq A$ is countable, $\mathcal{A} \models \neg Q^{cof}_\omega xy\varphi_n(x, y, \vec{a})$, $\varphi_n(\cdot, \cdot, \vec{a})^{\mathcal{A}}$ is a linear order without a last element, then let $h_n(Y, \vec{a})$ be an element of A which is an upper bound for Y in $\varphi_n(\cdot, \cdot, \vec{a})^{\mathcal{A}}$. Take any countable subset X of A. Let $B_0 = X$. If B_α is defined let $B_{\alpha+1}$ be a countable subset of A, containing each $h_n(B_\alpha, \vec{a})$, where $n < \omega$ and $\vec{a} \in B_\alpha^{<\omega}$, and closed under the functions g_n, f_n for all n. At limits we take unions. Now B_{ω_1} is an $L(Q^{cof}_\omega)$-elementary submodel of $\mathcal{A}$ of cardinality $\leq \omega_1$. Details on the blackboard. □

Axioms

Definition

1. The axioms of first order logic.
2. $Q^{cof}_{\omega} xy\varphi(x,y) \to$ "$\varphi(x,y)$ is a linear order without last element"
3. $\neg(Q^{cof}_{\omega} xy\psi(x,y) \wedge \neg Q^{cof}_{\omega} xy\varphi(x,y) \wedge$ "$\varphi(x,y)$ is a linear order without last element" $\wedge \forall x(\exists y\varphi(x,y) \to \exists y\exists v(\theta(v,y) \wedge \varphi(x,y))) \wedge \forall w(\exists v\psi(w,v) \to \exists x\forall y\forall v((\varphi(x,y) \wedge \theta(v,y)) \to \psi(w,v))))$.
4. Rules: Modus Ponens and Generalization.

Lemma

These axioms are valid.

Weak model

Definition

A weak model is a pair $(\mathcal{M}, q)$, where $\mathcal{M}$ is an ordinary model and $q \subseteq \mathcal{P}(M \times M)$. Then we define

$$(\mathcal{M}, q) \models Qxy\varphi(x, y) \Leftrightarrow_{def} \{(a, b) \in M \times M : (\mathcal{M}, q) \models \varphi(a, b)\} \in q.$$

The theory of weak models can be developed as for $L(Q_1)$, including compactness, omitting types theorems and Chain Lemma.

Idea of a proof of the Completeness Theorem

- Suppose the vocabulary has cardinality κ. W.l.o.g. $\kappa \geq \omega$. Suppose T is a consistent theory.
- We construct a model of T as the union $\mathcal{M}$ of an elementary chain of weak models $\mathcal{M}_\alpha$ of cardinality κ, $\alpha < \kappa^+$.
- If T thinks $\varphi(x, y, \vec{a})$, $\vec{a} \in M_\alpha$, defines a linear order R without last element, we make sure $\mathcal{M}_{\alpha+1}$ adds an element to R after all the elements of R in $\mathcal{M}_\alpha$.
- After $\mathcal{M}$ is constructed, we define a new elementary sequence $\mathcal{N}_n$, $n < \omega$, with $\mathcal{N}_0 = \mathcal{M}$.
- If T thinks $\varphi(x, y, \vec{a})$, $\vec{a} \in N_n$, defines a linear order R without last element and $Q^{cof}_\omega xy\varphi(x, y, \vec{a})$, and T also thinks $\psi(x, y, \vec{b})$, $\vec{b} \in N_n$, defines a linear order S without last element and $\neg Q^{cof}_\omega xy\psi(x, y, \vec{b})$, we make sure $\mathcal{N}_{n+1}$ adds an element to R after all the elements of R in $\mathcal{N}_n$ without adding any elements to S after all the elements of N_n.
- $\mathcal{N} = \bigcup_n \mathcal{N}_n$ is the model of T we desire.

Theorem (Omitting Types Theorem)

Suppose κ is an infinite cardinal, L is a vocabulary of cardinality $\leq \kappa$, T is a consistent first-order L-theory, and for each $\xi < \kappa$, Γ_ξ is a set of formulas of vocabulary L such that if $\Sigma(x)$ is any set of formulas $\psi(x)$ such that $|\Sigma(x)| < \kappa$ and $T \cup \Sigma(x)$ is consistent, then $T \cup \Sigma(x) \cup \{\neg\varphi(x)\}$ is consistent for some $\varphi(x) \in \Gamma_\xi$. Then there is a model $\mathcal{M}$ of T of cardinality $\leq \kappa$ which omits each Γ_ξ, $\xi < \kappa$, i.e. no element $a \in M$ satisfies in $\mathcal{M}$ all formulas of Γ_ξ.

Proof.

The usual proof works, as we see below: □

Let $\Gamma_\xi = \{\varphi^\xi_\alpha(x) : \alpha < \kappa\}$. We construct a sequence T_α, $\alpha < \kappa$, of consistent theories such that the canonical model of $\bigcup_{\alpha<\kappa} T_\alpha$ is the desired model which omits each Γ_ξ. We keep deciding sentences and adding Henkin axioms by means of a set C of constant symbols, $|C| = \kappa$. Suppose we have constructed T_α. We want $T_{\alpha+1}$ to contain $\neg\varphi^\xi_\gamma(c_\alpha)$ for some γ. We have $T_\alpha = T \cup S$, where $|S| < \kappa$. Suppose $T \cup S \cup \{\neg\varphi^\xi_\gamma(c_\alpha)\}$ is inconsistent for all $\gamma < \kappa$. Then for all γ there is a finite $S_\gamma \subseteq S$ such that $T \cup S_\gamma \cup \{\neg\varphi^\xi_\gamma(c_\alpha)\}$ is inconsistent. Let $\psi_\gamma(c_\alpha)$ be the conjunction of S_γ with all other C-constants changed to existentially quantified variables except c_α. Thus $T \cup \{\psi_\gamma(c_\alpha)\} \cup \{\neg\varphi^\xi_\gamma(c_\alpha)\}$ is inconsistent. Let $\Sigma(x) = \{\psi_\gamma(x) : \gamma < \kappa\}$. Note that $|Sigma(x)| < \kappa$. Since $T \cup \Sigma(x)$ is consistent, there is some $\varphi^\xi_\gamma(x)$ such that $T \cup \Sigma_\gamma(x) \cup \{\neg\varphi^\xi_\gamma(x)\}$ is consistent. This contradicts the fact that $T \cup \{\psi_\gamma(c_\alpha)\} \cup \{\neg\varphi^\xi_\gamma(c_\alpha)\}$ is inconsistent.

Not so great!

The problem with this Omitting Types Theorem is that it is rather difficult to satisfy the assumption on Γ_ξ, involving as it does infinite sets of formulas $\Sigma(x)$, seriously limiting the applicability of the result. In particular, Γ_ξ cannot be countable unless $\kappa = \aleph_0$, for else we could take $\Sigma(x) = \Gamma_\xi$.

Theorem (Chang)

Assume κ is an infinite cardinal. Let L be a vocabulary of cardinality $\leq \kappa$, T an L-theory, and for each $\xi < \kappa$, Γ_ξ is a set $\{\varphi^\xi_\alpha(x) : \alpha < \kappa\}$ of formulas in the vocabulary L. Assume that

1. *If $\alpha \leq \beta < \kappa$, then $T \vdash \varphi^\xi_\beta(x) \to \varphi^\xi_\alpha(x)$.*
2. *For every L-formula $\psi(x)$, for which $T \cup \{\psi(x)\}$ is consistent, and for every $\xi < \kappa$, there is an $\alpha < \kappa$ such that $T \cup \{\psi(x)\} \cup \{\neg\varphi^\xi_\alpha(x)\}$ is consistent.*

Then T has a model which omits each Γ_ξ.

Note: If $\varphi^\xi_\alpha(x)$ says $d_\alpha < x$ and T says that $<$ is a linear order with $d_\eta < d_\theta$ for all $\eta < \theta < \kappa$, then $\{\varphi^\xi_\alpha(x) : \alpha < \kappa\}$ satisfies (1) above.

Proof.

We show that the assumption of Theorem on slide 128 is valid. Suppose therefore that $\Sigma(x)$ is a set of formulas such that $|\Sigma(x)| < \kappa$ and $T \cup \Sigma(x)$ is consistent. We claim that $T \cup \Sigma(x) \cup \{\neg\varphi^{\xi}_{\alpha}(s)\}$ is consistent for some $\alpha < \kappa$. Otherwise there is for every $\alpha < \kappa$ some finite $\Sigma_{\alpha}(x) \subseteq \Sigma(x)$ such that $T \cup \Sigma_{\alpha}(x) \cup \{\neg\varphi^{\xi}_{\alpha}(s)\}$ has no models. Since $|\Sigma(x)| < \kappa$, there is a fixed finite $\Sigma^*(x) \subseteq \Sigma(x)$ such that $T \cup \Sigma^*(x) \cup \{\neg\varphi^{\xi}_{\alpha}(s)\}$ has no models for κ different α. Because of our assumption 1 above, $T \cup \Sigma^*(x) \cup \{\neg\varphi^{\xi}_{\alpha}(s)\}$ has no models for any $\alpha < \kappa$ whatsoever. Let $\psi(x) = \bigwedge \Sigma^*(x)$. Now ψ contradicts assumption 2 above. □

Compactness Theorem

Theorem (Shelah)

The logic $L(Q^{cof}_\omega)$ *satisfies the Completeness Theorem in arbitrary (even uncountable) vocabulary.*

Proof.

Suppose T is a consistent theory in $L(Q^{cof}_\omega)$. We use the idea presented above. Extending $\mathcal{N}_n$ to $\mathcal{N}_{n+1}$ is possible by means of the above omitting types theorem of Chang. Note that the type of an element extending a linear order satisfies condition (1) of Chang's theorem. □

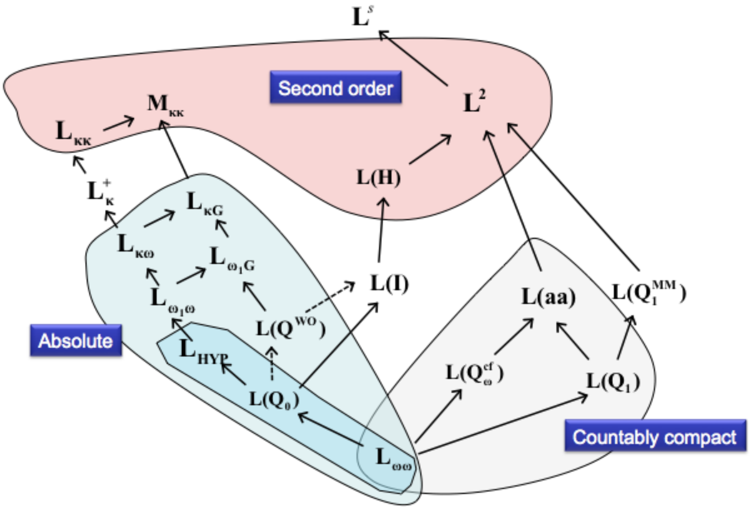
Corollary

$L(Q^{cof}_\omega)$ *is a compact logic.*

- $h_{L(Q_\omega^{cof})} = \aleph_0$ (by compactness!)
- $\Delta(L(Q_\omega^{cof})) \neq L(Q_\omega^{cof})$, but $Craig(L(Q_\omega^{cof}), L(aa))$ (Shelah).

Problem session 5

1. A model $\mathcal{A}$ with a unary predicate symbol P in its vocabulary, which is assumed to be of cardinality $\leq \lambda$, is said to be **of type** (κ, λ) if $|A| = \kappa$ and $|P^{\mathcal{A}}| = \lambda$. Show that every model of type (κ, λ) has for every κ' such that $\lambda \leq \kappa' \leq \kappa$ an ($L_{\omega\omega}$-) elementary submodel of type (κ', λ). [Don't be surprised if you find this problem almost too easy!]
2. Let Q_α be the generalized quantifier $Q_\alpha x\varphi(x, \vec{a}) \iff |\varphi(\cdot, \vec{a})| \geq \aleph_\alpha$. Let T be the theory $\{P(c_\beta) : \beta < \aleph_\alpha\} \cup \{\neg c_\beta = c_\gamma : \beta < \gamma < \aleph_\alpha\}$. Show that for every first order sentence φ with vocabulary τ there is an $L(Q_{\alpha+1})$-sentence φ' of vocabulary $\tau \cup \{P\}$ such that models of φ of type $(\aleph_{\alpha+1}, \aleph_\alpha)$ are exactly the reducts of models of $T \cup \{\varphi'\}$ of cardinality $\aleph_{\alpha+1}$ to the vocabulary τ. [Don't be surprised if you find this problem extremely easy!]
3. Show that for every vocabulary τ there is a vocabulary $\tau^* \supseteq \tau$ of cardinality $\aleph_0 \cdot |\tau|$ and a *first order* theory T in the vocabulary $\tau^* \cup P$, $P \notin \tau^*$, such that for every $L(Q_{\alpha+1})$-sentence φ of vocabulary τ there is a first order sentence φ^* with vocabulary $\tau^* \cup \{P\}$ such that φ has a model of cardinality $\aleph_{\alpha+1}$ if and only if the first order theory $T \cup \varphi^*$ has a model of type $(\aleph_{\alpha+1}, \aleph_\alpha)$. [To eliminate the quantifier $Q_{\alpha+1}$, use $|P| \leq \aleph_\alpha$, and use T to asset that there are enough definable mappings between definable subsets and either the predicate P or the whole domain of the model.]
4. Show that the following are equivalent:
   (A) If every finite subset of a first order theory T, $|T| \leq \aleph_\alpha$, has a model of type $(\aleph_{\alpha+1}, \aleph_\alpha)$, then the whole theory has such a model.
   (B) Suppose $T \subseteq L(Q_{\alpha+1})$ is of cardinality $\leq \aleph_\alpha$. If every finite subset of T has a model of size $\aleph_{\alpha+1}$, then so does T.
5. Let us say κ is **small for** $\aleph_\alpha$ if $\prod_{i\in I} \lambda_i < \aleph_\alpha$ whenever $|I| \leq \kappa$ and $|\lambda_i| < \aleph_\alpha$ for all $i \in I$. Show that if κ is small for $\aleph_\alpha$, then $L(Q_\alpha)$ is κ-compact. [Hint: Prove first Łoś Lemma for ultraproducts of κ models. Then use the proof of the Compactness Theorem of first order logic on slide 95.]


L^s
Second order
L^2
M_κκ
L_κκ
L_κ^+
L(H)
L_κG
L_κω
L_ω1G
L(I)
L(aa)
L(Q_1^MM)
L_ω1ω
L(Q^WO)
Absolute
L_HYP
L(Q_ω^cf)
L(Q_1)
L(Q_0)
Countably compact
L_ωω

Stationary logic $L(\mathtt{aa})$

- Like second order logic but second order variables s only for countable subsets of the domain.
- No second order existential or universal quantifiers, but instead $\mathtt{aa}\boldsymbol{s}\varphi$ (club many s satisfies φ) and $\mathtt{stat}\,\boldsymbol{s}\varphi$ (stationary many s satisfies φ).
- They play the role of second order existential and universal quantifier.
- $\mathcal{M} \models \mathtt{aa}\boldsymbol{s}\varphi(\boldsymbol{s}) \iff \{A \in [M]^\omega : \mathcal{M} \models \varphi(A)\}$ contains a club.
- $\mathtt{aa}\boldsymbol{s}\varphi \equiv \neg\mathtt{stat}\,\boldsymbol{s}\neg\varphi$.
- $\mathtt{stat}\,\boldsymbol{s}\varphi \equiv \neg\mathtt{aa}\boldsymbol{s}\neg\varphi$.

- Diagonal intersection of a family $\{C_a : a \in M\}$ of clubs is

$$\Delta_{a\in M} = \{s : \forall a \in s(s \in C_a)\},$$

which is also club
- Fodor Lemma: If f is a function such that for a stationary set S of s we have $f(s) \in s$, then there is a stationary $S' \subseteq S$ such that $f \restriction S'$ is constant.

Proofs on the blackboard!

Example

1. $Q_1 x\varphi(x) \equiv \neg \mathtt{aa}\, s \forall x(\varphi(x) \to s(x))$
2. $Q^{cof}_{\omega} xy\varphi(x,y) \equiv \mathtt{aa}\, s \forall x(\exists y \varphi(x,y) \to \exists y(s(y) \wedge \varphi(x,y))) \wedge \varphi(\cdot,\cdot)$ is a linear order without last element.
3. A definable set $C \subseteq \omega_1$ contains a club iff $(\omega_1, <)$ satisfies $\mathtt{aa}\, s(\sup(s) \in C)$.
4. A definable set $C \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is stationary iff $(\kappa, <)$ satisfies $\mathtt{stat}\, s(\sup(s) \in C)$.

Theorem

1. *If* $L(\mathtt{aa})$ *satisfies* $\mathrm{LST}(\aleph_1)$ *then every stationary* $A \subseteq \omega_2$ *of ordinals of cofinality* ω *reflects i.e.* $A \cap \alpha$ *is stationary for some* $\alpha < \omega_2$ *of cofinality* ω_1*. (Proof on the next slide.) Note that this implies* $V \neq L$ *(Jensen).*
2. *If* κ *is supercompact, then* $L(\mathtt{aa})$ *satisfies* $\mathrm{LST}(\kappa)$*. (Proof on a slide below.)*
3. *Assuming the consistency of a supercompact, it is consistent that* $L(\mathtt{aa})$ *satisfies* $\mathrm{LST}(\aleph_1)$*. (Ben-David) (Proof omitted).*
4. *Assuming the consistency of a supercompact, it is consistent that the first* κ *for which* $L(\mathtt{aa})$ *satisfies* $\mathrm{LST}(\kappa)$ *is the first supercompact. (Magidor) (Proof omitted).*

Suppose $S \subseteq \omega_2$ is stationary. We show that the set S' of countable $s \subseteq \omega_2$ such that $sup(s) \in S$ is stationary. Let C be a club of countable sets $s \subseteq \omega_2$. There is a function $f : \omega_2^{<\omega} \to \omega_2$ such that the the club C^* of s which are closed under f is contained in C. Let D be the set of $\alpha < \omega_2$ which are closed under f. Clearly D is club, whence there is $\delta \in D \cap S$. Since δ is closed under f, there is an unbounded $s \subseteq \delta$ such that $s \in C^*$. Thus $s \in C \cap S'$. Let $\mathcal{A} = (\omega_2, <, S)$. $\mathcal{A}$ satisfies the $L(\mathtt{aa})$-sentences φ which says that the cofinality of the model is uncountable and that there is a stationary set of countable s such that $sup(s) \in S$. If $LST(\aleph_1)$ holds for $L(\mathtt{aa})$, then there is $\mathcal{B} \subseteq \mathcal{A}$ such that $|B| \leq \aleph_1$ and $\mathcal{B} \models \varphi$. Let δ be the sup of B. There is a stationary set of countable s in $[B]^{\leq\omega}$ such that $sup(s) \in S$. It follows that $S \cap \delta$ is stationary

- Suppose $\mathcal{A}$ is a model of cardinality λ and $\mathcal{A} \models \varphi$, where $\varphi \in L(\mathtt{aa})$.
- Let $i : V \to M$ such that ${}^{\lambda^\omega}M \subseteq M$. Let $\mathcal{B}$ be the structure $i''\mathcal{A} \in M$.
- Claim: For all φ: $M \models$ "$\mathcal{B} \models \varphi(\vec{s}, \vec{a})$" $\iff$ $V \models$ "$\mathcal{B} \models \varphi(\vec{s}, \vec{a})$".
- Suppose $M \models$ "$\mathcal{B} \models \mathtt{aa}s\varphi(s, \vec{s'}, \vec{a})$". Let $C \in M$ be a club (in M) of countable (in M) subsets s of B such that $M \models \varphi(s, \vec{s'}, \vec{a})$. Then C is in V a club of countable subsets s such that (by Ind. Hyp.) $V \models$ "$\mathcal{B} \models \varphi(s, \vec{s'}, \vec{a})$". Conversely, suppose such a C exists in V. Since M is closed under countable subsets and $|C| \leq \lambda^\omega$, $C \in M$. Hence $M \models$ "$\mathcal{B} \models \mathtt{aa}s\varphi(s, \vec{s'}, \vec{a})$".
- Note that $M \models \mathcal{B} \subseteq i(\mathcal{A})$ and $V \models \mathcal{B} \cong \mathcal{A}$.
- Since $\mathcal{A} \models \varphi$, we have $\mathcal{B} \models \varphi$. By the above, $M \models$ "$\mathcal{B} \models \varphi$".
- Hence $M \models$ "$i(\mathcal{A})$ has a submodel $\mathcal{B}$ of cardinality $< i(\kappa)$ satisfying ψ". So $\mathcal{A}$ has a submodel of cardinality $< \kappa$ satisfying ψ.

Definition

The principle $\square_{\omega_1}$ says that there are sets C_α, $\alpha = \cup\alpha < \omega_2$ such that

1. Each C_α is club on α.
2. If $cof(\alpha) = \omega$, then $|C_\alpha| = \omega$.
3. If β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.

Lemma

Assume $\square_{\omega_1}$. Then there is a non-reflecting set $A \subseteq \omega_2$.

Proof.

Let S be the set of ordinals in ω_2 of cofinality ω. Let E_β be the set of $\alpha = \cup\alpha < \omega_2$ with C_α of order type β. Clearly $S = \bigcup_{\beta<\omega_2} E_\beta$. Hence there is β such that E_β is stationary. But E_β cannot be reflecting: Suppose α has cofinality ω_1 and $E_\beta \cap \alpha$ is a stationary subset of α. Suppose $\gamma \in E_\beta \cap lim(C_\alpha)$. Then the order type of C_γ is β. So there can be only one such γ, and $E_\beta \cap \alpha$ cannot be stationary. □

Axioms

Definition

1. $\mathtt{aa}\boldsymbol{s}\ s(x)$
2. $\mathtt{aa}\boldsymbol{s}'\forall x(s(x) \to s'(x))$
3. $(\mathtt{aa}\boldsymbol{s}\varphi \wedge \mathtt{aa}\boldsymbol{s}\psi) \to \mathtt{aa}\boldsymbol{s}(\varphi \wedge \psi)$
4. $\mathtt{aa}\boldsymbol{s}(\varphi \to \psi) \to (\mathtt{aa}\boldsymbol{s}\varphi \to \mathtt{aa}\boldsymbol{s}\psi)$
5. $\forall x \mathtt{aa}\boldsymbol{s}\varphi \to \mathtt{aa}\boldsymbol{s}\forall x(s(x) \to \varphi)$ (Fodor Lemma, or diagonal intersection axiom)

Lemma (Exercise)

The axioms are valid.

Completeness Theorem

Theorem

If L is a countable vocabulary and T is a finitely consistent set of sentences of stationary logic, then T has a model (of cardinality $\aleph_1$).

Corollary

Stationary logic is countably compact.

Idea of proof

- We construct a model for a consistent theory T as the union $\mathcal{M}$ of an elementary chain of countable models $(\mathcal{M}_\alpha)$, $\alpha < \omega_1$.
- If T thinks $\mathtt{aa}\, s\varphi(s)$, we make sure $\mathcal{M} \models \varphi(M_\alpha)$ for all α.
- If T thinks $\mathtt{stat}\, s\varphi(s)$, we make sure $\mathcal{M} \models \varphi(M_\alpha)$ for a stationary set of α.
- In the beginning we split ω_1 into $\aleph_1$ disjoint stationary sets X_α. We take $\aleph_1$ new constant symbols. We also list all formulas in a list φ_α, $\alpha < \omega_1$.
- If $\xi \in X_\alpha$ and T thinks $\mathtt{stat}\, s\varphi_\alpha(s)$, then we make sure $\mathcal{M} \models \varphi_\alpha(M_\xi)$. This guarantees $\mathcal{M} \models \mathtt{stat}\, s\varphi_\alpha(s)$.

Kaufmann-Shelah: (Proof omitted).

1. $h_{L(\mathtt{aa})} = \beth_{h_{L(\mathtt{aa})}}$
2. $V = L$ implies $h_{L(\mathtt{aa})} = h_{L^2}$.

Problem session 6

1. Prove that the axioms of $L(Q^{cof}_{\omega})$ are valid.
2. Prove $LST(\aleph_2)$ for $L(Q^{cof}_{\omega}, Q^{cof}_{\omega_1})$ i.e. for the extension of first order logic by the two generalized quantifiers Q^{cof}_{ω} and $Q^{cof}_{\omega_1}$, where $Q^{cof}_{\omega_1} xy\varphi(x, y, \vec{a})$ holds iff $\varphi(\cdot, \cdot, \vec{a})$ is a linear order of cofinality ω_1. [Hint: Imitate the proof of $LST(\aleph_1)$ for $L(Q^{cof}_{\omega})$.]
3. Prove that the axioms of stationary logic are valid.
4. Show that if $A \subseteq \omega_1$ is stationary, then for all $\alpha < \omega_1$ there is a closed set $B \subseteq A$ of order-type α. [Hint: Prove by induction on α the stronger claim that the given claim holds above every countable ordinal.]
5. Show that there is a stationary $A \subseteq \omega_1$ which has a stationary complement. [Hint: Let for each limit $\delta < \omega_1$ an increasing ω-sequence (δ_n) be chosen. Consider the regressive functions $\delta \to \delta_n$ for various n.]

Many-sorted logic MSL

- several “sorts" of variables
- vector spaces: scalars and vectors
- geometry: points and lines
- second order logic: individuals and subsets.
- elements and sequences of elements

Definition

A many-sorted structure (model)

$$\mathcal{M} = (\{M_1, \ldots, M_n\}, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l)$$

consists of

1. Universes (“sorts") $M_1, \ldots, M_n$, each $\neq \emptyset$
2. Relations $R_1, \ldots, R_m$ between elements of the universes
3. Functions $f_1, \ldots, f_k$ between elements of the universes
4. Distinguished constants $c_1, \ldots, c_l$ in the universes.

We call $\mathcal{M}$ *n*-sorted. $\mathcal{M}$ is “strict" if the M_i are disjoint, otherwise “lax". We allow $n = 0$, i.e. “no sorts" as a special case.

where ...

$$R_i \subseteq M_{i_1} \times ... \times M_{i_s} \qquad \mathfrak{s}(R_i) = \langle i_1, ..., i_s \rangle \in \{1, ..., n\}^s$$

$$f_i : M_{i_1} \times ... \times M_{i_s} \to M_r \quad \mathfrak{s}(f_i) = \langle i_1, ..., i_s, r \rangle \in \{1, ..., n\}^{s+1}$$

$$c_i \in M_j \qquad \mathfrak{s}(c_i) = j \in \{1, ..., n\}$$

Restriction: we do not allow relations between elements of sort s or sort s' and elements of sort s''.

Vector spaces

Example

A vector space is a strict 2-sorted structure

$$\mathcal{V} = (\{F, V\}, +_F, \cdot_F, 0_F, 1_F, +_V, \cdot_V, 0_V),$$

where

1. $(F, +_F, \cdot_F, 0_F, 1_F)$ is a field
2. $(V, +_V, 0_V)$ is an Abelian group
3. $\cdot_V : F \times V \to V$ satisfies
   - $a \cdot_V (u +_V w) = a \cdot_V u +_V a \cdot_V w$
   - $(a \cdot_F b) \cdot_V u = a \cdot_V (b \cdot_V u)$
   - $(a +_F b) \cdot_V u = a \cdot_V u +_V b \cdot_V u$
   - $1_F \cdot_V u = u$

Vector spaces (textbook notation)

Example

A vector space is a strict 2-sorted structure

$$\mathcal{V} = (F, V, +, \cdot, 0),$$

where

1. F is a field
2. $(V, +, 0)$ is an Abelian group
3. The function $(a, v) \mapsto av (= a \cdot v) : F \times V \to V$ satisfies
   - $a(u + w) = au + aw$
   - $(ab) \cdot u = a(bu)$
   - $(a + b)u = au + bu$
   - $1u = u$

Definition

A many-sorted structure

$$\mathcal{M} = (\{M_1, \ldots, M_n\}, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l)$$

is isomorphic to

$$\mathcal{M}' = (\{M'_1, \ldots, M'_n\}, R'_1, \ldots, R'_m, f'_1, \ldots, f'_k, c'_1, \ldots, c'_l)$$

if there is a bijection $\pi : \bigcup_s M_s \to \bigcup_s M'_s$ such that

1. $\pi \restriction M_i : M_i \to M'_i$ is bijection for $i = 1, ..., n$
2. $R_i(a_1, ..., a_s) \iff R'_i(\pi(a_1), ..., \pi(a_s))$
3. $\pi f_i(a_1, ..., a_s) = f'_i(\pi(a_1), ..., \pi(a_s))$
4. $\pi c_i = c'_i$

Definition

A reduct of a many-sorted structure

$$\mathcal{M} = (\{M_1, \ldots, M_n\}, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l)$$

is obtained by leaving out some sorts, relations, functions, and constants, as in

$$\mathcal{M}' = (\{M_1, \ldots, M_{n-1}\}, R_1, \ldots, R_{m'}, f_1, \ldots, f_{k'}, c_1, \ldots, c_{l'}).$$

Relations, functions, and constants which are not meaningful are at the same time dropped. Respectively then, $\mathcal{M}$ is an expansion of $\mathcal{M}'$. A reduct may also have the same sorts but fewer relations, etc.

Example

Reducts of a vector space $(F, V, +, \cdot, 0)$:

- The scalar field: $F = (F, +_F . \cdot_F, 0_F, 1_F)$
- The vector addition group: $(V, +, 0)$

Definition

The vocabulary $L = \mathrm{voc}(\mathcal{M})$ of a many-sorted structure (model)

$$\mathcal{M} = (\{M_1, \ldots, M_n\}, R_1^{\mathcal{M}}, \ldots, R_m^{\mathcal{M}}, F_1^{\mathcal{M}}, \ldots, F_k^{\mathcal{M}}, c_1^{\mathcal{M}}, \ldots, c_l^{\mathcal{M}})$$

consists of

1. Sort symbols $s_1, \ldots, s_n$
2. Relation symbols $R_1, \ldots, R_m$
3. Function symbols $F_1, \ldots, F_k$
4. Constant symbols $c_1, \ldots, c_l$

We write $L = \{s_1, \ldots, s_n, R_1, \ldots, R_m, F_1, \ldots, F_k, c_1, \ldots, c_l\}$, $\mathrm{sort}(L) = \{s_1, \ldots, s_n\}$, $\mathrm{rel}(L) = \{R_1, \ldots, R_m\}$, $\mathrm{fun}(L) = \{F_1, \ldots, F_k\}$, $\mathrm{con}(L) = \{c_1, \ldots, c_l\}$.

Definition

Each vocabulary L has an *arity-function*

$$\mathfrak{a}_L : \mathrm{rel}(L) \cup \mathrm{fun}(L) \to \mathbb{N}$$

which tells the arity of each predicate and function symbol, and a *sort-function* $\mathfrak{s}_L$:

$$\mathfrak{s}_L(R) \in \mathrm{sort}(L)^{\mathfrak{a}_L(R)}, \mathfrak{s}_L(f) \in \mathrm{sort}(L)^{\mathfrak{a}_L(f)} \times \mathrm{sort}(L), \mathfrak{s}_L(c) \in \mathrm{sort}(L).$$

We do not have symbols for abstract relations between elements of arbitrary sorts except identity $=$ in lax structures.

Definition

Suppose L is a many sorted vocabulary. The L-variables over many-sorted L-structures are denoted:

$$x^s, s \in \mathrm{sort}(L)$$

with the intuition (fulfilled in the semantics below) that x^s ranges over the universe M_s of an L-structure

$$\mathcal{M} = (\{M_1, \ldots, M_n\}, R_1^{\mathcal{M}}, \ldots, R_m^{\mathcal{M}}, F_1^{\mathcal{M}}, \ldots, F_k^{\mathcal{M}}, c_1^{\mathcal{M}}, \ldots, c_l^{\mathcal{M}}).$$

No variables ranging over “everything"! No variables ranging over elements of two different sort s (but this can be arranged by having a new sort for the “union").

Definition

Suppose L is a many sorted vocabulary. The L-terms are defined as follows:

1. Constants c in L are L-terms and $\mathfrak{s}(c)$ is already defined.
2. L-variables x^s are L-terms and $\mathfrak{s}(x^s) = s$.
3. If $t_1, ..., t_n$ are L-terms with $s_i = \mathfrak{s}(t_i)$ and $f \in L$ with $\mathfrak{s}(f) = \langle s_1, \ldots, s_n, s \rangle$, then $ft_1...t_n$ (or $f(t_1, ..., t_n)$) is an L-term and $\mathfrak{s}(ft_1...t_n) = s$.

Syntax

$$t = t'$$
$$Rt_1 \ldots t_n$$
$$\neg\varphi$$
$$\varphi \wedge \psi$$
$$\varphi \vee \psi$$
$$\exists x^s \varphi$$
$$\forall x^s \varphi$$

- Strict many-sorted logic: we allow $t = t'$ only if $\mathfrak{s}(t) = \mathfrak{s}(t')$.
- Lax many-sorted logic: we allow $t = t'$ for all terms.

Semantics

Assignments v into $\mathcal{M}$ are functions on $\mathrm{voc}(L)$-variables such that $v(x^s) \in M_s$ for each $s \in \mathrm{sort}(\mathcal{M})$. Modified assignment $v(a/x^s)$ as usual.

Definition

$\mathcal{M} \models_v \exists x^s \varphi \iff \mathcal{M} \models_{v(a/x^s)} \varphi$ for some $a \in M_s$.
$\mathcal{M} \models_v \forall x^s \varphi \iff \mathcal{M} \models_{v(a/x^s)} \varphi$ for all $a \in M_s$.

Absoluteness and truth

- What is truth?
- Given a sentence φ, what does the truth of φ mean?
- Given a sentence φ and a model $\mathfrak{A}$, what does the truth of $\mathfrak{A} \models \varphi$ mean?
- Given a sentence φ, a model $\mathfrak{A}$ and a model M of set theory such that $\varphi, \mathfrak{A} \in M$, what does the truth of $M \models$ "$\mathfrak{A} \models \varphi$" mean?
- Given a sentence φ, a model $\mathfrak{A}$, a model M of set theory such that $\varphi, \mathfrak{A} \in M$, and a model of set theory $\mathfrak{A}$ such that $\varphi, \mathfrak{A}, M \in \mathfrak{A}$ what does the truth of $M \models$ "$\mathfrak{A} \models \varphi$" in $\mathfrak{A}$ mean?
- Etc, etc, ad infinitum ("infinite regress").
- Absoluteness helps us break out. Absoluteness means that we get the same answer in different levels of metatheory. For absolute properties the difference between metatheory and object theory dissolves.

Definition

A formula of set theory is bounded if all of its quantifiers are bounded, i.e. of the form $\forall x \in y$ or $\exists x \in y$.

Definition

A class C of set theory is Σ_1 (-definable) if there is a bounded formula $\psi(x, y)$ such that for all a:

$$a \in C \iff \exists y \psi(\vec{s}, y).$$

We say that a formula $\varphi(\vec{x})$ is Σ_1^V (sometimes we omit "V") if the class it defines is. If the equivalence is provable in a theory T, then $\varphi(\vec{x})$ is said to be Σ_1^T. Similarly Π_1^V, Π_1^T, Δ_1^V, and Δ_1^T.

Definition

A model (M', E'), $E' \subseteq M'^2$, is an end-extension of (M, E), $E \subseteq M^2$, in symbols $(M, E) \subseteq_{end} (M', E')$, if $(M, E) \subseteq (M', E')$ and

$$\forall x \in M \forall x' \in M'(x' E' x \to x' \in M).$$

Remark

A special case is a transitive set which is any set M such that $(M, \in) \subseteq_{end} (V, \in)$.

Transitive closure

Lemma

Every set x is element in a smallest transitive set, called the transitive closure of x, in symbols $TC(x)$.

Proof.

Let $TC_0(x) = \{x\}$, $TC_{n+1}(x) = TC_n(x) \cup \{y : \exists x \in TC_n(x)(y \in x)\}$, $TC(x) = \bigcup_n TC_n(x)$. □

Definition (Recap)

The set of sets of hereditary cardinality $< \kappa$, $H(\kappa)$, is the set of sets x such that $|TC(x)| < \kappa$.

Remark

It is easy to see that for regular κ, $(H(\kappa), \in) \models ZFC^-$.

Theorem (Levy Reflection Prinicple)

If κ is regular, then $(H(\kappa), \in) \prec_1 (V, \in)$ i.e. if $\varphi(x)$ is $\Sigma_1^{ZFC^-}$ and $a \in H(\kappa)$, then

$$H(\kappa) \models \varphi(a) \iff V \models \varphi(a).$$

Proof.

To be given on the blackboard. □

Lemma (Mostowski's Collapsing Lemma, recap)

If $\mathcal{M}$ is a model in which $(M_s, E) \models \text{Axiom of Extensionality}$ and (M_s, E) is well-founded, then there is $\mathcal{M}' \cong \mathcal{M}$ such that M'_s is a transitive set and E' is the membership relation ($\in$) restricted to M'_s.

Proof.

Define for $x \in M_s$:

$$\pi(x) = \{\pi(y) : y \in M_s, yEx\}.$$

Since E is well-founded, this is a legitimate definition by transfinite recursion. The function π is one-one, because (M, E) satisfies the Axiom of Extensionality. Let $\mathcal{M}' \cong \mathcal{M}$ so that $\pi : (M_s, E) \cong (M'_s, \in)$. □

From relative truth to truth via absoluteness.

Lemma

If a formula $\varphi(x)$ of the language $\{\in\}$ (sort s) of set theory is Σ_1 in sort s and

$$\mathcal{M} \models \varphi(n),$$

where $\mathcal{M}$ is such that $(M_s, E) \models ZFC_n$ (a large finite fragment of ZFC) and (M_s, E) is well-founded, then $\varphi(n)$ is true.

Note: "well-founded" can be omitted if φ is Δ_1^{KP}, where KP is the Kripke-Platek set theory. (Proof on the blackboard.)

Proof.

Let $\mathcal{M}' \cong \mathcal{M}$ such that M'_s is transitive and $E' = \in$. Then $(M'_s, \in) \models \varphi(n)$. Since M'_s is transitive, $(M'_s, \in) \subseteq_{end} (V, \in)$, so $\varphi(n)$ is actually true. □

An injection of set theory to logic

Definition (Recap)

The Härtig-quantifier is the generalised quantifier

$$Ixy\varphi(x)\psi(y) \iff |\varphi(\cdot)| = |\psi(\cdot)|.$$

The extension of first order logic by I is denoted $L(I)$.

Theorem (Lindström)

The class of well-orders $(M, <)$ is the class of reducts of models of a sentence of $L(I)$. So is the class of linear orders which are not well-orders.

Proof.

Let 1 be a new sort and R a new predicate of sort $\langle 0, 1\rangle$. Let θ be the conjunction of:

$$\forall x^0, y^0(x^0 < y^0 \rightarrow \forall x^1(R(x^0, x^1) \rightarrow R(y^0, x^1)))$$

$$\forall x^0, y^0(x^0 < y^0 \rightarrow \neg I x^1 y^1 R(x^0, x^1) R(y^0, y^1))$$

A linear order $(M, <)$ is a well order iff $(\{M\}, <)$ is a reduct of a model $(\{M, N\}, <, R)$. □

Note: The class of single-sorted well-orders $(M, <)$ is not the class of reducts of models of a single-sorted sentence of $L(I)$. The reason: Consider countable models. Here I would be eliminable if $L_{\omega_1\omega}$ is used. But well-order is not definable in $L_{\omega_1\omega}$ even in countable models.

We are particularly interested in classes that are model classes, i.e. classes of models of the same vocabulary, closed under isomorphisms.

Theorem

Every Σ_1^V-definable (in set theory) model class is the class of reducts of models of a sentence of $L(I)$.

Proof

Suppose K is defined by the Σ_1-formula $\varphi(x)$. We assume the vocabulary $\{R\}$, R binary, is (single sorted and) in sort 0. Let E be a binary predicate of sort 1, f a new function symbol of sort $\langle 0, 1\rangle$, and c a new constant of sort 1. Let $\theta \in L(I)$ so that the reducts of models of θ in sort 1 and predicate E are exactly the well-founded binary structures. We can assume that sort 0 does not occur in θ. Let Φ be the conjunction of:

1. "$(M_1, E) \models ZFC_n$" (for large enough n)
2. θ (to make sure we deal with a well-founded model)
3. $\varphi(c)$
4. "$c \in M_1$ is an L-structure $(U^c, P^c)_E$" (i.e. in the sense of E).
5. "f is an isomorphism between sort 0 and c" (e.g. $\forall x^0 \forall y^0 (f(x^0) = f(y^0) \to x^0 = y^0)$, $\forall x^0 (f(x^0)\ E\ U^c)$ and $\forall x^0 \forall y^0 (R(x^0, y^0) \leftrightarrow \langle f(x^0), f(y^0)\rangle_E\ E\ P^c$, etc.)

Claim: $\mathfrak{A} \in K$ if and only if $\mathfrak{A}$ is the 0-reduct of a model of Φ.

Left to right: Suppose $\mathfrak{A} \in K$, i.e. $V \models \varphi(\mathfrak{A})$. Pick κ so that $\mathfrak{A} \in H(\kappa)$. By Levy Reflection, $H(\kappa) \models \varphi(\mathfrak{A})$. By using $(H(\kappa), \in)$ for sort 1 we can easily find $\mathcal{M} \models \Phi$ such that $(M_0, R) = \mathfrak{A}$. W.l.o.g. $H(\kappa) \prec_n V$.

Claim: $\mathfrak{A} \in K$ if and only if $\mathfrak{A}$ is the 0-reduct of a model of Φ.

Right to Left: Suppose $\mathcal{M} \models \Phi$ and $\mathfrak{A} = (M_0, R)$. Let $\mathcal{M}' \cong \mathcal{M}$ such that (M_1', E') is a transitive ϵ-structure $(M_1', \in)$. Since K is closed under $\cong$, it suffices to show that $\varphi(\mathfrak{A}')$ is true, where $\mathfrak{A}' = (M_0', R')$. Since M_1' is a transitive model of ZFC_n, the set $c^{\mathcal{M}'}$ is a real L-structure $\mathfrak{B}$. Clearly $f^{\mathcal{M}'}, \mathfrak{B} \cong \mathfrak{A}'$, so it suffices to prove $\varphi(\mathfrak{B})$. We know that $(M_1', \in) \models \varphi(\mathfrak{B})$. Since $\varphi(x)$ is Σ_1, we get $V \models \varphi(\mathfrak{B})$, as desired.

Definition

Let $Cd(x)$ be the predicate "x is a cardinal number" of set theory. Let $Pw(x, y)$ be the predicate "x is the power-set of y" of set theory. We use $\Sigma_1(Cd)^V$ to denote the set of formulas which are Σ_1^V in the extended language $\{\in, Cd\}$, respectively $\Sigma_1(Pw)^V$, $\Sigma_1(Pw)^T$, etc.

Remark

- *Both Cd and Pw are* Π_1*-predicates.*
- $\Sigma_1(Cd)^{ZFC} \subseteq \Sigma_1(Pw)^{ZFC} = \Sigma_2^{ZFC}$
- $\Sigma_1(Cd)^{ZFC+V=L} = \Sigma_1(Pw)^{ZFC+V=L}$. *(Proof omitted, or sketched on the blackboard).*
- *If a model class is* $\Sigma_1(Cd)^T$ *and* T *is true (i.e. assumed), then the model class is* $\Sigma_1(Cd)^V$*. (Trivial)*
- *Much more model classes are* $\Sigma_1(Cd)^{ZFC}$ *than are merely* Σ_1^{ZFC}*. (See below for examples.)*

Remark

The following classes are $\Sigma_1(Cd)^{ZFC}$*: The class of* (M, P)*,* $P \subseteq M$*, where*

- $|P|$ *is finite.*
- $|P|$ *is finite and even.*
- $|P|$ *is the code of a Turing machine that halts.*
- $|P|$ *is the Gödel number of a first order sentence true in* $(\mathbb{N}, +, \cdot, 0, 1)$*.*
- $|P|$ *is countable.*
- $|P| = \aleph_n$ *(any fixed n).*
- $|P| = \aleph_n$ *for some n.*
- $|P| = \aleph_{|P|}$*.*

Remark

- $(H(\kappa), \in) \models Cd(\alpha)$ *if and only if* α *is a real cardinal.*
- *We say that a transitive* $(M, \in)$ *is Cd-absolute if for all* $\alpha \in M$*:* $(M, \in) \models Cd(\alpha)$ *if and only if* α *is a real cardinal.*
- *If* $(M, \in)$ *is Cd-absolute, then for all* Σ_0*-formulas of the vocabulary* $\{\in, Cd\}$ *we have* $(M, \in) \models \varphi(a)$ *if and only if* $\varphi(a)$ *is true. In other words* $(M, \in) \prec_{\Sigma_0(Cd)} (V, \in)$
- *"x is transitive" is* Δ_1^{ZFC}
- *"x is Cd-absolute" is* $\Delta_1(Cd)^{ZFC}$

Theorem

Every $\Sigma_1^V(Cd)$-definable (in set theory) model class is the class of reducts of models of a sentence of $L(I)$.

Proof

Suppose K is defined by the $\Sigma_1(Cd)$-formula $\varphi(x)$. We assume the vocabulary $\{R\}$, R binary, is (single sorted and) in sort 0. Let E be a binary predicate of sort 1, f a new function symbol of sort $\langle 0, 1 \rangle$, and c a new constant of sort 1. Let $\theta \in L(I)$ so that the reducts of models of θ in sort 1 and predicate E are exactly the well-founded binary structures. We can assume that sort 0 does not occur in θ. Let Φ be the conjunction of:

1. "$(M_1, E) \models ZFC_n$"
2. θ (to make sure we deal with a well-founded model)
3. $\varphi(c)$
4. "$c \in M_1$ is an L-structure $(U^c, P^c)_E$" (i.e. in the sense of E).
5. "f is an isomorphism between sort 0 and c" (e.g. $\forall x^0 \forall y^0 (f(x^0) = f(y^0) \to x^0 = y^0)$, $\forall x^0 (f(x^0)\ E\ U^c)$ and $\forall x^0 \forall y^0 (R(x^0, y^0) \leftrightarrow \langle f(x^0), f(y^0)\rangle_E\ E\ P^c$, etc.)
6. "Cardinals of (M_1, E) are real cardinals" i.e. $\forall x^1$("x^1 is a cardinal$_E$" $\to \forall y^1 (y^1 E x^1 \to \neg I z^1 v^1 (z^1 E y^1)(v^1 E x^1))$

Claim: $\mathfrak{A} \in K$ if and only if $\mathfrak{A}$ is the 0-reduct of a model of Φ.

Left to right: Suppose $\mathfrak{A} \in K$, i.e. $V \models \varphi(\mathfrak{A})$. Suppose $\varphi(x)$ is Σ_n. Pick κ so that $\mathfrak{A} \in H(\kappa) \prec_n V$. By Levy Reflection, $H(\kappa) \models \varphi(\mathfrak{A})$. By using $(H(\kappa), \in)$ for sort 1 we can easily find $\mathcal{M} \models \Phi$ such that $(M_0, R) = \mathfrak{A}$.

Claim: $\mathfrak{A} \in K$ if and only if $\mathfrak{A}$ is the 0-reduct of a model of Φ.

Right to Left: Suppose $\mathcal{M} \models \Phi$ and $\mathfrak{A} = (M_0, R)$. Let $\mathcal{M}' \cong \mathcal{M}$ such that (M_1', E') is a transitive ϵ-structure $(M_1', \in)$, which is Cd-absolute. Since K is closed under $\cong$, it suffices to show that $\varphi(\mathfrak{A}')$ is true, where $\mathfrak{A}' = (M_0', R')$. Since M_1' is a transitive model of ZFC_n, the set $c^{\mathcal{M}'}$ is a real L-structure $\mathfrak{B}$. Clearly, $f^{\mathcal{M}'}$, $\mathfrak{B} \cong \mathfrak{A}'$. So it suffices to show $\varphi(\mathfrak{B})$ is true. We know that $(M_1', \in) \models \varphi(\mathfrak{B})$. Since $\varphi(x)$ is $\Sigma_1(Cd)$ and $(M_1', \in)$ is Cd-absolute, we get $V \models \varphi(\mathfrak{B})$, as desired.

Now the converse...

Theorem

If a model class is the class of reducts of a sentence of $L(I)$, then it is $\Sigma_1(Cd)^V$-definable in set theory.

Proof: Suppose C, a model class in a vocabulary $L = \{R\}$, where R is binary and in sort 0, is the class of reducts to 0 of models of $\varphi_0 \in L(I)$.

$$\mathfrak{A} \in C \iff \exists \mathcal{M}(\mathcal{M} \models \varphi_0 \wedge (\mathfrak{A} = (M_0, R))).$$

This already *looks* like Σ_1, but we have to work more.

Lemma

Suppose $(M, \in) \models ZFC^-$ *is transitive and Cd-absolute. Then for all* $\mathfrak{A} \in M$ *and* $\varphi \in L(I)$:

$$\mathfrak{A} \models \varphi \iff (M, \in) \models \text{“}\mathfrak{A} \models \varphi\text{”}.$$

Proof.

By induction on φ. □

Back to the proof of the theorem:

Let $\Phi(x, y, z)$ be the conjunction of the following formulas of set theory

1. x is an L-structure,
2. y is a structure,
3. x is the reduct of y to sort 0.
4. z is a Cd-absolute transitive set, $x, y \in z$.
5. $((z, \in) \models \text{“}y \models \varphi_0\text{”})$ in the language of set theory[6]

Claim: $\mathfrak{A} \in C \iff \exists y \exists z \Phi(\mathfrak{A}, y, z)$.

Left to right: Suppose $\mathfrak{A} \in C$. Thus $\mathfrak{B} \models \varphi_0$ for some $\mathfrak{B}$ such that $(B_0, R) = \mathfrak{A}$. Find κ such that $\mathfrak{B} \in H(\kappa)$. Then $H(\kappa) \models \text{“}\mathfrak{B} \models \varphi_0\text{”}$ since $H(\kappa)$ is Cd-absolute. So we have $\Phi(\mathfrak{A}, \mathfrak{B}, H(\kappa))$.

[6] i.e. “$y \models \varphi_0$” is written in the language of set theory and then $\forall$ and $\exists$ are relativized to z.

Proof contd.

Right to left: Suppose $\Phi(\mathfrak{A}, \mathfrak{B}, N)$, where N is a transitive Cd-absolute set such that

$$N \models \text{“}\mathfrak{B} \models \varphi_0\text{”}.$$

Since $(N, \in) \prec_{\Sigma_1(Cd)} (V, \in)$, we have $\mathfrak{B} \models \varphi_0$, hence $\mathfrak{A} \in C$.

Corollary

TFAE:

1. *C is the class of reducts of a sentence of $L(I)$.*
2. *C is $\Sigma_1^V(Cd)$-definable in set theory*

Corollary

TFAE:

1. *C and the complement of C are classes of reducts of sentences of $L(I)$.*
2. *C is $\Delta_1^V(Cd)$-definable in set theory*

Corollary

TFAE:

1. *C is the class of reducts of a sentence of* $L(I)$.
2. *C is* $\Sigma_1^V(Cd)$*-definable in set theory*

Corollary

TFAE:

1. *C and the complement of C are classes of reducts of sentences of* $L(I)$.
2. *C is* $\Delta_1^V(Cd)$*-definable in set theory*

Definition

A model class (a sentence) K is Δ-definable in a logic L^*, $K \in \Delta(L^*)$, if K is the class of reducts of a sentence of L^* and so is the complement of K.

Corollary

TFAE:

1. $C \in \Delta(L(I))$.
2. C *is* $\Delta^V_1(Cd)$*-definable in set theory*

Similarly for second order logic L^2:

Theorem

TFAE:

1. $C \in \Delta(L^2)$.
2. C *is* Δ_2^V*-definable in set theory*

Symbiosis

- For many (most?) logics L^* stronger than first order logic one can find a predicate P of set theory so that definability in $\Delta(L^*)$ is equivalent to $\Delta_1(P)^V$-definability in set theory.
- This is a **symbiosis** between model theory and set theory.

Problem session 7

1. Show that the following concepts are Σ_0: $x = \{y, z\}$, $x = (y, z)$, "f is a function", $x = \bigcup y$.
2. Show that $\Sigma_1^{ZFC^-}$ formulas are closed under $\wedge$, $\vee$, $\exists x$, $\forall x \in y$.
3. Show that "α is uncountable" is not absolute, in fact not $\Sigma_1^{ZFC^-}$. [Hint: Take a countable model M of ZFC^- such that $\omega_1 \in M$. Then take a Mostowski collapse.]
4. Show that "$x = \mathcal{P}(y)$" is not absolute, in fact not $\Sigma_1^{ZFC^-}$.
5. Prove $\Sigma_1(Pw)^{ZFC} = \Sigma_2^{ZFC}$.
6. Show that the the class of structures (M, P), $P \subseteq M$, where $|P| = \aleph_3$, is $\Sigma_1(Cd)^{ZFC^-}$.
7. Prove: $(H(\kappa), \in) \models Cd(\alpha)$ if and only if α is a cardinal.