MODEL THEORY

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In these notes I will look at that part of "pre-Morley's theorem era" model theory that I feel is most relevant for the current trends in model theory. Some of the topics are not chosen because of the theorems but because of the methods behind the proofs. Only the surface of each topic chosen is scratched, for more see [CK] or [Ho]. When we attach some names of persons to theorems, we just indicate the name of the theorem commonly used, the person(s) are not always the one(s) that actually proved the theorem originally. For the history, see the historical notes in [CK].

Contents

1. First-order logic

In this section we recall the basic definitions of the first-order logic. Most of the definitions are recursive. The basic theory of recursive definitions is recalled in the appendix.

1.1 Definition. *A vocabulary L is a collection of relation, function and constant symbols. Each relation symbol R and function symbol f come with the arity $\#R, \#f \in \mathbb{N} - \{0\}$. Relation symbols are also called predicates.*

We let $L = \{R_i, f_j, c_k |\ i \in I^*,\ j \in J^*,\ k \in K^*\}$ be a fixed but arbitrary vocabulary. So when we say e.g. a function symbol f_i we mean the symbol from L. By e.g. a function symbol $f \in L$ we obviously mean f_i for some $i \in J^*$.

1.2 Definition. *The collection of (L-)terms is defined as follows:*
(i) variables v_i, $i \in \mathbb{N}$, are terms,
(ii) constant symbols c_k are terms,
(iii) if $n = \#f_j$ and $t_1, ..., t_n$ are terms, then $f_j(t_1, ..., t_n)$ is a term.

1.3 Definition. *The collection of atomic (L-)formulas is defined as follows:*
(i) if t and u are terms, then $t = u$ is an atomic formula,
(ii) if $n = \#R_i$ and $t_1, ..., t_n$ are terms, then $R_i(t_1, ..., t_n)$ is an atomic formula,
(iii) $\top$ is an atomic formula.

The formula $\top$ is needed for the elimination of quantifiers in the case L does not contain constant symbols, see section 5.

1.4 Definition. *The collection of (L-)formulas is defined as follows:*
(i) atomic formulas are formulas,
(ii) if ϕ is a formula, then $\neg\phi$ is a formula,
(iii) if ϕ and ψ are formulas, then $(\phi \wedge \psi)$ is a formula,
(iv) if ϕ is a formula and $i \in \mathbb{N}$, then $\exists v_i \phi$ is a formula.
By $L_{\omega\omega}$ we denote the set of all L-formulas.

The following notation is used:

$$\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$$

$$\phi \to \psi = \neg\phi \vee \psi$$

$$\phi \leftrightarrow \psi = (\phi \to \psi) \wedge (\psi \to \phi)$$

$$\forall v_i \phi = \neg\exists v_i \neg\phi.$$

1.5 Definition. *The notion v_i is free in ϕ is defined as follows:*
(i) ϕ is atomic: v_i is free in ϕ if v_i appears in ϕ,
(ii) $\phi = \neg\psi$: v_i is free in ϕ if it is free in ψ,
(iii) $\phi = \psi \wedge \theta$: v_i is free in ϕ if it is free in ψ or θ,
(iv) $\phi = \exists v_j \psi$: v_i is free in ϕ if it is free in ψ and $i \neq j$.

A sentence is a formula in which no v_i is free. A(n L-)theory is a collection of (L-)sentences.

If $x = (x_1, ..., x_n)$ is a sequence of variables (when we write like this we assume that for $k \neq m$, $x_k \neq x_m$), then the notation $\phi(x)$ means that if v_i is free in ϕ then $v_i \in \{x_1, ..., x_n\}$. Similarly for a term t, $t(x)$ means that if v_i appears in t, then $v_i \in \{x_1, ..., x_n\}$. Often we split x into two or more (disjoint) sequences y and z and write $\phi(y, z)$ in place of $\phi(x)$.

1.6 Definition. *A (L-)structure (i.e. model) is a sequence*

$$\mathcal{A} = (\mathcal{A}, R_i^{\mathcal{A}}, f_j^{\mathcal{A}}, c_k^{\mathcal{A}})_{i \in I^*, j \in J^*, k \in K^*}$$

where

(i) $\mathcal{A}$ is a non-empty set (the universe of $\mathcal{A}$, when we want to make a distinction between the model and its universe, we write $dom(\mathcal{A})$ for the universe),

(ii) $R^{\mathcal{A}_i} \subseteq \mathcal{A}^{\# R_i}$,

(iii) $f_j^{\mathcal{A}} : \mathcal{A}^{\# f_j} \to \mathcal{A}$,

(iv) $c_k^{\mathcal{A}} \in \mathcal{A}$.

When there is no risk of confusion, we write just $R_i = R_i^{\mathcal{A}}$ etc.

1.7 Definition. *For a term $t(x)$, $x = (x_1, ..., x_n)$, structure $\mathcal{A}$ and $a = (a_1, ..., a_n) \in \mathcal{A}^n$, $t^{\mathcal{A}}(a)$ is defined as follows:*

(i) $t = v_i$: $t^{\mathcal{A}}(a) = a_m$, where m is such that $v_i = x_m$,

(ii) $t = c_k$: $t^{\mathcal{A}}(a) = c_k^{\mathcal{A}}$,

(iii) $t = f_j(t_1, ..., t_m)$: $t^{\mathcal{A}}(a) = f_j^{\mathcal{A}}(t_1^{\mathcal{A}}(a), ..., t_m^{\mathcal{A}}(a))$.

1.8 Definition (Tarski). *For a formula $\phi(x)$, $x = (x_1, ..., x_n)$, structure $\mathcal{A}$ and $a = (a_1, ..., a_n) \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ is defined as follows:*

(i) $\phi(x) = t(x) = u(x)$: $\mathcal{A} \models \phi(a)$ if $t^{\mathcal{A}}(a) = u^{\mathcal{A}}(a)$,

(ii) $\phi(x) = R_i(t_1(x), ..., t_m(x))$: $\mathcal{A} \models \phi(a)$ if $(t_1^{\mathcal{A}}(a), ..., t_m^{\mathcal{A}}(a)) \in R_i^{\mathcal{A}}$,

(iii) $\phi(x) = \top(x)$: $\mathcal{A} \models \phi(a)$ always,

(iv) $\phi(x) = \neg\psi(x)$: $\mathcal{A} \models \phi(a)$ if $\mathcal{A} \not\models \psi(a)$,

(v) $\phi(x) = \psi(x) \wedge \theta(x)$: $\mathcal{A} \models \phi(a)$ if $\mathcal{A} \models \psi(a)$ and $\mathcal{A} \models \theta(a)$,

(vi) $\phi(x) = \exists v_i \psi(v_i, x)$: $\mathcal{A} \models \phi(a)$ if for some $b \in \mathcal{A}$, $\mathcal{A} \models \psi(b, a_1, ..., a_n)$.

For a theory T, by $\mathcal{A} \models T$, we mean that $\mathcal{A} \models \phi$ for every $\phi \in T$.

1.9 Remark. *In Definition 1.8 (v) we assumed that $v_i \notin \{x_1, ..., x_n\}$. This can be done without loss of generality, see the course Matemaattinen logiikka. This sloppy notation will be used regularly in these notes.*

1.10 Fact. *For all $\phi(x)$, $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and $z = (z_1, ..., z_m)$, there is $\psi(y)$ such that for all $\mathcal{A}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{A} \models \psi(a)$. Further more, we can choose ψ so that no z_i, $1 \leq i \leq m$, is bounded in ϕ (i.e. $\exists z_i$ does not appear in ψ).*

Proof. See the course Matemaattinen logiikka. □

1.11 Definition. *For a structure $\mathcal{A}$, a relation $R \subseteq \mathcal{A}^n$ is definable (in the vocabulary L), if there are a formula $\phi(x,y)$, $x = (x_1,...,x_n)$ and $y = (y_1,...,y_m)$, and $b \in \mathcal{A}^m$ such that for all $a \in \mathcal{A}^n$, $a \in R$ iff $\mathcal{A} \models \phi(a,b)$ (i.e. $R = \phi(\mathcal{A},b)$). The elements of b are called the parameters of the definition. If the parameters are not needed, we say that R is definable without parameters. A function $f : \mathcal{A}^n \to \mathcal{A}$ is definable if the relation $\{(a_1,...,a_{n+1}) \in \mathcal{A}^{n+1} | \ f(a_1,...,a_n) = a_{n+1}\}$ is definable. An element $a \in \mathcal{A}$ is definable if the relation $\{a\}$ is definable.*

1.12 Exercise. *Show that the set of integers is definable without parameters in $(\mathbf{C},+,\times,exp,0,1)$, where $+$ and $\times$ are the addition and multiplication of complex numbers and $exp(x) = e^x$.*

In Section 6 we will see that the set of integers is not definable in $(\mathbf{C},+,\times,0,1)$ without parameters.

1.13 Definition. *Suppose $L' \subseteq L$, $\mathcal{A}$ is an L-structure and $X \subseteq \mathcal{A}$.*
(i) We say that X is L'-closed if $c^{\mathcal{A}} \in X$ for all constants $c \in L'$ and for all $n \in \mathbb{N} - \{0\}$, $a_1,...,a_n \in X$ and n-ary function symbols $f \in L'$, $f^{\mathcal{A}}(a_1,...,a_n) \in X$.
(ii) Suppose X is L'-closed. By $\mathcal{A} \restriction X, L'$ we mean the L'-structure $\mathcal{B}$ such that $dom(\mathcal{B}) = X$, for all constants $c \in L'$, $c^{\mathcal{B}} = c^{\mathcal{A}}$, n-ary function symbols $f \in L'$, $f^{\mathcal{B}} = f^{\mathcal{A}} \restriction X^n$ and n-ary relation symbols $R \in L'$, $R^{\mathcal{B}} = R^{\mathcal{A}} \cap X^n$. If $X = \mathcal{A}$, we write just $\mathcal{A} \restriction L'$ for $\mathcal{A} \restriction X, L'$ and if $L' = L$ we write just $\mathcal{A} \restriction X$ for $\mathcal{A} \restriction X, L'$.

So e.g. $(\mathbf{C},+,\times,exp,0,1) \restriction \mathbf{R}, \{+,\times,0,1\} = (\mathbf{R},+,\times,0,1)$.

1.14 Exercise. *Let $\mathcal{A}$ be an $L \cup \{R\}$-structure, $\#R = n$, and $X \subseteq \mathcal{A}^m$ be definable in $\mathcal{A}$. Show that if $R^{\mathcal{A}}$ is definable in $\mathcal{A} \restriction L$, then X is definable in $\mathcal{A} \restriction L$.*

1.15 Exercise. *Let $L' \subseteq L$ be vocabularies, $\phi(v_0)$ be an L-formula and T an L-theory such that for all $\mathcal{A} \models T$, $\phi(\mathcal{A})$ is L'-closed. Show that for all L'-formulas $\psi(x)$, there is an L-formula $\psi^*(x)$, $x = (x_1,...,x_n)$, such that for all $\mathcal{A} \models T$ and $a \in (\phi(\mathcal{A}))^n$, $\mathcal{A} \models \psi^*(a)$ iff $\mathcal{A} \restriction \phi(\mathcal{A}), L' \models \psi(a)$.*

1.16 Definition. *Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures. We say that $F : \mathcal{A} \to \mathcal{B}$ is an isomorphism if F is one-to-one and onto, for all constants $c \in L$, $F(c^{\mathcal{A}}) = c^{\mathcal{B}}$, for all n-ary function symbols and $a = (a_1,...,a_n) \in \mathcal{A}^n$, $F(f^{\mathcal{A}}(a)) = f^{\mathcal{B}}(F(a)) (= f^{\mathcal{B}}((F(a_1),...,F(a_n))))$ and n-ary relation symbols $R \in L$ and $a \in \mathcal{A}^n$, $a \in R^{\mathcal{A}}$ iff $F(a) \in R^{\mathcal{B}}$. Isomorphisms $F : \mathcal{A} \to \mathcal{A}$ are called automorphisms (of $\mathcal{A}$). The set of all automorphisms of $\mathcal{A}$ is denoted by $Aut(\mathcal{A})$.*

1.17 Fact.
(i) Suppose $F : \mathcal{A} \to \mathcal{B}$ is an isomorphism. Then for all formulas $\phi(x)$, $x = (x_1,...,x_n)$, and $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(F(a))$.
(ii) If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are isomorphisms, then so are $F^{-1} : \mathcal{B} \to \mathcal{A}$ and $G \circ F : \mathcal{A} \to \mathcal{C}$.

Proof. See the course Matemaattinen logiikka. □

1.18 Exercise. *Suppose* $F : \mathcal{A} \to \mathcal{A}$ *is an automorphism. Show that* $X \subseteq \mathcal{A}^n$ *is definable iff* $F(X)\,(=\{F(a)|\ a \in X\})$ *is definable and that if* X *is definable without parameters, then* $F(X) = X$. *Conclude that* $2\pi i$ *is not definable without parameters in* $(\mathbf{C}, +, \times, exp, 0, 1)$ *(see Exercise 1.12).*

1.19 Exercise. *Suppose* L *consists of unary predicates* P *and* S *and a binary predicate* I. *Let* $\mathcal{A}\,(=A_2(\mathbf{R}))$ *be an* L*-structure such that its universe is* $P^{\mathcal{A}} \cup S^{\mathcal{A}}$, $P^{\mathcal{A}} = \mathbf{R}^2$, $S^{\mathcal{A}} = \{l_{abc}|\ (a,b) \in \mathbf{R}^2 - \{(0,0)\},\ c \in \mathbf{R}\}$, *where* $l_{abc} = \{(x,y) \in \mathbf{R}^2|\ ax + by + c = 0\}$ *and* $(p,l) \in I^{\mathcal{A}}$ *if* $p \in P^{\mathcal{A}}$, $l \in S^{\mathcal{A}}$ *and* $p \in l$. *Let* $l_x = l_{010}$ *and* $\mathbf{0} = (0,0)$. *Show that using* l_x *and* $\mathbf{0}$ *as parameters the relation*

$$R = \{((a,0),(b,0),(c,0)) \in l_x^3|\ c = a + b\}$$

can be defined. Show also that R *is not definable using just* l_x *as a parameter.*

2. On ordinals and cardinals

In this section we recall some facts from set theory that are needed throughout these notes.

2.1 Definition. *Suppose* $R \subseteq X^2$. *We say that* (X,R) *is a well-ordering and alternatively that* R *well-orders* X *if*

(i) (X,R) *is a linear ordering i.e. for all* $x,y,z \in X$, *(a)-(c) below holds:*
(a) if $(x,y) \in R$ *and* $(y,z) \in R$ *then* $(x,z) \in R$,
(b) if $(x,y) \in R$, *then* $(y,x) \notin R$,
(c) $(x,y) \in R$ *or* $x = y$ *or* $(y,x) \in R$,
(ii) there are no $x_n \in X$, $n \in \mathbb{N}$, *such that for all* $n \in \mathbb{N}$, $(x_{n+1}, x_n) \in R$.

Notice that if $(X,<)$ is a well-ordering and $Y \subseteq X$ is not empty, then there is a $<$-least element in Y.

Collections of sets definable (with parameters) in the vocabulary $\{\in\}$ of set theory are called classes.

2.2 Definition. *We say that a set* α *is an ordinal if* $(\alpha, \in)$ *is a well-ordering and* α *is transitive i.e. for all* x *and* y, *if* $y \in x \in \alpha$, *then* $y \in \alpha$. *The class of all ordinals is denoted by* On.

2.3 Fact.

(i) $\in$ *well-orders the class* On *(for ordinals* α *and* β, *instead of writing* $\alpha \in \beta$ *we write* $\alpha < \beta$*).*

(ii) If α *is an ordinal and* $x \in \alpha$, *then* x *is an ordinal i.e.* $\alpha = \{\beta \in On|\ \beta < \alpha\}$.

(iii) For every well-ordering (X,R) *there are a unique ordinal* α *and a unique bijection* $\pi : X \to \alpha$ *such that for all* $x,y \in X$, $(x,y) \in R$ *iff* $\pi(x) < \pi(y)$.

(iv) For ordinals α *and* β, $\alpha \subseteq \beta$ *iff* $\alpha = \beta$ *or* $\alpha < \beta$.

(v) $\emptyset$ *is an ordinal (usually denoted by* 0*), if* α *is an ordinal then* $\alpha \cup \{\alpha\}$ *is the least ordinal strictly greater than* α *(usually denoted by* $\alpha + 1$*) and if* α_i, $i \in I$, *are ordinals, then* $\bigcup_{i \in I} \alpha_i$ *is the least ordinal greater or equal to every* α_i.

Proof. Basic set theory course or [Je]. □

Finite ordinals and natural numbers are often thought as the same, i.e. $0 = \emptyset$, $1 = 0+1 = \emptyset \cup \{\emptyset\} = \{\emptyset\}$, $2 = 1+1 = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$ etc. The set of all finite ordinals is called ω (by Fact 2.3 (v), ω is an ordinal) and so $\omega = \mathbb{N}$. An ordinal α is a successor ordinal if there is an ordinal β such that $\alpha = \beta + 1$. Otherwise α is called a limit ordinal.

Many of our constructions and proofs are based on the following recursion and induction principles.

2.4 Fact.

(i) Suppose G is a function from sets to sets definable in the vocabulary of set theory (i.e. a class function). Then there is a unique class function F from On to sets such that for all ordinals α, $F(\alpha) = G(F \restriction \alpha)$.

(ii) Suppose that P and $X \subseteq On$ are classes. Then $X \subseteq P$ if for all ordinals $\alpha \in X$ the following holds:

(*) if for all $\beta \in X \cap \alpha$, $\beta \in P$, then $\alpha \in P$.

Proof. Basic set theory course or [Je]. □

2.5 Fact (Schröder-Bernstein). *If there are one-to-one functions (injections) $f : X \to Y$ and $g : Y \to X$, then there is a one-to-one and onto function (bijection) $\pi : X \to Y$.*

Proof. Basic set theory course or [Je]. □

2.6 Definition. *We say that an ordinal α is a cardinal if for all $\beta < \alpha$, there is no injection ($\Leftrightarrow$ bijection by Schröder-Bernstein) from α to β.*

Notice that every finite ordinal is a cardinal as well as ω and that infinite cardinals are limit ordinals.

2.7 Definition. *For all ordinals α, a cardinal ω_α (often also called $\aleph_\alpha$) is defined to be the least infinite cardinal strictly greater than ω_β for any $\beta < \alpha$.*

Notice that the class function $\alpha \mapsto \omega_\alpha$ exists by Fact 2.4 (i) and that $\omega_0 = \omega$ ($= \mathbb{N}$). For a cardinal κ, by κ^+ we denote the least cardinal $> \kappa$ (so if $\kappa = \omega_\alpha$, $\kappa^+ = \omega_{\alpha+1}$).

2.8 Fact.

(i) For every set X there are a cardinal κ and a bijection $\pi : X \to \kappa$. Furthermore such κ is unique and is denoted by $|X|$ and called the cardinality of X (or just the size or power of X).

(ii) Suppose that at least one of X and Y is infinite and that neither is empty. Then $|X \cup Y| = |X \times Y| = max\{|X|, |Y|\}$ (usually this cardinal is denoted by $|X| + |Y|$).

(iii) Suppose I is infinite and X_i, $i \in I$, are non-empty and distinct. Then $|\bigcup_{i\in I} X_i| = max\{\kappa, |I|\}$, where $\kappa = \bigcup_{i\in I} |X_i|$.

(iv) Suppose X is infinite and let $P(X)$ be the set of all subsets of X and X^X be the set of all function from X to X. Then $|P(X)| = |X^X| > |X|$. We denote $|P(X)|$ by $2^{|X|}$ (so 2^ω is the cardinality of the continuum).

(v) If κ is a cardinal and $\alpha < \kappa^+$, then there is no function $f : \alpha \to \kappa^+$ such that $\cup_{i<\alpha} f(i) = \kappa^+$ (i.e. successor cardinals are regular).

Proof. Basic set theory course or [Je]. □

If $\mathcal{A}$ is a structure, then by the cardinality $|\mathcal{A}|$ of $\mathcal{A}$ we mean the cardinality of the universe of $\mathcal{A}$.

For an L-structure $\mathcal{A}$, by $Th(\mathcal{A})$ we mean the set of all L-sentences true in $\mathcal{A}$.

2.9 Exercise. *Let $\mathcal{A} = (\mathbb{N}, S, 0)$, where $S : \mathbb{N} \to \mathbb{N}$ is such that $S(x) = x+1$, for all $x \in \mathbb{N}$ and $T = Th(\mathcal{A})$. Show that if $\mathcal{B}, \mathcal{C} \models T$ and $|\mathcal{B}| = |\mathcal{C}| > \omega$, then $\mathcal{B}$ and $\mathcal{C}$ are isomorphic. Hint: Think the equivalence classes of the equivalence relation xEy if for some $n \in \mathbb{N}$, $S^n(x) = y$ or $S^n(y) = x$ (where $S^0 = id$ and $S^{n+1} = S \circ S^n$).*

2.10 Exercise. *Let $\mathcal{A} = (\omega^2, R)$, where ($\omega^2 = \omega \times \omega$ and) $R \subseteq (\omega^2)^2$ is a binary relation on ω^2 such that $(n,m)R(n',m')$ if $m < m'$. Show that there is a unary relation $X \subseteq \omega^2$ such that it is not definable in $\mathcal{A}$ but for all $F \in Aut(\mathcal{A})$, $F(X) = X$. Hint: Start by calculating the number of definable relations in $\mathcal{A}$.*

3. Compactness

3.1 Definition. *Let $F \subseteq P(X)$.*

(i) F has the finite intersection property if for all $X_i \in F$, $i < n$, $\bigcap_{i<n} X_i \neq \emptyset$.

(ii) F is a filter if $X \in F$, $\emptyset \notin F$, if $Z, Y \in F$, then $Z \cap Y \in F$ and if $Z \in F$ and $Z \subseteq Y \subseteq X$, then $Y \in F$.

(iii) F is an ultrafilter if it is a filter and for all $Y \subseteq X$, either $Y \in F$ or $X - Y \in F$.

3.2 Lemma. *Suppose $F \subseteq P(X)$ has the finite intersection property. Then there is an ultrafilter $U \subseteq P(X)$ such that $F \subseteq U$.*

Proof. Let X_i, $i < \alpha$, enumerate all elements of $P(X)$. By recursion we define an increasing sequence of subsets U_i of $P(X)$ with the finite intersection property:

$i = 0$: $U_i = F$.

$i = j+1$: If $U_j \cup \{X_j\}$ has the finite intersection property, we let $U_i = U_j \cup \{X_j\}$. Otherwise $U_j \cup \{(X - X_j)\}$ has the finite intersection property and we let this be U_i. (If $(\bigcap_{k<n} Y_j) \cap X_i = \emptyset$ and $(\bigcap_{k<m} Z_j) \cap (X - X_i) = \emptyset$ then $(\bigcap_{k<n} Y_j) \cap (\bigcap_{k<m} Z_j) = \emptyset$.)

i is limit: $U_i = \bigcup_{j<i} U_j$.

It is easy to see that $U = \bigcup_{i<\alpha} U_i$ is as wanted (exercise) □

Suppose $\mathcal{A}_\eta$, $\eta \in X$, are models and $U \subseteq P(X)$ is an ultrafilter. By $\Pi_{\eta \in X} \mathcal{A}_\eta$ we mean the set of all $f : X \to \cup_{\eta \in X} \mathcal{A}_\eta$ such that for all $\eta \in X$, $f(\eta) \in \mathcal{A}_\eta$. Then $f \equiv g \ mod\ U$ if $\{\eta \in X |\ f(\eta) = g(\eta)\} \in U$ is an equivalence relation (exercise). By

f/U we mean the equivalent class of f and let $\Pi_{\eta\in X}\mathcal{A}_\eta/U$ be the set of all these equivalence classes. We make $\Pi_{\eta\in X}\mathcal{A}_\eta/U$ into an L-structure $\mathcal{A}$ (also denoted by $\Pi_{\eta\in X}\mathcal{A}_\eta/U$) by adding the following interpretations:

$(g_1/U,...,g_n/U)\in R_i^{\mathcal{A}}$ if $\{\eta\in X|\ (g_1(\eta),...,g_n(\eta))\in R_i^{\mathcal{A}_\eta}\}\in U$, where $n=\#R_i$,

$f_j^{\mathcal{A}}(g_1/U,...,g_n/U)=g/U$, where $n=\#f_j$ and $g(\eta)=f_j^{\mathcal{A}_\eta}(g_1(\eta),...,g_n(\eta))$,

$c_k^{\mathcal{A}}=g/U$, where $g(\eta)=c_k^{\mathcal{A}_n}$.

We notice that these definitions do not depend on the representatives of the equivalence classes g_1/U,...,g_n/U (exercise).

3.3 Lemma. *For all terms $t=t(x)$, $x=(x_1,...,x_n)$, $\mathcal{A}=\Pi_{\eta\in X}\mathcal{A}_\eta/U$ and $a_i\in\Pi_{\eta\in X}\mathcal{A}_\eta$, $1\le i\le n$, $t^{\mathcal{A}}(a_1/U,...,a_n/U)=g/U$, where g is such that $g(\eta)=t^{\mathcal{A}_\eta}(a_i(\eta),...,a_n(\eta))$.*

Proof. By induction on t: For variables and constants the claim is the definition of the interpretation. So suppose $t(x)=f(t_1(x),...,t_m(x))$, $f\in L$ a function symbol. By the induction assumption, for $1\le k\le m$, $t_k^{\mathcal{A}}(a_1/U,...,a_n/U)=g_k/U$, where $g_k(\eta)=t_k^{\mathcal{A}_\eta}(a_1(\eta),...,a_n(\eta))$. Then $t^{\mathcal{A}}(a_1/U,...,a_n/U)=f^{\mathcal{A}}(g_1/U,...,g_n/U)$. By the definition of $f^{\mathcal{A}}$,

$f^{\mathcal{A}}(g_1/U,...,g_n/U)=g/U$, where

$g(\eta)=f^{\mathcal{A}_\eta}(g_1(\eta),...,g_m(\eta))=$

$f^{\mathcal{A}_\eta}(t_1^{\mathcal{A}_\eta}(a_1(\eta),...,a_n(\eta)),...,t_m^{\mathcal{A}_\eta}(a_1(\eta),...,a_n(\eta)))=t^{\mathcal{A}_\eta}(a_i(\eta),...,a_n(\eta))$. □

3.4 Theorem (Łos). *For all formulas ϕ, $\Pi_{\eta\in X}\mathcal{A}_\eta/U\models\phi(g_1/U,...,g_n/U)$ iff $\{\eta\in X|\ \mathcal{A}_\eta\models\phi(g_1(\eta),...,g_n(\eta))\}\in U$.*

Proof. Easy induction on ϕ (exercise)□

3.5 Definition. *We recall that a collection of sentences is called a theory. If T is a theory, we say that it is consistent if it has a model i.e. there is a structure $\mathcal{A}$ such that $\mathcal{A}\models T$. If $x=(x_1,...,x_n)$ and Σ is a collection of formulas of the form $\psi(x)$, then we write that $\Sigma\models\phi(x)$ if for all $\mathcal{A}$ and $a\in\mathcal{A}^n$, $\mathcal{A}\models\phi(a)$ if $\mathcal{A}\models\psi(a)$ for all $\psi\in\Sigma$. In particular, for a theory T and a sentence ϕ, we write $T\models\phi$ if every model of T is a model of ϕ. If $T=\emptyset$, we write just $\models\phi$.*

3.6 Compactness theorem. *If every finite $T'\subseteq T$ is consistent, then T is consistent.*

Proof. We prove this by induction on $|T|$. For finite T the claim is trivial. So we may assume that $\kappa=|T|$ is infinite. Let ϕ_i, $i<\kappa$, enumerate T. By the induction assumption, for all $i<\kappa$, there is $\mathcal{A}_i$ such that $\mathcal{A}_i\models\phi_j$ for all $j\le i$. Let $F=\{\kappa-\alpha|\ \alpha<\kappa\}$. Then F has the finite intersection property and thus there is an ultrafilter U extending F by Lemma 3.2. By Łos, since for all $\alpha<\kappa$, $\kappa-\alpha\subseteq\{i<\kappa|\ \mathcal{A}_i\models\phi_\alpha\}$ and $\kappa-\alpha\in U$, for all $\alpha<\kappa$, $\Pi_{\eta\in X}\mathcal{A}_\eta/U\models\phi_\alpha$. □

Notice that compactness theorem implies the following: if $\Sigma\models\phi$, then there is finite $\Sigma'\subseteq\Sigma$ such that $\Sigma'\models\phi$ (exercise, see fact 4.4 below).

In the following example, we let $L = \{+, \times, -, 0, 1\}$, where $+$ and $\times$ are 2-ary function symbols, $-$ is 1-ary function symbol and 0 and 1 are constant symbols. In stead of $+(x, y)$ we write $x + y$ and the same for $\times$. (The function symbol $-$ is included for convenience for section 6, it is not really needed.) We let T_f consist of the following sentences:

$\forall v_0 \forall v_1 \forall v_2 (v_0 + (v_1 + v_2) = (v_0 + v_1) + v_2)$,
$\forall v_0 \forall v_1 (v_0 + v_1 = v_1 + v_0)$,
$\forall v_0 (v_0 + 0 = v_0)$,
$\forall v_0 (v_0 + (-v_0) = 0)$,
$\forall v_0 \forall v_1 \forall v_2 (v_0 \times (v_1 \times v_2) = (v_0 \times v_1) \times v_2)$,
$\forall v_0 \forall v_1 (v_0 \times v_1 = v_1 \times v_0)$,
$\forall v_0 (v_0 \times 1 = v_0)$,
$\neg 0 = 1$,
$\forall v_0 \exists v_1 ((v_0 = 0) \vee (v_0 \times v_1 = 1))$,
$\forall v_0 \forall v_1 \forall v_2 (v_0 \times (v_1 + v_2) = (v_0 \times v_1) + (v_0 \times v_2))$.

So $\mathcal{A} \models T_f$ iff $\mathcal{A}$ is a field.

For $n \in \mathbb{N}$, the notation nt, t a term, is defined as follows: $0t = 0$ (here the first 0 is the natural number and the second is the constant) and $(n+1)t = nt + t$. Similarly $t^0 = 1$ and $t^{n+1} = t^n \times t$. Then we let

$T_{f_0} = T_f \cup \{\neg p1 = 0 |\ p \text{ a prime}\}$

and for primes p,

$T_{f_p} = T_f \cup \{p1 = 0\}$.

Then $\mathcal{A} \models T_{f_0}$ iff $\mathcal{A}$ is a field of characteristic 0 and $\mathcal{A} \models T_{f_p}$ iff $\mathcal{A}$ is a field of characteristic p.

3.7 Example. *For all L-sentences ϕ, if $T_{f_0} \models \phi$ then there is n such that $T_{f_p} \models \phi$ for all $p > n$.*

Proof. Suppose not. Let ϕ witness this. Then $X = \{p \in \mathbb{N} |\ T_{f_p} \not\models \phi\}$ is infinite. For all $p \in X$ choose $\mathcal{A}_p \models T_{f_p}$ so that $\mathcal{A}_p \not\models \phi$. Let F consist of sets $\{p \in X |\ p > m\}$, $m \in \mathbb{N}$. Then F has the finite intersection property and can be extended to an ultrafilter U. By Łos, $\mathcal{A} = \Pi_{p \in X} \mathcal{A}_p / U \models T_f$ and $\mathcal{A} \not\models \phi$. Also for all primes q, the set of $p \in X$ such that $\mathcal{A}_p \models q1 = 0$ contains at most q. Thus by Łos, $\mathcal{A} \models \neg q1 = 0$ for all primes q i.e. $\mathcal{A} \models T_{f_0}$, a contradiction. □

3.8 Exercise (Finite Ramsay's theorem). *Show using infinite Ramsay's theorem (Theorem 10.3 below, see also Definition 10.2) that for all $n, k, p \in \mathbb{N} - \{0\}$, there is $m \in \mathbb{N}$ such that for all $f : [m]^n \to k$, there is $X \subseteq m$ such that $f \restriction [X]^n$ is constant and $|X| \geq p$. Hint: Suppose that the claim is not true for some n, k and p and use compactness to construct a counter example for $\omega \to (\omega)^n_k$.*

The claim in the following exercise is not provable in Peano arithmetic.

3.9 Exercise (Paris-Harrington). *We say that a non-empty set $X \subseteq \mathbb{N}$ is fat if $|X| > min(X)$. Show using infinite Ramsay's theorem that for all $n, k, p \in$*

$\mathbb{N}-\{0\}$, *there is* $m\in\mathbb{N}$ *such that for all* $f:[m]^n\to k$, *there is fat* $X\subseteq m$ *such that* $f\restriction[X]^n$ *is constant and* $|X|\ge p$. *Hint: As Exercise 3.8.*

3.10 Exercise. *Let* B *be a set,* α *an ordinal and for all* $i<\alpha$, *let* A_i *be a finite set and* F_i *a finite non-empty set of functions* $f:A_i\to B$. *Let* $A=\bigcup_{i<\alpha}A_i$ *and suppose*

(i) if $A_i\subseteq A_j$ *and* $f\in F_j$, *then* $f\restriction A_i\in F_i$,

(ii) for all $i,j<\alpha$ *there is* $k<\alpha$ *such that* $A_i\cup A_j\subseteq A_k$ *(i.e. the sets form a directed system).*

Show that there is a function $g:A\to B$ *such that for all* $i<\alpha$, $g\restriction A_i\in F_i$. *Hint: Start by choosing an ultrafilter* $U\subseteq P(\alpha)$ *so that for all* $i<\alpha$, $\{j<\alpha|\ A_i\subseteq A_j\}\in U$ *and then for all* $i<\omega$, *pick some* $f_i\in F_i$.

4. Tarski-Vaught

4.1 Definition. *Suppose* $\mathcal{A}$ *and* $\mathcal{B}$ *are structures,* $X\subseteq\mathcal{A}$ *and* $f:X\to\mathcal{B}$.

(i) We say that f *is a partial isomorphism from* $\mathcal{A}$ *to* $\mathcal{B}$ *if for all atomic* $\phi(x)$, $x=(x_1,...,x_n)$ *and* $a\in X^n$, $\mathcal{A}\models\phi(a)$ *iff* $\mathcal{B}\models\phi(f(a))$.

(ii) We say that f *is a partial elementary map from* $\mathcal{A}$ *to* $\mathcal{B}$ *if for all formulas* $\phi(x)$, $x=(x_1,...,x_n)$ *and* $a\in X^n$, $\mathcal{A}\models\phi(a)$ *iff* $\mathcal{B}\models\phi(f(a))$.

(iii) f *is embedding if it is a partial isomorphism and* $X=\mathcal{A}$.

(iv) f *is elementary embedding if it is a partial elementary map and* $X=\mathcal{A}$.

(v) $\mathcal{A}$ *is a submodel of* $\mathcal{B}$ *(denoted* $\mathcal{A}\subseteq\mathcal{B}$*) if the identity function* $id_{\mathcal{A}}$ *is an embedding.* $\mathcal{A}$ *is an elementary submodel of* $\mathcal{B}$ *(denoted* $\mathcal{A}\preceq\mathcal{B}$*) if* $id_{\mathcal{A}}$ *is an elementary embedding.*

Abusing the notation, if 4.1 (i) (4.1 (ii)) holds we write also that $f:\mathcal{A}\to\mathcal{B}$ is a partial isomorphism (elementary map) although the domain of f may not be the whole $\mathcal{A}$. Notice that if $f:\mathcal{A}\to\mathcal{B}$ is a partial isomorphism (partial elementary map), then also $f^{-1}:\mathcal{B}\to\mathcal{A}$ is a partial isomorphism (partial elementary map).

4.2 Exercise. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* L*-structures.*

(i) $\mathcal{A}$ *is a submodel of* $\mathcal{B}$ *iff dom*$(\mathcal{A})\subseteq$ *dom*$(\mathcal{B})$ *is* L*-closed in* $\mathcal{B}$ *and* $\mathcal{A}=\mathcal{B}\restriction$ *dom*$(\mathcal{A})$.

(ii) Suppose $f:\mathcal{A}\to\mathcal{B}$ *is onto. Show (using Fact 1.17) that the following are equivalent*

(a) f *is an isomorphism,*

(b) f *is an elementary embedding,*

(c) f *is an embedding.*

(iii) Show that $f:\mathcal{A}\to\mathcal{B}$ *is an (elementary) embedding iff* f *is an isomorphism between* $\mathcal{A}$ *and some (elementary) substructure of* $\mathcal{B}$.

(iv) Show that if $\mathcal{A}\subseteq\mathcal{B}$ *and* $a\in\mathcal{A}^n$ *and* $\phi(x)$, $x=(x_1,...,x_n)$ *is quantifier free (i.e. no quantifier appear in* ϕ*), then* $\mathcal{A}\models\phi(a)$ *iff* $\mathcal{B}\models\phi(a)$.

(v) Suppose $A\subseteq\mathcal{A}$ *and* $B=\{t^{\mathcal{A}}(a)|\ t(x_1,...,x_n)$ is a term and $a\in A^n\}$. *Show that* $\mathcal{A}\restriction B$ *is the least submodel of* $\mathcal{A}$ *containing* A *(i.e. the submodel generated by* A*).*

(vi) Suppose $U \subseteq P(X)$ is an ultrafilter and for all $i \in X$, $\mathcal{A}_i = \mathcal{A}$ for some structure $\mathcal{A}$. Show that $\mathcal{A}$ is isomorphic with an elementary submodel of $\Pi_{i\in X}\mathcal{A}_i/U$.

(vii) Suppose $f : \mathcal{B} \to \mathcal{A}$ is a partial isomorphism, $\mathcal{C}$ is the submodel generated by $dom(f)$ and define $g : \mathcal{C} \to \mathcal{A}$ so that for all terms $t(x_1, ..., x_n)$ and $a \in dom(f)^n$, $g(t^{\mathcal{B}}(a)) = t^{\mathcal{A}}(f(a))$. Show that g is well-defined and an embedding.

4.3 Definition. *Let $\mathcal{A}$ be a structure and $A \subseteq \mathcal{A}$. By $L(A)$ we mean a vocabulary we get from L by adding new constants $\underline{a}$ for all $a \in A$. $(\mathcal{A}, A)$ means an $L(A)$-structure we get from $\mathcal{A}$ by interpreting $\underline{a}^{\mathcal{A}} = a$. Recall that by $Th(\mathcal{A})$ we mean the set of all L-sentences true in $\mathcal{A}$ and so by $Th(\mathcal{A}, A)$ we mean the set of all $L(A)$-sentences true in $(\mathcal{A}, A)$.*

For a formula ϕ and constant c, $\phi(c/x_i)$ is defined as follows:

(i) if ϕ is atomic then $\phi(c/x)$ is what we get from ϕ by replacing x by c everywhere,

(ii) if $\phi = \neg\psi$ then $\phi(c/x) = \neg(\psi(c/x))$,

(iii) if $\phi = \psi \wedge \theta$ then $\phi(c/x) = \psi(c/x) \wedge \theta(c/x)$,

(iv) if $\phi = \exists v_i\psi$, then $\phi(c/x) = \phi$ if $v_i = x$ and otherwise $\phi(c/x) = \exists v_i(\psi(c/x))$.

For $\phi(y, x)$, $x = (x_1, ..., x_n)$ and $a = (a_1, ..., a_n) \in A^n$, we write $\phi(y, \underline{a})$ instead of $\phi(\underline{a_1}/x_1)...(\underline{a_n}/x_n)$.

4.4 Fact. *For all formulas $\phi(x)$, $x = (x_1, ..., x_n)$, and $a \in A^n$, $\mathcal{A} \models \phi(a)$ iff $(\mathcal{A}, A) \models \phi(\underline{a})$.*

Proof. The course Matemaattinen logiikka. □

4.5 Lemma. *If $\mathcal{A}$ is infinite and κ is a cardinal, then there is $\mathcal{B}$ of cardinality $\geq \kappa$ such that $\mathcal{A} \preceq \mathcal{B}$. (In particular, the Hanf number of the first-order logic is ω.)*

Proof. Let $L^* = L(\mathcal{A}) \cup \{c_i |\ i < \kappa\}$, where c_i are new constant symbols. Let $T^* = Th(\mathcal{A}, \mathcal{A}) \cup T$ where $T = \{\neg c_i = c_j |\ i < j < \kappa\}$. Since $\mathcal{A}$ is infinite, for all finite $T' \subseteq T$, we can interpret the constants c_i in $(\mathcal{A}, \mathcal{A})$ so that T' is true in that model. Since $(\mathcal{A}, \mathcal{A}) \models Th(\mathcal{A}, \mathcal{A})$, by the compactness theorem T^* has a model $\mathcal{B}^*$. Clearly $|\mathcal{B}^*| \geq \kappa$. By renaming the elements of $\mathcal{B}^*$ (i.e. by taking an isomorphic copy of $\mathcal{B}^*$), we may assume that for all $a \in \mathcal{A}$, $\underline{a}^{\mathcal{B}} = a$. Let $\mathcal{B} = \mathcal{B}^* \restriction L$ (see Definition 1.13). We are left to show that $\mathcal{A} \preceq \mathcal{B}$. But $\mathcal{A} \models \phi(a)$ iff $(\mathcal{A}, \mathcal{A}) \models \phi(\underline{a})$ iff $\phi(\underline{a}) \in T^*$ iff $\mathcal{B}^* \models \phi(\underline{a})$ iff $\mathcal{B} \models \phi(a)$. □

4.6 Theorem (Tarski-Vaught). *$\mathcal{A} \preceq \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all formulas $\phi(v_i, x)$, $x = (x_1, ..., x_n)$ and $a \in \mathcal{A}^n$ the following holds: If $\mathcal{B} \models \exists v_i\phi(v_i, a)$, then there is $b \in \mathcal{A}$ such that $\mathcal{B} \models \phi(b, a)$.*

Proof. By induction on $\psi(y)$, $y = (y_1, ..., y_m)$, we show that for all $b \in \mathcal{A}^m$, $\mathcal{A} \models \psi(b)$ iff $\mathcal{B} \models \psi(b)$.

1. ψ is atomic: Immediate since $\mathcal{A} \subseteq \mathcal{B}$.
2. $\psi = \neg\phi$ or $\phi \wedge \theta$: Immediate by the induction assumption.
3. $\psi = \exists v_i\phi(v_i, y)$: Two directions:

"$\Rightarrow$": If $\mathcal{A} \models \psi(b)$, then there is $c \in \mathcal{A}$ such that $\mathcal{A} \models \phi(c,b)$. By the induction assumption, $\mathcal{B} \models \phi(c,b)$ and thus $\mathcal{B} \models \psi(b)$.

"$\Leftarrow$": If $\mathcal{B} \models \psi(b)$, then by the assumption, there is $c \in \mathcal{A}$ such that $\mathcal{B} \models \phi(c,b)$. By the induction assumption, $\mathcal{A} \models \phi(c,b)$ and thus $\mathcal{A} \models \psi(b)$. □

Suppose that for all $\gamma < \beta < \alpha$, $\mathcal{A}_\gamma \subseteq \mathcal{A}_\beta$. Then $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$ is the structure $\mathcal{B}$ such that $dom(\mathcal{B}) = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$, $R_i^{\mathcal{B}} = \bigcup_{\beta<\alpha} R_i^{\mathcal{A}_\beta}$, $f_j^{\mathcal{B}} = \bigcup_{\beta<\alpha} f_j^{\mathcal{A}_\beta}$ and $c_k^{\mathcal{B}} = c_k^{\mathcal{A}_0}$.

We say that $(\mathcal{A}_\beta)_{\beta<\alpha}$ is an elementary chain if for all $\gamma < \beta < \alpha$, $\mathcal{A}_\gamma \preceq \mathcal{A}_\beta$.

4.7 Corollary. *Suppose that $(\mathcal{A}_\beta)_{\beta<\alpha}$ is an elementary chain and let $\mathcal{B} = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$. Then for all $\gamma < \alpha$, $\mathcal{A}_\gamma \preceq \mathcal{B}$. Furthermore, if for all $\beta < \alpha$, $\mathcal{A}_\beta \preceq \mathcal{C}$, then $\mathcal{B} \preceq \mathcal{C}$.*

Proof. We repeat the proof of Tarski-Vaught and proof by induction on $\phi(x)$, $x = (x_1, ..., x_n)$, that for all $a \in \mathcal{B}^n$, $\mathcal{B} \models \phi(a)$ iff $\mathcal{A}_\gamma \models \phi(a)$ for all $\gamma < \alpha$ such that $a \in \mathcal{A}_\gamma^n$. We prove the case $\phi = \exists v_i \psi(v_i, x)$, the other cases are left as an exercise:

"$\Rightarrow$": Then there is $b \in \mathcal{B}$ such that $\mathcal{B} \models \psi(b,a)$. But then we can find $\gamma \leq \beta < \alpha$ such that $b \in \mathcal{A}_\beta$. By the induction assumption, $\mathcal{A}_\beta \models \psi(b,a)$ and thus $\mathcal{A}_\beta \models \phi(a)$. Since $\mathcal{A}_\gamma \preceq \mathcal{A}_\beta$, $\mathcal{A}_\gamma \models \phi(a)$.

"$\Leftarrow$": Exactly as the direction "$\Rightarrow$" in the proof of Tarski-Vaught.

The furthermore part follows immediately from Tarski-Vaught (exercise). □

Recall that $L_{\omega\omega}$ is the set of all (L-)formulas. Notice that $|L_{\omega\omega}| = |L| + \omega$.

4.8 Lemma. *Suppose $A \subseteq \mathcal{A}$. Then there is $A \subseteq \mathcal{B} \preceq \mathcal{A}$ such that $|\mathcal{B}| \leq |A| + |L_{\omega\omega}|$ (i.e. the Löwenheim-Skolem number of the first-order logic is $|L_{\omega\omega}|$).*

Proof. For every formula $\phi(v_i, x)$, $x = (x_1, ..., x_n)$ we define a function $g_{\phi(v_i,x)} : \mathcal{A}^n \to \mathcal{A}$ so that if $\mathcal{A} \models \exists v_i \phi(v_i, a)$, then $\mathcal{A} \models \phi(g_{\phi(v_i,x)}(a), a)$. (Such functions are called Skolem functions, see Section 12.) Then we close $C = A \cup \{c^{\mathcal{A}} |\ c \in L,\ c \text{ is a constant symbol}\}$ under these function and under the functions $f^{\mathcal{A}}$, for all function symbols $f \in L$ (we could drop the elements $c^{\mathcal{A}}$ from C and forget the functions $f^{\mathcal{A}}$ and still get the same set) i.e. we let $B \subseteq \mathcal{A}$ be the intersection of all $D \subseteq \mathcal{A}$ such that $C \subseteq D$ and if $a \in D^n$ and g is an n-ary function as above, then $g(a) \in D$ (see Appendix). Then $|B| \leq |A| + |L_{\omega\omega}|$ (use Fact 2.8 or see [Je]). Then we let $\mathcal{B} = \mathcal{A} \restriction B$. By Tarski-Vaught, $\mathcal{B} \preceq \mathcal{A}$. □

4.9 Löwenheim-Skolem theorem. *If T is a theory and it has an infinite model, then it has a model in every cardinality $\kappa \geq |L_{\omega\omega}|$.*

Proof. By Lemma 4.5 T has a model $\mathcal{A}$ of cardinality $\geq \kappa$. By 4.8 we can find $\mathcal{B} \preceq \mathcal{A}$ of cardinality κ. Then $\mathcal{B} \models T$. □

4.10 Exercise. *Assume $\mathcal{A}$ is a structure.*

(i) Suppose $\phi(v_0, x)$, $x = (x_0, ..., x_n)$, is a formula, $a \in \mathcal{A}^n$ and $\phi(\mathcal{A}, a)$ is infinite. Show that there is $\mathcal{B} \succeq \mathcal{A}$ such that $\phi(\mathcal{B}, a) \not\subseteq \mathcal{A}$.

(ii) Suppose $X \subseteq \mathcal{A}$ is infinite. Show that there is $\mathcal{B} \succeq \mathcal{A}$ such that X is not definable in $\mathcal{B}$. Hint: Using (i), build a suitable elementary chain.

We say that a formula is $\forall\exists$-formula if it is of the form $\forall x_0 ... \forall x_n \exists y_0 ... \exists y_m \phi$, where ϕ is quantifier free. $\forall$-, $\exists$- and $\exists\forall$-formulas are defined similarly. Notice that all these classes are closed under disjunctions and conjunctions (i.e. if e.g. ϕ and ψ are $\forall\exists$-formula, then $\phi \wedge \psi$ is equivalent to a $\forall\exists$-formula). Also a negation of a $\forall\exists$-formula is equivalent to a $\exists\forall$-formula and similarly for the other classes.

We say that a theory T is $\forall$-axiomatizable if there is a theory T' that consists of $\forall$-sentences and has exactly the same models as T has. $\forall\exists$-axiomatizability is defined similarly. We say that T is closed under submodels if $\mathcal{A} \subseteq \mathcal{B} \models T$ implies $\mathcal{A} \models T$.

4.11 Exercise. *Show that T is closed under submodels iff T is $\forall$-axiomatizable. Hint: For the non-trivial direction, show that if T' is the set of all $\forall$-sentences true in every model of T, then every model of T' is a submodel of a model of T.*

4.12 Theorem. *The following are equivalent:*
(i) T is closed under unions, see Definition 5.3 below,
(ii) T is $\forall\exists$-axiomatizable.

Proof. . (ii)$\Rightarrow$(i): Easy (exercise).
(i)$\Rightarrow$(i): Let T' be the set of all $\forall\exists$-sentences true in every model of T. For a contradiction suppose that there are $\mathcal{A}_0 \models T'$ and $\theta \in T$ such that $\mathcal{A}_0 \models \neg\theta$.

4.12.1 Claim. *There is $\mathcal{B} \models T$ such that for all $\exists\forall$-sentences ϕ, if $\mathcal{A}_0 \models \phi$, then $\mathcal{B} \models \phi$.*

Proof. If not, by compactness, there are $\exists\forall$-sentences $\phi_0, ..., \phi_n$ true in $\mathcal{A}_0$ such that $T \models \neg\phi_0 \vee ... \vee \neg\phi_0$. Since this disjunction is equivalent to a $\forall\exists$-formula, this contradicts the choice of $\mathcal{A}_0$. □ Claim 4.12.1.

4.12.2 Claim. *There are $\mathcal{A}_0 \subseteq \mathcal{B}_0 \subseteq \mathcal{A}_1$ such that $\mathcal{A}_0 \preceq \mathcal{A}_1$ and $\mathcal{B}_0 \equiv \mathcal{B}$ (i.e. $Th(\mathcal{B}_0) = Th(\mathcal{B})$)*

Proof. Let T_0 be the set of all $\forall$-sentences in vocabulary $L(\mathcal{A}_0)$ true in the model $(\mathcal{A}_0, \mathcal{A}_0)$. Then by Claim 4.12.1, $T_0 \cup Th(\mathcal{B})$ is consistent and we let $\mathcal{B}_0$ be a model of this theory. As before we can choose $\mathcal{B}_0$ so that $\mathcal{A}_0 \subseteq \mathcal{B}_0$. Let T_1 be the set of all quantifier free sentences in the vocabulary $L(\mathcal{B}_0)$ true in $(\mathcal{B}_0, \mathcal{B}_0)$. Then by the choice of $\mathcal{B}_0$, $T_1 \cup Th(\mathcal{A}_0, \mathcal{A}_0)$ is consistent (exercise) and we let $\mathcal{A}_1$ be a model of this theory. Again we can choose $\mathcal{A}_1$ so that $\mathcal{B}_0 \subseteq \mathcal{A}_1$ and then $\mathcal{A}_0 \preceq \mathcal{A}_1$. □ Claim 4.12.2.

By applying Claim 4.12.2 ω many times we can find $\mathcal{A}_i$, $\mathcal{B}_i$, $i < \omega$, so that for all $i < \omega$, $\mathcal{A}_i \subseteq \mathcal{B}_i \subseteq \mathcal{A}_{i+1}$, $\mathcal{A}_i \preceq \mathcal{A}_{i+1}$ and $\mathcal{B}_i \equiv \mathcal{B}$. Let $\mathcal{C} = \cup_{i<\omega} \mathcal{B}_i = \cup_{i<\omega} \mathcal{A}_i$. Then $\mathcal{C} \models T$ by (i) and the fact that $T \subseteq Th(\mathcal{B})$ but by Corollary 4.7, $\mathcal{C} \models \neg\theta$, a contradiction. □

5. Completeness and elimination of quantifiers

5.1 Definition.

(i) We say that a theory T is complete if for all sentences ϕ, either $T \models \phi$ or $T \models \neg\phi$.

(ii) We say that T is κ-categorical if upto isomorphisms T has exactly one model of cardinality κ.

Often when one talks about complete theories, one assumes also that T is consistent (inconsistent theories are not usually considered interesting). In fact unless otherwise stated, whenever we talk about a theory T, T is assumed to be consistent.

5.2 Lemma (Łos-Vaught). *If T is κ-categorical for some $\kappa \geq |L_{\omega\omega}|$ and T does not have finite models, then T is complete.*

Proof. Suppose not. Let ϕ be a sentence that witnesses this. Then by Löwenheim-Skolem both $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ have a model of size κ. This contradicts the assumption that T is κ-categorical. □

5.3 Definition. *We say that T is closed under unions if for all $\mathcal{A}_i \models T$, $i < \alpha$, the following holds: If for all $i < j < \alpha$, $\mathcal{A}_i \subseteq \mathcal{A}_j$, then $\bigcup_{i<\alpha} \mathcal{A}_i \models T$.*

5.4 Exercise. *Give an example of an $\exists\forall$-axiomatizable theory that is not closed under unions.*

5.5 Definition.

(i) T has quantifier free set amalgamation (AP for short) if for all $\mathcal{A}, \mathcal{B} \models T$ and partial isomorphism $f : \mathcal{A} \to \mathcal{B}$ there are $\mathcal{B} \subseteq \mathcal{C} \models T$ and an embedding $g : \mathcal{A} \to \mathcal{C}$ such that $g \restriction dom(f) = f$.

(ii) T has quantifier free joint embedding (JEP for short) if for all $\mathcal{A}, \mathcal{B} \models T$ there are $\mathcal{B} \subseteq \mathcal{C} \models T$ and an embedding $f : \mathcal{A} \to \mathcal{C}$.

5.6 Lemma. *If T has AP and there is $\mathcal{A}$ (not necessarily a model of T) such that for all $\mathcal{B} \models T$ there is an embedding $f : \mathcal{A} \to \mathcal{B}$, then T has JEP.*

Proof. Exercise. □

5.7 Lemma. *Suppose $\kappa \geq |L_{\omega\omega}|$ and T has JEP and is closed under unions. Then there is a model $\mathcal{A}$ of T such that for all $\mathcal{B} \models T$ of power $\leq \kappa$, there is an embedding $f : \mathcal{B} \to \mathcal{A}$.*

Proof. Let $\mathcal{B}_i$, $i < \alpha$, list all models of T of power $\leq \kappa$ i.e. if $\mathcal{B}$ is a model of T of power $\leq \kappa$, then $\mathcal{B}$ is isomorphic with some $\mathcal{B}_i$, $i < \alpha$. By recursion on $i \leq \alpha$, we define models $\mathcal{A}_i$ of T as follows:

$i = 0$: $\mathcal{A}_i = \mathcal{B}_0$.

$i = j + 1$: By JEP we choose $\mathcal{A}_j \subseteq \mathcal{A}_i \models T$ such that there is an embedding $f_j : \mathcal{B}_j \to \mathcal{A}_i$. Notice that earlier constructed embeddings $f_k : \mathcal{A}_k \to \mathcal{A}_{k+1}$, $k < j$, are also embeddings to $\mathcal{A}_i$ since $\mathcal{A}_{k+1} \subseteq \mathcal{A}_i$ (see also the limit case below).

i is limit: $\mathcal{A}_i = \bigcup_{j<i} \mathcal{A}_j$.

Clearly $\mathcal{A} = \mathcal{A}_\alpha$ is as wanted. □

5.8 Lemma. *Suppose T has AP and is closed under unions and $\mathcal{A} \models T$. Then there is $\mathcal{A} \subseteq \mathcal{B} \models T$ such that for all partial isomorphisms $f : \mathcal{A} \to \mathcal{A}$ there is an embedding $g : \mathcal{A} \to \mathcal{B}$ such that $g \restriction dom(f) = f$.*

Proof. Let f_i, $i < \alpha$, list all partial isomorphism from $\mathcal{A}$ to $\mathcal{A}$. We define models $\mathcal{B}_i$ of T, $i \le \alpha$, as follows:

$i = 0$: $\mathcal{B}_i = \mathcal{A}$.

$i = j + 1$: Now f_j is a partial isomorphism from $\mathcal{A}$ also to $\mathcal{B}_j$ since $\mathcal{A} \subseteq \mathcal{B}_j$ and thus by AP there is $\mathcal{B}_j \subseteq \mathcal{B}_i \models T$ and an embedding $g_j : \mathcal{A} \to \mathcal{B}_i$ such that $g_j \restriction dom(f_j) = f_j$. As in the proof of Lemma 5.7, notice that the embeddings $g_k : \mathcal{A} \to \mathcal{B}_{k+1}$ constructed earlier for $k < j$, are also embeddings from $\mathcal{A}$ to $\mathcal{B}_i$ since $\mathcal{B}_{k+1} \subseteq \mathcal{B}_i$.

i is limit: $\mathcal{B}_i = \bigcup_{j<i} \mathcal{B}_j$.

Clearly $\mathcal{B} = \mathcal{B}_\alpha$ is as wanted. □

5.9 Definition. *We say that $\mathcal{A} \models T$ is existentially closed if for all $\mathcal{A} \subseteq \mathcal{B} \models T$, atomic or negated atomic formulas $\phi_i(v_k, x)$, $i < n$ and $x = (x_1, ..., x_m)$, and $a \in \mathcal{A}^m$, the following holds: if $\mathcal{B} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$ then $\mathcal{A} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$.*

5.10 Lemma. *If $\mathcal{A} \models T$ is existentially closed, then for all $\mathcal{A} \subseteq \mathcal{B} \models T$, quantifier free formula $\phi(v_k, x)$, $x = (x_1, ..., x_m)$, and $a \in \mathcal{A}^m$, the following holds: if $\mathcal{B} \models \exists v_k \phi(v_k, a)$ then $\mathcal{A} \models \exists v_k \phi(v_k, a)$.*

Proof. Exercise. (Hint: By e.g. the course Logiikka I, every quantifier free formula $\phi(x)$ is equivalent with a formula of the form $\vee_{i<n} \wedge_{j<m} \phi_{ij}(x)$, where each ϕ_{ij} is atomic or negated atomic formula.) □

5.11 Theorem. *Suppose $\kappa \ge |L_{\omega\omega}|$ and T has AP, JEP and is closed under unions. Then there is a model $\mathcal{A}$ of T such that for all $\mathcal{B} \models T$ of power $\le \kappa$, there is an embedding $f : \mathcal{B} \to \mathcal{A}$ and for all partial isomorphisms $f : \mathcal{A} \to \mathcal{A}$ of power $\le k$, there is an automorphism g of $\mathcal{A}$ such that $g \restriction dom(f) = f$. Furthermore, such a model is existentially closed.*

Proof. By recursion on $i \le \kappa^+$ we define models $\mathcal{A}_i$ of T as follows:

$i = 0$: We let $\mathcal{A}_i$ be as given by Lemma 5.7.

$i = j + 1$: By Lemma 5.8 we can find $\mathcal{A}_j \subseteq \mathcal{A}_i \models T$ such that every partial isomorphism from $\mathcal{A}_j$ to $\mathcal{A}_j$ extends to an embedding from $\mathcal{A}_j$ to $\mathcal{A}_i$.

i is limit: $\mathcal{A}_i = \bigcup_{j<i} \mathcal{A}_j$.

We show that $\mathcal{A} = \mathcal{A}_{\kappa^+}$ is as wanted. Clearly $\mathcal{A}$ has the first of the required properties. For the second, let $f : \mathcal{A} \to \mathcal{A}$ be a partial isomorphism of power $\le \kappa$. Then by Fact 2.8 (v), there is $\alpha < \kappa^+$ such that $dom(f) \cup rng(f) \subseteq \mathcal{A}_\alpha$. By recursion on $\alpha \le i < \kappa^+$ we define an increasing sequence of partial isomorphism $f_i : \mathcal{A}_{i+1} \to \mathcal{A}_{i+1}$ as follows:

$i = \alpha$: $f_i = f$.

$i = j + 1$: Since f_j is a partial isomorphism from $\mathcal{A}_i \to \mathcal{A}_i$, by the choice of $\mathcal{A}_{i+1}$, there is a an embedding $f_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$.

i is limit: Let $f_i^* = \bigcup_{\alpha \le j < i} f_j$. Then $(f_i^*)^{-1}$ is a partial isomorphism from $\mathcal{A}_i$ to $\mathcal{A}_i$ and thus there is an embedding $g : \mathcal{A}_i \to \mathcal{A}_{i+1}$ such that $g \restriction rng(f_i^*) = (f_i^*)^{-1}$. We let $f_i = g^{-1}$.

Then $g = \bigcup_{\alpha \le i < \kappa^+} f_i$ is a partial isomorphism, $dom(g) = \mathcal{A}$ and $rng(g) = \mathcal{A}$ i.e. g is an isomorphism and clearly it extends f.

To prove the furthermore part, let $\mathcal{B}$, $a = (a_1, ..., a_m)$ and $\phi_i(v_k, x)$, $i < n$, be as in the definition of existentially closed. Choose $b \in \mathcal{B}$ such that $\mathcal{B} \models \wedge_{i<n} \phi_i(b, a)$ and by Lemma 4.8 choose $\mathcal{C} \preceq \mathcal{B}$ of power $\le \kappa$ such that $a \in \mathcal{C}^m$ and $b \in \mathcal{C}$. Then $\mathcal{C} \models T$ and there is an embedding $f : \mathcal{C} \to \mathcal{A}$. Now $g = (f \restriction \{a_1, ..., a_m\})^{-1}$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{A}$ and thus there is an automorphism h' of $\mathcal{A}$ such that $h' \restriction dom(g) = g$. Let $h = h' \restriction rng(f)$. Then $h \circ f$ is an embedding of $\mathcal{C}$ to $\mathcal{A}$ and for all $1 \le j \le m$, $(h \circ f)(a_j) = a_j$. Since ϕ_i, $i < n$, are atomic or negated atomic, $\mathcal{A} \models \phi_i((h \circ f)(b), (h \circ f)(a))$. Thus $\mathcal{A} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$. □

5.12 Definition. *Let $\mathcal{K}$ be a class of L-structure. We say that $\mathcal{K}$ has the elimination of quantifiers if for all formulas $\phi(x)$, $x = (x_1, ..., x_n)$, there is a quantifier free formula $\psi(x)$ such that for all $\mathcal{A} \in \mathcal{K}$, $\mathcal{A} \models \forall x_1 ... \forall x_n (\phi(x) \leftrightarrow \psi(x))$. If the class of all models of T has the elimination of quantifiers, we say that T has the elimination of quantifiers.*

5.13 Theorem. *Suppose T has AP, JEP and is closed under unions. If T^* is such a theory that its models are exactly the existentially closed models of T, then T^* is complete and it has the elimination of quantifiers.*

Proof. Let $\kappa \ge |L_{\omega\omega}|$ and $\mathcal{A}$ be as in Theorem 5.11. Since $\mathcal{A}$ is existentially closed, $\mathcal{A} \models T^*$. The completeness of T^* follows easily from the elimination of quantifiers and the existence of $\mathcal{A}$ (exercise). To prove the elimination of quantifiers, we prove by a simultaneous induction on $\phi(x)$, $x = (x_1, ..., x_n)$ that

(i) if $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B} \models T^*$ and $a \in \mathcal{B}^n$, then $\mathcal{B} \models \phi(a)$ iff $\mathcal{A} \models \phi(a)$,

(ii) there is quantifier free $\psi(x)$ such that $T^* \models \forall x_1 ... \forall x_n (\phi(x) \leftrightarrow \psi(x))$.

The steps ϕ atomic, $\phi = \neg\theta$ and $\phi = \theta \wedge \theta'$ are trivial. So we assume that $\phi(x) = \exists v_i \theta(v_i, x)$.

Proof of (i): If $\mathcal{B} \models \phi(a)$ then for some $b \in \mathcal{B}$, $\mathcal{B} \models \theta(b, a)$ and so by (i) of the induction assumption, $\mathcal{A} \models \theta(b, a)$, so $\mathcal{A} \models \phi(a)$. Then suppose $\mathcal{A} \models \phi(a)$. By (ii) in the induction assumption, let ψ be quantifier free such that

(*) $T^* \models \forall v_i \forall x_1 ... \forall x_n (\theta(v_i, x) \leftrightarrow \psi(v_i, x))$.

Now $\mathcal{A} \models \exists v_i \psi(v_i, a)$ and since $\mathcal{B}$ is existentially closed $\mathcal{B} \models \exists v_i \psi(v_i, a)$. By (*), $\mathcal{B} \models \phi(a)$.

Proof of (ii): For $a \in \mathcal{A}^n$ and $x = (x_1, ..., x_n)$, let

$$t_{at}^x(a/\emptyset; \mathcal{A}) = \{\theta(x) | \ \theta \text{ atomic or negated atomic, } \mathcal{A} \models \theta(a)\}.$$

We write $(\mathcal{B}, b) \models t_{at}^x(a/\emptyset; \mathcal{A})$ if $\mathcal{B} \models \theta(b)$ for all $\theta(x) \in t_{at}^x(a/\emptyset; \mathcal{A})$.

1 Claim. *Suppose $\mathcal{A} \models \phi(a)$, $\mathcal{B} \models T^*$ and $(\mathcal{B}, b) \models t_{at}^x(a/\emptyset; \mathcal{A})$. Then $\mathcal{B} \models \phi(b)$.*

Proof of Claim 1: Suppose not. Let $\mathcal{B}$ and b witness this. By Lemma 4.8 we may assume that $|\mathcal{B}| \leq \kappa$. By the choice of $\mathcal{A}$, there is an embedding of $\mathcal{B}$ to $\mathcal{A}$ and so we may assume that $\mathcal{B} \subseteq \mathcal{A}$. Then $(\mathcal{A}, b) \models t^x_{at}(a/\emptyset; \mathcal{A})$ and thus $b_i \mapsto a_i$, $1 \leq i \leq n$, is a partial isomorphism and thus there is an automorphism f of $\mathcal{A}$ such that $f(b_i) = a_i$ for all $1 \leq i \leq n$. But then $\mathcal{A} \models \neg\phi(a)$, since $\mathcal{A} \models \neg\phi(b)$ by (i) and the choice of $\mathcal{B}$ and b, a contradiction. □ Claim 1.

2 Claim. *Suppose $\mathcal{A} \models \phi(a)$. Then there is finite $q \subseteq t^x_{at}(a/\emptyset; \mathcal{A})$ such that $T^* \models \forall x_1 ... \forall x_n((\wedge q) \rightarrow \phi)$.*

Proof of Claim 2: By Claim 1, $T^* \cup t^x_{at}(a/\emptyset; \mathcal{A}) \models \phi(x)$ and thus the claim follows from compactness. □ Claim 2.

Let p_i, $i < \alpha$, enumerate the set $\{t^x_{at}(a/\emptyset; \mathcal{A}) | \ \mathcal{A} \models \phi(a)\}$. Let $q_i \subseteq p_i$ be as in Claim 2 and $\psi_i(x) = \wedge q_i$.

Claim 3. *If $\mathcal{B} \models T^*$ and $\mathcal{B} \models \phi(b)$, then for some $i < \alpha$, $\mathcal{B} \models \psi_i(b)$.*

Proof of Claim 3: Suppose not. Then as in the proof of Claim 1, we can find $\mathcal{B}$ and b witnessing this so that $\mathcal{B} \subseteq \mathcal{A}$. Then $\mathcal{A} \models \neg\psi_i(b)$ for all $i < \alpha$ and by (i), $\mathcal{A} \models \phi(b)$. This contradicts the fact that p_i, $i < \alpha$, enumerates the set $\{t^x_{at}(a/\emptyset; \mathcal{A}) | \ \mathcal{A} \models \phi(a)\}$. □ Claim 3.

So $T^* \cup \{\neg\psi_i(x) | \ i < \alpha\} \models \neg\phi(x)$. By compactness, there is finite $X \subseteq \alpha$ such that $T^* \models \forall x_1 ... \forall x_n(\wedge_{i \in X} \neg\psi_i(x) \rightarrow \neg\phi(x))$ i.e. $T^* \models \forall x_1 ... \forall x_n(\phi(x) \rightarrow \vee_{i \in X} \psi_i(x))$. Since for all $i \in X$, $T^* \models \forall x_1 ... \forall x_n(\psi_i(x) \rightarrow \phi(x))$,

$$T^* \models \forall x_1 ... \forall x_n(\phi(x) \leftrightarrow \vee_{i \in X} \psi_i(x)).$$

□

5.14 Definition. *We say that T is model complete if for all $\mathcal{A}, \mathcal{B} \models T$, $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{A} \preceq \mathcal{B}$.*

5.15 Lemma. *If T has the elimination of quantifiers, then T is model complete.*

Proof. Exercise. □

5.16 Exercise.

(i) In section 1 we said that the atomic formula $\top$ is needed in the proof of Theorem 5.13. Where was it needed in the proof?

(ii) Show that in Lemma 5.2 the assumption that T does not have finite models is necessary.

5.17 Exercise. *Let $L = \{<\}$, $<$ is a 2-ary predicate symbols, and let T_{lo} (lo for linear ordering) consist of the following sentences:*

$\forall v_0 \forall v_1 \forall v_2((v_0 < v_1 \wedge v_1 < v_2) \rightarrow v_0 < v_2)$

$\forall v_0 \forall v_1(v_0 < v_1 \rightarrow \neg v_1 < v_0)$

$\forall v_0 \forall v_1(v_0 < v_1 \vee v_0 = v_1 \vee v_1 < v_0)$.

Show that T_{lo} has AP, JEP and is closed under unions and find a theory T so that the models of T are exactly the existentially closed models of T_{lo}.

5.18 Exercise. *Let $L = \{E\}$, E is a 2-ary predicate symbols, and let T_{gr} (gr for graph) consist of the following two sentences:*
$\forall v_0 \neg E(v_0, v_0)$
$\forall v_0 \forall v_1 (E(v_0, v_1) \to E(v_1, v_0))$.
Show that T_{gr} has AP, JEP and is closed under unions and find a theory T so that the models of T are exactly the existentially closed models of T_{gr}.

Next example shows that from AP, JEP and closed under unions alone one can not deduce that the class of existentially closed models of T has the elimination of quantifiers. And so there may not be a theory T^* whose models are exactly the existentially closed models of T.

5.19 Example. *Let L consist of unary relation symbols P_i, $i < 3$, binary relation symbols R and Q and constants c_i, $i < \omega$. Let T be an L-theory such that $\mathcal{A} \models T$ if*
(a) $P_i^{\mathcal{A}}$, $i < 3$, form a partition of $dom(\mathcal{A})$,
(b) $c_i^{\mathcal{A}}$, $i < \omega$, are distinct and belong to $P_0^{\mathcal{A}}$,
(c) if $(a, b) \in Q^{\mathcal{A}}$, then $a \in P_0^{\mathcal{A}}$ and $b \in P_1^{\mathcal{A}}$,
(d) if $(a, b) \in R^{\mathcal{A}}$, then $a \in P_1^{\mathcal{A}}$ and $b \in P_2^{\mathcal{A}}$,
(e) for all $i < \omega$ and $a \in P_1^{\mathcal{A}}$, if $(c_i^{\mathcal{A}}, a) \in Q^{\mathcal{A}}$, then $(a, b) \in R^{\mathcal{A}}$ for all $b \in P_2^{\mathcal{A}}$.
It is easy to see (exercise) that T has AP, JEP and is closed under unions. Also if $\mathcal{A}$ is an existentially closed model of T and $a \in \mathcal{A}$, then a satisfies the formula $\phi = \forall v_1 (P_2(v_1) \to R(v_0, v_1))$ iff for some $i < \omega$, $(c_i^{\mathcal{A}}, a) \in Q^{\mathcal{A}}$ (exercise). But then, in the class of existentially closed models of T, one can not eliminate quantifiers from ϕ (exercise).

5.20 Exercise. *Let $L = \{P_0, P_1, R\}$ be a vocabulary such that P_0 and P_1 are unary relation symbols and R is a binary relation symbol. Let $\mathcal{A}$ be an L-structure such that $dom(\mathcal{A}) = \omega$, $P_0^{\mathcal{A}} = \{0\}$, $P_1^{\mathcal{A}} = \{1\}$ and $R^{\mathcal{A}} = \{(n, 0) |\ 1 < n < \omega \text{ even}\} \cup \{(n, 1) |\ 1 < n < \omega \text{ odd}\}$. Show that $Th(\mathcal{A})$ is model complete but does not have the elimination of quantifiers.*

5.21 Exercise. *Find a theory T such that it is model complete but not complete.*

Suppose $\mathcal{A} \subseteq \mathcal{B}$. We say that $\mathcal{A}$ is strongly existentially closed in $\mathcal{B}$ if for all $\exists$-formulas $\phi(x)$ the following holds: if $a \in \mathcal{A}^n$ and $\mathcal{B} \models \phi(a)$, then $\mathcal{A} \models \phi(a)$. We say that $\mathcal{A}$ is a strongly existentially closed model of a theory T, if $\mathcal{A} \models T$ and for all $\mathcal{A} \subseteq \mathcal{B} \models T$, $\mathcal{A}$ is strongly existentially closed in $\mathcal{B}$.

5.22 Exercise. *Suppose T and T^* satisfy the assumptions of Theorem 5.13. Show that every existentially closed model of T is a strongly existentially closed closed model of T.*

5.23 Exercise.
(i) Suppose $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C}$ are models of T, $\mathcal{A}$ is strongly existentially closed in $\mathcal{B}$ and $f : \mathcal{A} \to \mathcal{C}$ is an embedding. Show that there are $\mathcal{C} \subseteq \mathcal{D} \models T$ and an embedding $g : \mathcal{B} \to \mathcal{D}$ such that $f \subseteq g$.

(ii) Show that if $\mathcal{A} \subseteq \mathcal{B}$ are models of T, $\mathcal{B}$ is a strongly existentially closed model of T and $\mathcal{A}$ is strongly existentially closed in $\mathcal{B}$, then $\mathcal{A}$ is a strongly existentially closed model of T.

(iii) Suppose T is closed under unions, $\mathcal{B} \models T$ and $\mathcal{A} \subseteq \mathcal{B}$ is strongly existentially closed. Show that $\mathcal{A} \models T$.

We say that $\mathcal{A} \models T$ is κ-existentially closed model of T if for all $A \subseteq \mathcal{A} \subseteq \mathcal{B} \models T$ and $A \subseteq \mathcal{C} \subseteq \mathcal{B}$ the following holds: if $|\mathcal{C}| < \kappa$, then there is and embedding $f : \mathcal{C} \to \mathcal{A}$ such that $f \restriction A = id_A$. Notice that if $\kappa > |L_{\omega\omega}|$, then κ-existentially closed models of T are strongly existentially closed models of T.

5.24 Exercise. *Suppose $\kappa > |L_{\omega\omega}|$ and T is closed under unions and has JEP. Show that there is $\mathcal{A} \models T$ such that:*

(i) $\mathcal{A}$ is a κ-existentially closed model of T,

(ii) for all $\mathcal{B} \models T$ of power $< \kappa$, there is an embedding $f : \mathcal{B} \to \mathcal{A}$,

(iii) if $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ are strongly existentially closed models of T of power $< \kappa$ and $f : \mathcal{B} \to \mathcal{C}$ is an isomorphism, then there is an automorphism g of $\mathcal{A}$ such that $f \subseteq g$.

Hint: Notice that for every $\mathcal{B} \models T$, there is $\mathcal{B} \subseteq \mathcal{C} \models T$ such that $\mathcal{C}$ is a κ-existentially closed model of T. Then just repeat the proof of Theorem 5.11.

5.25 Exercise. *Let κ, T and $\mathcal{A}$ be as in Exercise 5.24. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a partial isomorphism such that $|dom(f)| < \kappa$ and for all $\exists$-formulas $\phi(x)$ and $a \in dom(f)^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{A} \models \phi(f(a))$. Show that there is an automorphism g of $\mathcal{A}$ such that $f \subseteq g$. Hint: Find first a strongly existentially model $\mathcal{C} \subseteq \mathcal{A}$ such that $rng(f) \subseteq \mathcal{C}$ and $|\mathcal{C}| < \kappa$. Then using compactness, find $\mathcal{A} \subseteq \mathcal{B} \models T$ such that there are $\mathcal{D} \subseteq \mathcal{B}$ and an isomorphism $h : \mathcal{D} \to \mathcal{C}$ such that $f \subseteq h$.*

6. Example: Algebraically closed fields

We return to the example from section 3. So in this section $L = \{+, \times, -, 0, 1\}$, and we study the theory T_{f_0}. Instead of 0 we could work also with any positive characteristic p, only changes needed would be that we should replace $\mathbf{Q}$ and $\mathbf{Z}$ by the p element field F_p.

6.1 Lemma. *For all polynomials $P \in \mathbf{Z}[x_1, ..., x_n]$ there is a term $t(x_1, ..., x_n)$ such that for all $\mathcal{A} \models T_{f_0}$ and $a \in \mathcal{A}^n$, $P(a) = t^{\mathcal{A}}(a)$ and vice versa.*

Proof. Exercise. □

So for every atomic formula $\phi(x)$, $x = (x_1, ..., x_n)$, there is a polynomial $P \in \mathbf{Z}[x_1, ..., x_n]$ such that for all $\mathcal{A} \models T_{f_0}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ iff $P(a) = 0$.

T_{acf_0} is the theory we get from T_{f_0} by adding the sentences

$\forall v_0 ... \forall v_n (\neg v_n = 0 \rightarrow \exists v_{n+1} \Sigma_{i=0}^{i=n} v_i \times v_{n+1}^i = 0)$,

for all $n \in \mathbb{N} - \{0\}$. Then $\mathcal{A} \models T_{acl_0}$ if $\mathcal{A}$ is an algebraically closed closed field of characteristic 0.

We need few facts from algebra.

6.2 Fact.

(i) Every field $\mathcal{A}$ can be extended to an algebraically closed field $\mathcal{B}$. Furthermore, this can be done so that there is $a \in \mathcal{B}$ such that for all non-zero $P \in \mathcal{A}[X]$, $P(a) \neq 0$ (i.e. a is not algebraic over $\mathcal{A}$).

(ii) If $\mathcal{A}, \mathcal{B} \models T_{f_0}$, $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{B}$ (i.e. $\mathcal{C}$ and $\mathcal{D}$ are subrings) and $f : \mathcal{C} \to \mathcal{D}$ is an isomorphism, then there is an isomorphism g between the fields generated by $\mathcal{C}$ and $\mathcal{D}$ such that $g \restriction \mathcal{C} = f$.

(iii) If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \models T_{f_0}$, $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{B}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are algebraic over $\mathcal{C}$ and $\mathcal{D}$, respectively, $P \in \mathcal{C}[X]$ is the minimal polynomial of a, $f : \mathcal{C} \to \mathcal{D}$ is an isomorphism and $f(P)(b) = 0$, then there is an isomorphism $g : \mathcal{C}(a) \to \mathcal{D}(b)$ such that $g \restriction \mathcal{C} = f$ and $g(a) = b$. (Here $\mathcal{C}(a)$ is the field generated by $\mathcal{C} \cup \{a\}$ and $f(\Sigma_{i=0}^{i=n} c_i X^i) = \Sigma_{i=0}^{i=n} f(c_i) X^i$.)

(iv) If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \models T_{f_0}$, $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{B}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are not algebraic over $\mathcal{C}$ and $\mathcal{D}$ respectively and $f : \mathcal{C} \to \mathcal{D}$ is an isomorphism, then there is an isomorphism $g : \mathcal{C}(a) \to \mathcal{D}(b)$ such that $g \restriction \mathcal{C} = f$ and $g(a) = b$.

Proof. See the course Algebra II. □

6.3 Lemma. *T_{f_0} has AP, JEP and is closed under unions.*

Proof. T_{f_0} is closed under unions by Lemma 5.4 and since for all $\mathcal{A} \models T_{f_0}$, there is an embedding $f : (\mathbf{Z}, +, \times, -, 0, 1) \to \mathcal{A}$, JEP follows from AP and Lemma 5.6. So we are left to prove AP.

So suppose $\mathcal{A}, \mathcal{B} \models T_{f_0}$ and $f : \mathcal{A} \to \mathcal{B}$ is a partial isomorphism. By recursion on ordinals i we define subfields $\mathcal{A}_i$ of $\mathcal{A}$, algebraically closed fields $\mathcal{C}_i \supseteq \mathcal{B}$ and embeddings $f_i : \mathcal{A}_i \to \mathcal{C}_i$ as follows:

$i = 0$: We start by letting $C = \{t^{\mathcal{A}}(a) |\ t(x_1, ..., x_n)$ a term, $a \in dom(f)^n\}$ and $D = \{t^{\mathcal{B}}(b) |\ t(x_1, ..., x_n)$ a term, $b \in rng(f)^n\}$. When equipped with the induced structure, C and D are subrings of $\mathcal{A}$ and $\mathcal{B}$, respectively, and since f is a partial isomorphism $g(t^{\mathcal{A}}(a)) = t^{\mathcal{B}}(f(a))$ is an isomorphism from C to D such that $g \restriction dom(f) = f$ (by Exercise 4.2 (vii)). Let $\mathcal{A}_0$ be the subfield generated by C and $\mathcal{D}$ be the subfield generated by D. By Fact 6.2 (ii), there is an isomorphism $f_0 : \mathcal{A}_0 \to \mathcal{D}$ such that $f_0 \restriction C = g$. By Fact 6.2 (i), we let $\mathcal{C}_0$ be any algebraically closed field containing $\mathcal{B}$.

$i = j + 1$: If $\mathcal{A}_j = \mathcal{A}$, then f_j is the require embedding of $\mathcal{A}$ to $\mathcal{C}_j \supseteq \mathcal{B}$.

So suppose $a \in \mathcal{A} - \mathcal{A}_j$. There are two cases:

(a) a is algebraic over $\mathcal{A}_j$: We let $\mathcal{C}_i = \mathcal{C}_j$, $\mathcal{A}_i = \mathcal{A}_j(a)$ and $P \in \mathcal{A}_j[X]$ be the minimal polynomial of a over $\mathcal{A}_j$. Since $\mathcal{C}_i$ is algebraically closed, there is $b \in \mathcal{C}_i$ such that $f(P)(b) = 0$. By Fact 6.2 (iii) there is an isomorphism $f_i : \mathcal{A}_i \to rng(f_j)(b)$ such that $f_i \restriction \mathcal{A}_j = f_j$.

(b) a is not algebraic: Let $\mathcal{A}_i = \mathcal{A}_j(a)$ and choose an algebraically closed $\mathcal{C}_i \supseteq \mathcal{C}_j$ such that some $b \in \mathcal{C}_i$ is not algebraic over $\mathcal{C}_j$. Then b is not algebraic over $rng(f_j)$ and thus by Fact 6.2 (iv) there is an isomorphism $f_i : \mathcal{A}_i \to rng(f_j)(b)$ such that $f_i \restriction \mathcal{A}_j = f_j$.

i is limit: Let $\mathcal{A}_i = \bigcup_{j<i} \mathcal{A}_j$, $f_i = \bigcup_{j<i} f_j$, $\mathcal{C}_i = \bigcup_{j<i} \mathcal{C}_j$. □

6.4 Theorem. *T_{acf_0} is complete and has the elimination of quantifiers.*

Proof. By Lemma 6.3 and Theorem 5.13, it is enough to show that for all $\mathcal{A} \models T_{f_0}$, $\mathcal{A}$ is existentially closed iff $\mathcal{A} \models T_{acf_0}$. By Fact 6.2 (i), the claim from left to right is clear. So we prove the other direction. So suppose $\mathcal{A} \models T_{acf_0}$, $\mathcal{A} \subseteq \mathcal{B} \models T_{f_0}$, $\phi_i(v_k, x)$, $i < n$, are atomic or negated atomic formulas, $a \in \mathcal{A}^m$ and $\mathcal{B} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$. Let $b \in \mathcal{B}$ be such that $\mathcal{B} \models \wedge_{i<n}\phi_i(b, a)$. There are two cases:

1. b is algebraic over $\mathcal{A}$: Since $\mathcal{A}$ is algebraically closed, $b \in \mathcal{A}$ (exercise, see Fact 6.2 (iii)) and thus $\mathcal{A} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$.

2. b is not algebraic over $\mathcal{A}$: Then w.o.l.g. each $\phi_i(v_k, a)$ is of the form $\neg P_i(v_k, a) = 0$ (or $0 = 0$), where $P_i(v_k, a) = \Sigma_{i=0}^{i=l} b_i v_k^i$ where each $b_i \in \{a_1, ..., a_m\}$. Since each polynomial $P_i(v_k, a)$ has only finitely many roots and $\mathcal{A}$ is infinite, there is $c \in \mathcal{A}$ such that $P_i(c, a) \neq 0$ for all $i < n$. Thus $\mathcal{A} \models \exists v_k \wedge_{i<n} \phi_i(v_k, a)$. □

6.5 Remark. *So $\{\phi |\ (\mathbf{C}, +, \times, -, 0, 1) \models \phi\}$ is recursive (i.e. computer can be programmed to tell for all sentences ϕ, whether $(\mathbf{C}, +, \times, -, 0, 1) \models \phi$ or not, see the course Matemaattinen logiikka). Similarly one (Tarski) can show that $\{\phi |\ (\mathbf{R}, +, \times, -, 0, 1) \models \phi\}$ is recursive. The following is a famous open question: Is $\{\phi |\ (\mathbf{R}, +, \times, -, exp, 0, 1) \models \phi\}$ recursive? Schanuel's conjecture (see Section 11) implies that the answer is yes. By Exercise 1.12, $\{\phi |\ (\mathbf{C}, +, \times, -, exp, 0, 1) \models \phi\}$ knows which Diophantine equations $P(X_1, ..., X_n) = 0$, $P(X_1, ..., X_n) \in \mathbf{Z}[X_1, ..., X_n]$, have an integer root and thus by the negative answer to Hilbert's 10th problem (due to M.Davis, Y.Matiyasevich, H.Putnam and J.Robinson) $\{\phi |\ (\mathbf{C}, +, \times, -, exp, 0, 1) \models \phi\}$ is not recursive.*

6.6 Exercise. *Let F be an algebraically closed field of characteristic 0. $C \subseteq F^n$ is called (Zariski) closed a.k.a. affine variety if it is a zero set of some finite number of polynomials from $F[X_1...X_n]$. Complements of closed sets are called open. Show that if $X \subseteq F^n$ is definable, then it is a finite union of sets of the form $U \cap C$, where U is open and C is closed.*

6.7 Exercise. *Let $L = \{+, 0\} \cup \{f_q |\ q \in \mathbf{Q}\}$, where $+$ is a 2-ary function symbol and f_q are 1-ary function symbols and 0 is a constant (instead of $+(t, u)$ we write $t + u$ and insted of $f_q(t)$ we write qt). Let T_{qv} consist of the following sentences:*

$$\forall v_0 \forall v_1 \forall v_2 ((v_0 + v_1) + v_2 = v_0 + (v_1 + v_2))$$

$$\forall v_0 \forall v_1 (v_0 + v_1 = v_1 + v_0)$$

$$\forall v_0 \exists v_1 (v_0 + v_1 = 0)$$

$$\forall v_0 (v_0 + 0 = v_0)$$

$$\exists v_0 \neg (v_0 = 0)$$

for all $q, r \in \mathbf{Q}$,

$$\forall v_0 \forall v_1 (q(v_0 + v_1) = qv_0 + qv_1)$$

$$\forall v_0((q+r)v_0 = qv_0 + rv_0)$$

$$\forall v_0((qr)v_0 = q(rv_0))$$

$$\forall v_0(1v_0 = v_0).$$

(I.e. the models of T_{qv} are the (non-trivial) vector spaces over $\mathbf{Q}$*.) Show that T_{qv} is complete and has elimination of quantifiers.*

7. Ehrenfeucht-Fraïssé games

7.1 Definition.
(i) We say that ϕ is a relational atomic formula if it is of the form $R_i(x_1, ..., x_n)$ or $x_p = f_j(x_1, ..., x_n)$ or $f_j(x_1, ..., x_n) = x_p$ or $x_1 = c_k$ or $c_k = x_1$ or $x_1 = x_2$.
(ii) Relational formulas are defined as follows: Relational atomic formulas are relational formulas and if ϕ and ψ are relational then so are $\neg\phi$, $\phi \wedge \psi$ and $\exists x\phi$.

Notice that if L is finite, then for all $n \in \mathbb{N} - \{0\}$, the number of relational atomic formulas of the form $\phi(v_1, ..., v_n)$ is finite (the same is true for atomic formulas only if L does not contain function symbols).

7.2 Definition.
(i) For terms t, the relationality rank $rr(t)$ is defined as follows: If $t = v_i$, then $rr(t) = 0$, if $t = c_k$, then $rr(t) = 1$ and if $t = f_j(t_1, ..., t_n)$, then $rr(t) = max\{rr(t_1), ..., rr(t_n)\} + 1$.
(ii) For atomic formulas ϕ, the relationality rank $rr(\phi)$ is defined as follows: If $\phi = R_i(t_1, ..., t_n)$, then $rr(\phi) = max\{rr(t_1), ..., rr(t_n)\}$ and if $\phi = t = u$, then $rr(\phi) = rr(t) + rr(u) - 1$.

Notice that atomic ϕ is relational iff $rr(\phi) \leq 0$.

7.3 Lemma. *For all atomic formulas $\phi(x)$, $x = (x_1, ..., x_n)$, there is a relational formula $\psi(x)$ such that for all $\mathcal{A}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a) \leftrightarrow \psi(a)$ (i.e. $\phi(x)$ and $\psi(x)$ are equivalent).*

Proof. Easy induction on $rr(\phi)$: If $rr(\phi) \leq 0$, the claim is clear and for $rr(\phi) = p + 1 > 0$ e.g. if $\phi = R_i(t_1(x), ..., t_m(x))$, we observe that ϕ is equivalent with $\exists y_1 ... \exists y_m(R_i(y_1, ..., y_m) \wedge \bigwedge_{1 \leq j \leq m} y_j = t_j)$ and that the relationality ranks of formulas $y_j = t_j$ are $\leq p$ and thus the claim follows from the induction assumption. □

7.4 Lemma. *For all formulas $\phi(x)$, there is a relational formula $\psi(x)$ such that it is equivalent with $\phi(x)$.*

Proof. By Lemma 7.3, trivial induction on ϕ. □.

7.5 Definition. *$f : A \to \mathcal{B}$ is a relational partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ if $A \subseteq \mathcal{A}$ and for all relational atomic formulas $\phi(x_1, ..., x_n)$ and $a \in A^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(f(a))$.*

The definition of an Ehrenfeucht-Fraïssé game below is not the most common one, in Section 13 the common definition is given.

7.6 Definition. *Suppose* $a \in \mathcal{A}^n$ *and* $b \in \mathcal{B}^n$. *In order to simplify the notation, we assume that* $\mathcal{A} \cap \mathcal{B} = \emptyset$.

(i) Ehrenfeucht-Fraïssé game $EF_k((\mathcal{A},a),(\mathcal{B},b))$ *of length* $k \leq \omega$ *is a game played by two players,* I *and* II. *At each round* $m < k$, *first* I *chooses* $c_m \in \mathcal{A} \cup \mathcal{B}$ *and then* II *chooses a relational partial isomorphism* $f_m : \mathcal{A} \to \mathcal{B}$ *such that* $c_m \in dom(f_m) \cup rng(f_m)$, *for all* $1 \leq i \leq n$, $a_i \in dom(f_m)$, $f_m(a_i) = b_i$ *and if* $m > 0$, *then* $f_m \restriction dom(f_{m-1}) = f_{m-1}$. *For* $k = 0$, II *wins if* $a_i \mapsto b_i$ *is a relational partial isomorphism and for* $k > 0$, *the first who breaks the rules looses and if neither break the rules,* II *wins.*

(ii) A strategy for a player II *in* $EF_k((\mathcal{A},a),(\mathcal{B},b))$ *is a sequence* $(g_i)_{i<k}$ *such that for all* $i < k$, g_i *is an* $i+1$*-ary function from* $\mathcal{A} \cup \mathcal{B}$ *to partial maps from* $\mathcal{A}$ *to* $\mathcal{B}$. *The strategy is winning if* II *always wins the game by choosing* $g_i(c_0,..,c_i)$ *on each round* i.

(iii) We say that II *wins* $EF_k((\mathcal{A},a),(\mathcal{B},b))$ *(*$II \uparrow EF_k((\mathcal{A},a),(\mathcal{B},b))$*) if* II *has a winning strategy in the game.*

If $a = b = \emptyset$, we write $EF_k(\mathcal{A},\mathcal{B})$ for $EF_k((\mathcal{A},a),(\mathcal{B},b))$.

Notice that in (i) above we could require that $|dom(f_m)| = n + m + 1$ and this would not change the winner of the game.

7.7 Definition. *The quantifier rank* $qr(\phi)$ *of a formula* ϕ *is defined as follows: If* ϕ *is atomic,* $qr(\phi) = 0$, $qr(\neg\psi) = qr(\psi)$, $qr(\psi \wedge \theta) = max\{qr(\psi), qr(\theta)\}$ *and* $qr(\exists x\psi) = qr(\psi) + 1$.

Notice that the proof of the following theorem does not use the assumption that L is finite in the direction (i) implies (ii).

7.8 Theorem. *Suppose* L *is finite,* $a \in \mathcal{A}^n$ *and* $b \in \mathcal{B}^n$, *Then for all* $k \in \mathbb{N}$ *the following are equivalent:*

(i) $II \uparrow EF_k((\mathcal{A},a),(\mathcal{B},b))$.

(ii) For all relational formulas $\phi(x)$, $x = (x_1,...,x_n)$, *of quantifier rank* $\leq k$, $\mathcal{A} \models \phi(a)$ *iff* $\mathcal{B} \models \phi(b)$.

Proof. (i)$\Rightarrow$(ii): We prove this by induction on k. The case $k = 0$ is immediate by the definitions. So we assume that $k = p + 1 > 0$. We prove (ii) by induction on ϕ. The cases when ϕ is atomic or $\neg\psi$ or $\psi \wedge \theta$ are trivial. So we suppose $\phi(x) = \exists y\psi(y,x)$. Clearly we may assume that $qr(\psi) \leq p$. By symmetry it is enough to show that if $\mathcal{A} \models \phi(a)$ then $\mathcal{B} \models \phi(b)$. Since $\mathcal{A} \models \phi(a)$, there is $c \in \mathcal{A}$ such that $\mathcal{A} \models \psi(c,a)$. We let this c be the first choice of I in $EF_k((\mathcal{A},a),(\mathcal{B},b))$. Let f_0 be the answer given by the winning strategy of II and let $d = f_0(c)$. Then $II \uparrow EF_p((\mathcal{A},c,a),(\mathcal{B},d,b))$ and so by the induction assumption $\mathcal{A} \models \psi(c,a)$ iff $\mathcal{B} \models \psi(d,b)$. So $\mathcal{B} \models \psi(d,b)$ thus $\mathcal{B} \models \phi(b)$.

(ii)$\Rightarrow$(i): Clearly it is enough to prove the following claim:

1 Claim. *Suppose $k \in \mathbb{N}$. For all $n \in \mathbb{N}$ there is a finite set F_n^k of relational formulas $\phi(x)$, $x = (v_1, ..., v_n)$, (so for $n = 0$ the formulas are sentences) of quantifier rank $\leq k$ such that*

(a) for all $\mathcal{A}$ and $a \in \mathcal{A}^n$ there is $\phi(x) \in F_n^k$ such that $\mathcal{A} \models \phi(a)$

(b) if $\mathcal{A} \models \phi(a)$ and $\phi \in F_n^k$, then the following holds: For all $\mathcal{B}$ and $b \in \mathcal{B}^n$

(*) $II \uparrow EF_k((\mathcal{A}, a), (\mathcal{B}, b))$ iff $\mathcal{B} \models \phi(b)$.

Proof. By induction on k.

$k = 0$: Let $\psi_i(x)$, $x = (v_1, ..., v_n)$ and $i < m$, list all relational atomic formulas in which only variables $v_1, ..., v_n$ appear. For $Y \subseteq m$, let $\phi_Y(x) = \bigwedge_{i \in Y} \psi_i(x) \wedge \bigwedge_{i \in m-Y} \neg\psi_i(x)$. Let $F_n^k = \{\phi_Y | \ Y \subseteq m\}$. Clearly F_n^k is as required.

$k = p+1$: Let $\psi_i(x, v_{n+1})$, $x = (v_1, ..., v_n)$ and $i < m$, enumerate the set F_{n+1}^p. For all non-empty $Y \subseteq m$, let

$$\phi_Y(x) = \bigwedge_{i \in Y} \exists v_{n+1}\psi_i(x, v_{n+1}) \wedge \forall v_{n+1} \bigvee_{i \in Y} \psi_i(x, v_{n+1}).$$

We show that $F_n^k = \{\phi_Y(x) | \ Y \subseteq m, \ Y \neq \emptyset\}$ is as required.

By the induction assumption, each ϕ_Y is relational and of quantifier rank $\leq p + 1 = k$. So let $\mathcal{A}$ and $a \in \mathcal{A}^n$ be given. Let Y be the set of all $i < m$ such that $\mathcal{A} \models \exists v_{n+1}\psi_i(a, v_{n+1})$. By the induction assumption, $Y \neq \emptyset$ and so $\phi_Y \in F_n^k$. Furthermore, $\mathcal{A} \models \phi_Y(a)$. Thus (a) holds.

For (b), suppose $\mathcal{A} \models \phi_Y(a)$, $Y \subseteq m$ non-empty. By (i)$\Rightarrow$(ii), if $II \uparrow EF_k((\mathcal{A}, a), (\mathcal{B}, b))$, then $\mathcal{B} \models \phi_Y(b)$.

For the other direction in (*), suppose $\mathcal{B} \models \phi_Y(b)$ and we describe a winning strategy for II. Let $c_0 \in \mathcal{A} \cup \mathcal{B}$ be the first move of I. We suppose $c_0 \in \mathcal{B}$, the other case is similar. Since $\mathcal{B} \models \phi_Y(b)$, there is $i \in Y$ such that $\mathcal{B} \models \psi_i(b, c_0)$. Since $\mathcal{A} \models \exists v_{n+1}\psi_i(a, v_{n+1})$, there is $d \in \mathcal{A}$ such that $\mathcal{A} \models \psi_i(a, d)$. The first move of II is f_0, where $dom(f_0) = \{a_1, ..., a_n, d\}$, $f_0(a_i) = b_i$ for $1 \leq i \leq n$ and $f_0(d) = c_0$. By the induction assumption $II \uparrow EF_p((\mathcal{A}, a, d), (\mathcal{B}, b, c_0))$ and thus the rest of the moves, II can play according to this winning strategy and win the game. □ Claim 1.

□

7.9 Definition. *We say that $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent ($\mathcal{A} \equiv \mathcal{B}$) if for all sentences ϕ, $\mathcal{A} \models \phi$ iff $\mathcal{B} \models \phi$.*

7.10 Corollary. *The following are equivalent:*

(i) $\mathcal{A} \equiv \mathcal{B}$.

(ii) For all finite $L^* \subseteq L$ and $k \in \mathbb{N}$, $II \uparrow EF_k(\mathcal{A} \restriction L^*, \mathcal{B} \restriction L^*)$.

Proof. Immediate by Theorem 7.8, Lemma 7.4 and the fact that every L-formula is L^*-formula for some finite $L^* \subseteq L$. □

The following exercise shows that the restriction to finite vocabularies is necessary in Corollary 7.10.

7.11 Exercise. *Show that there are a vocabulary L and L-structures $\mathcal{A}$ and $\mathcal{B}$ such that L does not contain function symbols and II wins $EF_n(\mathcal{A}\restriction L^*,\mathcal{B}\restriction L^*)$ for all finite $L^*\subseteq L$ and $n<\omega$ (i.e. $\mathcal{A}\equiv\mathcal{B}$) but II does not win $EF_1(\mathcal{A},\mathcal{B})$.*

7.12 Exercise. *Let $\mathcal{A}=(\mathbb{N},S)$ where $S(a)=a+1$ for all $a\in\mathbb{N}$. Show that if $X\subseteq\mathcal{A}$ is definable, then either X is finite or $\mathcal{A}-X$ is finite. Hint: Suppose $f:\mathcal{A}\to\mathcal{A}$ is a partial map with finite domain containing 0. For all $a\in dom(f)$, by $pre(a,dom(f))$ we denote largest element in the set $\{a'\in dom(f)|\ a'<a\}$ (if exists) and by $succ(a,dom(f))$ we denote smallest element in the set $\{a'\in dom(f)|\ a'>a_i\}$ (if exists) and similarly for $rng(f)$ and $a\in rng(f)$. We say that f is n-good if for all $a,a'\in dom(f)$, $a<a'$ iff $f(a)<f(a')$ and for all $a\in dom(f)$, either $a-prec(a,dom(f))=f(a)-prec(f(a),rng(f))$ or $a-prec(a,dom(f)),f(a)-prec(f(a),rng(f))\geq 2^{n+1}$ and either $succ(a,dom(f))-a=succ(f(a),rng(f))-f(a)$ or $succ(a,dom(f))-a,succ(f(a),rng(f))-f(a)\geq 2^{n+1}$. Start by showing that if f is $n+1$-good and $a\in\mathcal{A}$, then there is $b\in\mathcal{A}$ such that $f\cup\{(a,b)\}$ is n-good.*

7.13 Exercise. *In the proof of Exercise 7.12, the use of n-good functions can be replaced by the use of ultraproducts. How is this done? Hint: Exercise 2.9.*

Next exercise explains why in the definition of Eherenfeucht-Fraïssé games, relational partial isomorphisms were used.

7.14 Exercise. *Let $L=\{f\}$ where f is a unary function symbol. Show that there are L-structures $\mathcal{A}$ and $\mathcal{B}$ and $a\in\mathcal{A}$ such that $\mathcal{A}\equiv\mathcal{B}$ but there is no partial isomorphism $h:\mathcal{A}\to\mathcal{B}$ with $a\in dom(h)$. Hint: Let $\mathcal{B}$ consist of n-cycles so that for each $n\in\omega-\{0,1\}$, $\mathcal{B}$ contains exactly one n-cycle.*

7.15 Exercise. *Show directly from the definitions (i.e. without using Theorem 7.8) that if $II\uparrow EF_k(\mathcal{A},\mathcal{B})$ and $II\uparrow EF_k(\mathcal{B},\mathcal{C})$, then $II\uparrow EF_k(\mathcal{A},\mathcal{C})$.*

8. Types and saturated models

8.1 Lemma. *Suppose T is a complete theory and $\mathcal{A},\mathcal{B}\models T$.*

(i) If $f:\mathcal{A}\to\mathcal{B}$ is an partial elementary map, then there is $\mathcal{C}\models T$ and an elementary embedding $g:\mathcal{A}\to\mathcal{C}$ such that $\mathcal{B}\preceq\mathcal{C}$ and $g\restriction dom(f)=f$.

(ii) There are $\mathcal{B}\preceq\mathcal{C}$ and an elementary embedding $f:\mathcal{A}\to\mathcal{C}$.

Proof. Since T is complete, the empty function from $\mathcal{A}$ to $\mathcal{B}$ is elementary and thus (ii) follows from (i) (compare Lemma 5.6). We prove (i). Without loss of generality, we may assume that $\mathcal{A}\cap\mathcal{B}=\emptyset$ (this simplifies the notation).

As e.g. in the proof of Lemma 4.5, it suffices to show that

$$T^*=Th(\mathcal{A},\mathcal{A})\cup Th(\mathcal{B},\mathcal{B})\cup\{\underline{d}=\underline{e}|\ d\in dom(f),\ e=f(d)\}$$

is consistent (exercise). Let $X\subseteq dom(f)$ be finite and $\phi(\underline{c},\underline{b})\in Th(\mathcal{B},\mathcal{B})$ be such that $\phi=\phi(x,y)$, $c\in(\mathcal{B}-rng(f))^m$ and $b=(b_1,...,b_n)$ is such that $f(X)=\{b_1,...,b_n\}$. Let

$$\psi(x,y)=\phi(x,y)\wedge\bigwedge_{1\leq i\leq m}\bigwedge_{1\leq j\leq n}\neg x_i=y_j.$$

Notice that $\psi(\underline{c}, \underline{b}) \in Th(\mathcal{B}, \mathcal{B})$ Let $a \in X^n$ be such that $f(a_i) = b_i$ for all $1 \leq i \leq n$. Since $Th(\mathcal{B}, \mathcal{B})$ is closed under conjunctions, by compactness it is enough to prove that

$$T' = Th(\mathcal{A}, \mathcal{A}) \cup \{\psi(\underline{c}, \underline{b})\} \cup \{\underline{d} = \underline{e} |\ d \in X,\ e = f(d)\}$$

is consistent. Now $\mathcal{B} \models \exists x_1 ... \exists x_m \psi(x, b)$ and since f is elementary,

$$\mathcal{A} \models \exists x_1 ... \exists x_m \psi(x, a).$$

Let $c' = (c'_1, ..., c'_m) \in \mathcal{A}^m$ be such that $\mathcal{A} \models \psi(c', a)$. Let $\mathcal{C}$ be a model we get from $(\mathcal{A}, \mathcal{A})$ by adding the interpretations for $\underline{e}$, $e \in \mathcal{B}$, as follows: If $e = f(d)$ for some $d \in X$, then $\underline{e}^{\mathcal{C}} = d$, if $e = c_i$ for some $1 \leq i \leq m$, then $\underline{e}^{\mathcal{C}} = c'_i$ and otherwise we choose the interpretations freely. Clearly $\mathcal{C} \models T'$. □

8.2 Definition. *Suppose $A \subseteq \mathcal{A}$ and $n > 0$.*

(i) $L_{\omega\omega}(A, n)$ is the set of all $\phi(x, a)$, where $\phi(x, y)$ is a formula, $x = (v_1, ..., v_n)$ and a is a sequence of elements of A.

(ii) An n-type over A is a non-empty subset of $L_{\omega\omega}(A, n)$.

(iii) An n-type p over A is complete if for all $\phi \in L_{\omega\omega}(A, n)$, $\phi \in p$ or $\neg\phi \in p$.

(iv) $b \in \mathcal{A}^n$ realizes an n-type p over A if $\mathcal{A} \models \phi(b, a)$ for all $\phi(x, a) \in p$. $t(b/A) = t(b/A; \mathcal{A})$ is the set of all $\phi(x, a) \in L_{\omega\omega}(A, n)$ such that $\mathcal{A} \models \phi(b, a)$ i.e. the unique complete n-type over A realized by b.

(v) An n-type p over A is consistent (in $\mathcal{A}$) if there is $\mathcal{B}$ and $b \in \mathcal{B}^n$ such that $\mathcal{A} \preceq \mathcal{B}$ and b realizes p.

(vi) $S_n(A) = S_n(A; \mathcal{A})$ is the set of all complete consistent n-types over A.

8.3 Lemma. *Suppose $A \subseteq \mathcal{A}$ and p is an n-type over A. Then the following are equivalent.*

(i) p is consistent.

(ii) For all $\phi_i(x, a^i) \in p$, $i < m \in \mathbb{N}$, $\mathcal{A} \models \exists v_1 ... \exists v_n \bigwedge_{i<m} \phi_i(x, a^i)$.

(iii) $T = Th((\mathcal{A}, A)) \cup \{\phi(c, \underline{a}) |\ \phi(x, a) \in p\}$ is consistent, where $c = (c_1, ..., c_n)$ are new constant symbols.

Proof. (i)$\Rightarrow$(ii): Let $\mathcal{B}$ and b witness the consistency. Then $\mathcal{B} \models \bigwedge_{i<m} \phi_i(b, a^i)$ and thus $\mathcal{B} \models \exists v_1 ... \exists v_n \bigwedge_{i<m} \phi_i(x, a^i)$. Since $\mathcal{A} \preceq \mathcal{B}$, $\mathcal{A} \models \exists v_1 ... \exists v_n \bigwedge_{i<m} \phi_i(x, a^i)$.

(ii)$\Rightarrow$(iii): Let $\phi_i(c, \underline{a^i}) \in \{\phi(c, \underline{a}) |\ \phi(x, a) \in p\}$, $i < m \in \mathbb{N}$. By compactness, it suffices to show that $T' = Th((\mathcal{A}, A)) \cup \{\phi_i(c, \underline{a^i}) |\ i < m\}$ is consistent. By (ii), there is $b \in \mathcal{A}^n$ such that $(\mathcal{A}, \mathcal{A}) \models \phi_i(b, \underline{a^i})$ for all $i < m$. Thus by interpreting c_i to b_i, we get a model for T'.

(iii)$\Rightarrow$(i): Let $\mathcal{B}^*$ be a model of T and $\mathcal{B} = \mathcal{B}^* \restriction L$. Clearly we may choose $\mathcal{B}^*$ so that for all $a \in A$, $\underline{a}^{\mathcal{B}^*} = a$. Then the identity function $id : A \to \mathcal{B}$ is an elementary partial map from $\mathcal{A}$ to $\mathcal{B}$ and thus by Lemma 8.1 (i), we may assume in addition that $\mathcal{A} \preceq \mathcal{B}$. Letting $b_i = c_i^{\mathcal{B}^*}$ for $1 \leq i \leq n$, $b = (b_1, ..., b_n)$ realizes p in $\mathcal{B}$. □

8.4 Remark.

(i) Suppose $A \subseteq \mathcal{A}$*. Letting the sets* $\{p \in S_n(A) | \ \phi(x,a) \in p\}$ *for* $\phi(x,a) \in L_{\omega\omega}(A)$*, be a basis for a topology on* $S_n(A)$*, we get a Hausdorff topology which is by Lemma 8.3 also compact (exercise). This space is called a Stone space.*

(ii) If T *is complete and* $\mathcal{A}, \mathcal{B} \models T$*, then* $S_n(\emptyset; \mathcal{A}) = S_n(\emptyset; \mathcal{B})$ *(by Lemma 8.3 (iii)) and thus when* T *is given, we can talk about* $S_n(\emptyset)$ *without need to specify the model.*

8.5 Definition.

(i) We say that $\mathcal{A}$ *is* κ*-saturated if for all* $n \in \mathbb{N} - \{0\}$*,* $A \subseteq \mathcal{A}$ *of power* $< \kappa$ *and* $p \in S_n(A)$*, some* $a \in \mathcal{A}^n$ *realizes* p*. We say that* $\mathcal{A}$ *is saturated if it is* $|\mathcal{A}|$*-saturated.*

(ii) We say that $\mathcal{A}$ *is strongly* κ*-homogeneous if for all partial elementary maps* $f : \mathcal{A} \to \mathcal{A}$ *with* $dom(f)$ *of power* $< \kappa$*, there is an automorphism* g *of* $\mathcal{A}$ *such that* $g \restriction dom(f) = f$*.*

(iii) We say that $\mathcal{A}$ *is* κ*-universal if for all* $\mathcal{B} \models Th(\mathcal{A})$ *of power* $< \kappa$*, there is an elementary embedding* $f : \mathcal{B} \to \mathcal{A}$*.*

8.6 Lemma.

(i) Suppose $\kappa \geq \omega$*. If for all* $A \subseteq \mathcal{A}$ *of power* $< \kappa$ *and* $p \in S_1(A)$*, some* $a \in \mathcal{A}$ *realizes* p*, then* $\mathcal{A}$ *is* κ*-saturated.*

(ii) If a model is finite, then it is κ*-saturated for all* κ*.*

Proof. Exercise. □

8.7 Theorem. *Suppose* T *is a complete theory and* κ *is a cardinal. Then there is* $\mathcal{A} \models T$ *which is* κ*-saturated,* κ*-universal and strongly* κ*-homogeneous.*

Proof. By Lemma 8.1 and Corollary 4.7, the proof is the same as that of Theorem 5.11, verbatim, except that one needs to replace $\subseteq$ by $\preceq$, partial isomorphisms by partial elementary maps and quantifier free formulas by complete types (exercise). □

Theorem 8.7 follows also from Theorem 5.11 by using Morleyzation: For each formula $\phi(x)$, $x = (v_1, ..., v_n)$, choose a new n-ary relation symbol $R_{\phi(x)}$. Let L^* be L together with these new predicate symbols and T^* the theory T together with the sentences

$$\forall v_1 ...., \forall v_n (\phi(x) \leftrightarrow R_{\phi(x)}(x)).$$

Then for all $\mathcal{A} \models T$ there is unique $\mathcal{A}^* \models T^*$ such that $\mathcal{A}^* \restriction L = \mathcal{A}$. Now from this and Lemma 8.1 it follows easily that T^* has AP, JEP and is closed under unions. Also every model of T^* is existentially closed (exercise). Thus by Theorem 5.13 T^* is complete and has elimination of quantifiers. Let now $\mathcal{A}^*$ be as in Theorem 5.11 for T^*. Then $\mathcal{A} = \mathcal{A}^* \restriction L$ is as required in Theorem 8.7 assuming $\kappa > |L_{\omega\omega}|$ (exercise, for κ-saturation show using an idea from the proof of Theorem 5.11, that κ-universality together with strong κ-homogeneity imply κ-saturation assuming $\kappa > |L_{\omega\omega}|$).

8.8 Remark.

(i) It is not an accident that the proofs of 8.7 and 5.11 are the same, see the work on abstract elementary classes.

(ii) Suppose $\mathcal{A}$ is strongly κ-homogeneous, $A \subseteq \mathcal{A}$ is of power $< \kappa$ and $a, b \in \mathcal{A}^n$. Then $t(a/A) = t(b/A)$ iff there is $f \in Aut(\mathcal{A}/A)$ such that $f(a) = b$. Thus the types over A can also be viewed as orbits of the natural action $fa = (f(a_1), ..., f(a_n))$ of $Aut(\mathcal{A}/A)$ on $\mathcal{A}^n$. (Exercise)

8.9 Lemma. *If T is a complete theory and $\mathcal{A}, \mathcal{B} \models T$ are infinite saturated models of the same size, then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

Proof. Let us enumerate $\mathcal{A} = \{a_i |\ i < \kappa\}$ and $\mathcal{B} = \{b_i |\ i < \kappa\}$. By induction on $i \leq \kappa$, we construct partial elementary maps $f_i, g_i : \mathcal{A} \to \mathcal{B}$ so that

(i) for $i < j \leq \kappa$, $f_i \subseteq g_j \subseteq f_j$,

(ii) for all $i < \kappa$, $a_i \in dom(g_{i+1})$ and $b_i \in rng(f_{i+1})$,

(iii) for all $i < \kappa$, $|dom(f_i)|, |dom(g_i)| < |i|^+ + \omega$.

For $i = 0$, we let $f_i = g_i = \emptyset$ (these are elementary because T is complete) and for limit i we let $g_i = f_i = \bigcup_{j<i} f_j$ $(= \bigcup_{j<i} g_j$ by (i)).

Suppose $i = j + 1$. Let $A = dom(f_j)$ and $p = t(a_j/A; \mathcal{A})$. Let $f_j(p) = \{\phi(x, f_j(a'_1), ..., f_j(a'_n))|\ \phi(x, a'_1, ..., a'_n) \in p\}$. Since f_j is elementary, by Lemma 8.3 (ii), $f_j(p)$ is consistent. Since $\mathcal{B}$ is saturated and f_j satisfies (iii), $f_j(p)$ is realized in $\mathcal{B}$ by some b. Let g_i be such that $dom(g_i) = dom(f_j) \cup \{a_j\}$, $g_i \restriction A = f_j$ and $g_i(a_j) = b$. Clearly (i)-(iii) are satisfied. f_i can be found similarly, only start from g_i and look the inverses.

Then f_κ is the isomorphism claimed to exist. □

8.10 Example. *Every uncountable model of T_{acf_0} is saturated and thus T_{acf_0} is κ-categorical for all $\kappa > \omega$.*

Proof. Exercise. □

We write $\kappa^{<\kappa}$ for the cardinality of the set $\{f : \alpha \to \kappa |\ \alpha < \kappa\}$.

8.11 Lemma. *Suppose $\kappa^{<\kappa} = \kappa > |L_{\omega\omega}|$ and T is a complete theory with infinite models. Then T has a saturated model of cardinality κ.*

Proof. Let $\mathcal{A} \models T$ be of power κ and $A \subseteq \mathcal{A}$ be of cardinality $< \kappa$. Then $|S(A; \mathcal{A})| \leq 2^{|A|+\omega} \leq \kappa^{<\kappa} = \kappa$ and we can enumerate it as $\{p_i |\ i < \kappa\}$. Now for each $i < \kappa$ choose $\mathcal{A}_i \models T$ of power κ so that $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_j \prec \mathcal{A}_i$ for $j < i$ and $\mathcal{A}_{i+1}$ realizes p_i. Then $\mathcal{A}(A) = \cup_{i<\kappa} \mathcal{A}_i$ realizes every type in $S(A; \mathcal{A})$.

Let $P_{<\kappa}(X)$ be the set of all subsets of X of power $< \kappa$. Then $|P_{<\kappa}(\mathcal{A})| \leq \kappa^{<\kappa} = \kappa$ and so we can enumerate it as $\{A_i |\ i < \kappa\}$. For all $i < \kappa$, choose $\mathcal{A}^*_i$ of power $\leq \kappa$ so that $\mathcal{A}^*_0 = \mathcal{A}$, $\mathcal{A}^*_j \prec \mathcal{A}^*_i$ for $j < i$ and $\mathcal{A}^*_{i+1} = \mathcal{A}^*_i(A_i)$. Then $\mathcal{A}^+ = \cup_{i<\kappa} \mathcal{A}^*_i$ realizes every type from $S(A; \mathcal{A})$ for all $A \subseteq \mathcal{A}$ of power $< \kappa$.

Finally for all $i < \kappa$, choose models $\mathcal{B}_i \models T$ of power κ so that $\mathcal{B}_0 = \mathcal{A}$, $\mathcal{B}_j \prec \mathcal{B}_i$ for $j < i$ and $\mathcal{B}_{i+1} = \mathcal{B}^+_i$. Since $\kappa^{<\kappa} = \kappa$ implies that κ is regular (see e.g. [Je]), $\mathcal{B} = \cup_{i<\kappa} \mathcal{B}_i$ is saturated. □

The cardinality assumption in Lemma 8.11 can not be made weaker: Let $T_{dlo} = Th((\mathbf{Q},<))$. Then T_{dlo} has a saturated model of power κ iff $\kappa^{<\kappa} = \kappa \geq \omega$ (exercise, keep Exercise 5.17 in mind and see Exercise 8.13 below). So T_{dlo} has a saturated model of power $\aleph_1$ iff the continuum hypothesis holds and it has never a saturated model of power $\aleph_\omega$. In fact, it is consistent that upto isomorphism, $(\mathbf{Q} <)$ is the only saturated model of T_{dlo}.

8.12 Lemma.
(i) If $\mathcal{A}$ is κ-saturated, then it is κ^+-universal.
(ii) If $\mathcal{A}$ is saturated then it is strongly $|\mathcal{A}|$-homogeneous.

Proof. Exercise. □

8.13 Exercise. *Show that if $\mathcal{A} \models T_{dlo}$ is saturated and $\kappa = |\mathcal{A}|$, then $\kappa^{<\kappa} = \kappa$. Hint: Suppose not. Let μ be the least cardinal such that $\kappa^\mu > \kappa$ and let T be the set of all functions $f : \alpha \to \kappa$, $0 < \alpha < \mu$. Order T by $f < g$ if $g \subsetneq f$ or if there is $x \in dom(f) \cap dom(g)$ such that $f(x) \neq g(x)$ and if y is the least such then $f(y) < g(y)$. Notice that $(T,<) \models T_{dlo}$ and that every branch in the tree $(T,\subseteq)$ determines a type over a subset of T of size μ.*

8.14 Exercise. *Suppose $|L_{\omega\omega}| = \omega$, $\mathcal{A}_i$, $i < \omega$, are infinite structures and $U \subseteq P(\omega)$ is an ultrafilter such that for all $n \in \omega$, $\{x \in \omega |\ x > n\} \in U$. Show that $\Pi_{i<\omega}\mathcal{A}_i/U$ is ω_1-saturated.*

8.15 Exercise. *Let T, κ and $\mathcal{A}$ be as in Theorem 5.11. Show that $Th(\mathcal{A})$ has elimination of quantifiers iff $\mathcal{A}$ is κ-saturated. Hint for the direction from right to left: Show that if $\mathcal{A} \models \forall v_1 ... \forall v_n (\bigwedge_{i<\alpha} \phi_i \to \phi)$, ($\alpha \leq |L_{\omega\omega}|$), then the same is true in every model of $Th(\mathcal{A})$.*

The result in the next exercise is not the best possible.

8.16 Exercise. *Let T be a complete theory with infinite models and $\kappa \geq |L_{\omega\omega}|$ a regular cardinal. Suppose that for every $\mathcal{A} \models T$ of power κ, $|S(\mathcal{A};\mathcal{A})| = \kappa$ (i.e. T is κ-stable). Show that T has a saturated model of power κ.*

9. Henkin constructions and omitting types

9.1 Definition.
(i) We say that a theory T locally omits an n-type p over $\emptyset$, if the following holds: For every formula $\phi(x)$, $x = (v_1, ..., v_n)$, if $T \not\models \forall v_1 ... \forall v_n \neg\phi(x)$, then there is $\theta(x) \in p$ such that $T \not\models \forall v_1 ... \forall v_n (\phi(x) \to \theta(x))$.
(ii) We say that $\mathcal{A}$ omits an n-type p over $\emptyset$ if no $a \in \mathcal{A}^n$ realize p.

In the following theorem, it is crucial that the vocabulary is countable (i.e. of size ω or finite) and that we claim only that $\mathcal{A}$ is countable, see Remark 9.3. In fact for uncountable vocabularies there are no known methods, anywhere as powerful as 9.2, to omit types.

9.2 Omitting types theorem. *Suppose L is countable, T is a consistent theory and D is a countable collection of types over $\emptyset$. If T locally omits every $p \in D$ then T has a countable model $\mathcal{A}$ which omits every $p \in D$.*

Proof. This proof is a modification of the usual proof of the completeness theorem, see the course Matemaattinen logiikka. For simplicity, we assume that D is a singleton and that the only type p in D is a 1-type (exercise: what changes are needed to prove the general case?). Let c_i, $i < \omega$, be new constants and denote $L^* = L \cup \{c_i | \ i < \omega\}$. Let ϕ_i, $i < \omega$, enumerate all L^*-sentences so that if c_j appears in ϕ_i then $j < i$. By recursion on $k < \omega$, we construct an increasing sequence of consistent L^*-theories T_k so that

(i) $T_k - T$ is finite and if c_i appears in some $\theta \in T_k$, then $i \leq k$,

(ii) $\phi_k \in T_{k+1}$ or $\neg\phi_k \in T_{k+1}$,

(iii) if $\phi_k = \exists x\theta(x) \in T_{k+1}$, then $\theta(c_{k+1}) \in T_{k+1}$,

(iv) there is $\theta(v_1) \in p$ such that $\neg\theta(c_k) \in T_{k+1}$.

We let $T_0 = T$.

For T_{k+1}, first we choose L-formula $\phi(v_1, x_0, ..., x_{k-1})$, so that

$$\models \phi(c_k, c_0, ..., c_{k-1}) \leftrightarrow \bigwedge\{\theta | \ \theta \in T_k - T\}$$

(if $k = 0$, we let $\phi = v_1 = v_1$). Now clearly $T \not\models \forall v_1 \neg\exists x_0 ... \exists x_{k-1}\phi$ (since $T \cup \{\phi(c_k, c_0, ..., c_{k-1})\}$ is consistent) and thus $T \not\models \phi(c_k, c_0, ..., c_{k-1}) \rightarrow \theta(c_k)$ for some $\theta(v_1) \in p$ because otherwise (exercise or see the course Matemaattinen logiikka)

$$T \models \forall v_1(\exists x_0 ... \exists x_{k-1}\phi(v_1, x_0, ..., x_{k-1}) \rightarrow \theta(v_1))$$

for all $\theta(v_1) \in p$ contradicting the assumption that T locally omits p. So there is $\theta(v_1) \in p$ such that $T^*_{k+1} = T_k \cup \{\neg\theta(c_k)\}$ is consistent. This takes care of (iv).

Clearly either $T^*_{k+1} \cup \{\phi_k\}$ or $T^*_{k+1} \cup \{\neg\phi_k\}$ is consistent and let T^{**}_{k+1} be the one of these that is consistent. This takes care of (ii).

Unless $\phi_k = \exists x\psi(x)$ for some ψ and $\phi_k \in T^{**}_{k+1}$, we let $T_{k+1} = T^{**}_{k+1}$. Otherwise, since c_{k+1} does not appear in T^{**}_{k+1}, $T_{k+1} = T^{**}_{k+1} \cup \{\psi(c_{k+1})\}$ is consistent (exercise or see the course Matemaattinen logiikka). This takes care of (iii). Clearly (i) holds.

Then $T^* = \cup_{k<\omega} T_k$ is consistent and it has a model, say $\mathcal{B}^*$. Let $\mathcal{B} = \mathcal{B}^* \restriction L$ and $A = \{c_i^{\mathcal{B}^*} | \ i < \omega\}$.

1 Claim. *For all constants $c \in L$, $c^{\mathcal{B}} \in A$ and for all $n-ary$ function symbols $f \in L$ and $a \in A^n$, $f^{\mathcal{B}}(a) \in A$.*

Proof. Exercise. □ Claim 1.

By Claim 1 we can let $\mathcal{A}$ be the L-model such that $dom(\mathcal{A}) = A$, for all $R \in L$, $R^{\mathcal{A}} = R^{\mathcal{B}} \cap A^{\#R}$, for all $f \in L$, $f^{\mathcal{A}} = f^{\mathcal{B}} \restriction A^{\#f}$ and for all $c \in L$, $c^{\mathcal{A}} = c^{\mathcal{B}}$ i.e. $\mathcal{A} = \mathcal{B} \restriction A$. Then $\mathcal{A}$ is a substructure of $\mathcal{B}$ and by (iii) and Tarski-Vaught, $\mathcal{A} \preceq \mathcal{B}$ (if $\mathcal{B} \models \exists x\psi(x, c^{\mathcal{B}^*})$, $c = (c_{i_1}, ..., c_{i_n})$ and $c^{\mathcal{B}^*} = (c_{i_1}^{\mathcal{B}^*}, ..., c_{i_n}^{\mathcal{B}^*})$, then $\exists x\psi(x, c) \in T^*$ and so $\psi(c_k, c) \in T^*$ for some k i.e. $\mathcal{B} \models \psi(c_k^{\mathcal{B}^*}, c^{\mathcal{B}^*})$). So $\mathcal{A} \models T$ and by (iv), $\mathcal{A}$ omits p. □

9.3 Remark. *Let us look at the theory* T_{f_0}*. Let*

$$p = \{\neg(\Sigma_{i=0}^{i=n} a_i v_1^i = 0) |\ n > 0,\ a_i \in \mathbf{Z}, a_n \neq 0\}.$$

Then every model of T_{f_0} *which omits* p *is countable. Using this, one can find* L'*,* T' *and a type* p' *such that* T' *locally omits* p' *but no model of* T' *omit* p' *(exercise. Hint: Use the fact that there is a model* $\mathcal{A} \models T_{acf_0}$ *which omits* p *and contains the field of rational numbers and keep in mind Fact 6.2 (iii) and Theorem 6.4.)*

9.4 Definition. *Assume* T *is complete.*

(i) We say that $p \in S_n(\emptyset)$ *is isolated if* T *does not locally omit* p *i.e. there is* $\phi(v_1, ..., v_n) \in p$ *such that for all* $\psi(v_1, ..., v_n) \in p$*,* $T \models \forall v_1 ... \forall v_n(\phi(v_1, ..., v_n) \to \psi(v_1, ..., v_n))$*. When this happens, we say that* ϕ *isolates* p*.*

(ii) $\mathcal{A} \models T$ *is atomic if for all* $n \in \mathbb{N}$ *and* $a \in \mathcal{A}^n$*,* $t(a/\emptyset)$ *is isolated.*

(iii) We say that $\phi(v_1, ..., v_n)$ *is complete if it isolates some* $p \in S_n(\emptyset)$*.*

(iv) We say that T *is atomic if for all* $\phi(v_1, ..., v_n)$ *either* $T \models \forall v_1 ... \forall v_n \neg\phi$ *or there is complete* $\psi(v_1, ..., v_n)$ *such that* $T \models \forall v_1 ... \forall v_n(\psi \to \phi)$*.*

9.5 Lemma. *Suppose* $|L_{\omega\omega}| = \omega$ *and* T *is a complete theory. Then the following are equivalent.*

(i) T *is atomic.*

(ii) T *has an atomic model.*

Proof. (ii)$\Rightarrow$(i): Exercise.

(i)$\Rightarrow$(ii): For all $n \in \mathbb{N} - \{0\}$, let

$$p_n = \{\neg\phi(v_1, ..., v_n) |\ \phi \text{ is complete}\}.$$

Since T is atomic, T locally omits every p_n. Thus by Theorem 9.2, T has a model $\mathcal{A}$ which omits every p_n. Clearly $\mathcal{A}$ is atomic. □

9.6 Exercise.

(i) Let T *be a consistent complete theory and* L *countable. Suppose that* $\phi(x)$*,* $x = (v_1, ..., v_n)$*, is such that* $T \not\models \forall v_1 ... \forall v_n \neg\phi$ *and there is no complete* $\psi(x)$ *with* $T \models \forall v_1 ... \forall v_n(\psi \to \phi)$*. Show that there are uncountably many* $p \in S_n(\emptyset)$ *such that* $\phi \in p$*. Conclude that if for all* $n \in \mathbb{N} - \{0\}$*,* $S_n(\emptyset)$ *is countable, then* T *has an atomic model.*

(ii) Using (i), show that T_{acf_0} *has an atomic model and describe it.*

(iii) Show that every model of T_{dlo} *is atomic (see Exercise 8.13 and Exercise 5.17).*

A theory can have an atomic model while having a lot of types over $\emptyset$: For all $\eta \in 2^{<\omega} = \{\xi : n \to 2 |\ n < \omega\}$, let P_η be a unary relation symbol and $L = \{P_\eta |\ \eta \in 2^{<\omega}\}$. Let $\mathcal{A}$ be an L-structure such that $dom(\mathcal{A}) = 3^\omega$ and for all $\xi \in dom(\mathcal{A})$, $\xi \in P_\eta^{\mathcal{A}}$ if $\xi \restriction dom(\eta) = \eta$. Let $T = Th(\mathcal{A})$. For all $\eta \in 2^\omega$, $p_\eta = \{P_{\eta \restriction n}(v_1) |\ n < \omega\}$ is a consistent type and thus $|S_1(\emptyset; T)| = 2^\omega$. However, it is not hard to see that T

has elimination of quantifiers and that $\mathcal{B} = \mathcal{A} \restriction (3^\omega - 2^\omega)$ is an elementary submodel of $\mathcal{A}$ (think of restrictions to finite subsets of L). Thus if $\xi \in \mathcal{B}$ and $n < \omega$ is the least such that $\xi(n) = 2$, then $P_{\xi\restriction n}(v_1) \wedge \neg P_{\xi\restriction n\cup\{(n,0)\}}(v_1) \wedge \neg P_{\xi\restriction n\cup\{(n,1)\}}(v_1)$ isolates $t(\xi/\emptyset)$. This generalizes to bigger arities and thus $\mathcal{B}$ is an atomic model of T.

9.7 Exercise. *Find a (consistent) theory that does not have an atomic model.*

9.8 Lemma. *Assume T is complete (with infinite models) and $\mathcal{A}$ and $\mathcal{B}$ are countable atomic models of T. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

Proof. This proof is essentially the same as that of Lemma 8.9: Let $\{a_i|\ i < \omega\}$ and $\{b_i|\ i < \omega\}$ be enumerations of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then we construct an increasing sequences of finite partial elementary maps $f_i, g_i : \mathcal{A} \to \mathcal{B}$, $i < \omega$, as in the proof of Lemma 8.9. We let $f_0 = g_0 = \emptyset$ and g_{i+1} is found as follows: Let $\{a'_1, ..., a'_n\} = dom(f_i)$. Then $t((a'_1, ..., a'_n, a_i)/\emptyset)$ is isolated, say by $\phi(v_1, ..., v_{n+1})$. Since f_i is elementary, $\mathcal{B} \models \exists v_{n+1}\phi(f_i(a'_1), ..., f_i(a'_n), v_{n+1})$. So there is $b \in \mathcal{B}$ such that $\mathcal{B} \models \phi(f_i(a'_1), ..., f_i(a'_n), b)$. Since ϕ isolates $t((a'_1, ..., a'_n, a_i)/\emptyset)$, $t((f_i(a'_1), ..., f_i(a'_n), b)/\emptyset) = t((a'_1, ..., a'_n, a_i)/\emptyset)$. This means that g_{i+1} is elementary when $dom(g_{i+1}) = dom(f_i) \cup \{a_i\}$, $g_{i+1} \restriction dom(f_i) = f_i$ and $g_{i+1}(a_i) = b$. f_{i+1} is found similarly. $\cup_{i<\omega} f_i$ is the required isomorphism. □

9.9 Lemma. *Assume T is complete (with infinite models) and $\mathcal{A}$ is a countable atomic model of T. Then $\mathcal{A}$ is a prime model i.e. for all $\mathcal{B} \models T$, there is an elementary embedding $f : \mathcal{A} \to \mathcal{B}$.*

Proof. As the previous lemma (exercise). □

9.10 Exercise. *Assume T is a complete theory in a countable language L. Show that $\mathcal{A}$ is a prime model of T iff $\mathcal{A}$ is a countable atomic model of T. Hint: One direction is Lemma 9.9 and for the other direction start by showing that if T does not have an atomic model, then every model of T realizes a type that can be omitted.*

9.11 Theorem (Ryll-Nardzewski). *Assume L is countable and T is complete and has infinite models. Then the following are equivalent:*
(i) T is ω-categorical,
(ii) for all $n \in \mathbb{N} - \{0\}$, $S_n(\emptyset)$ is finite.

Proof. (ii)$\Rightarrow$(i): If $S_n(\emptyset)$ is finite, then every $p \in S_n(\emptyset)$ is isolated (if $S_n(\emptyset) = \{p_0, ..., p_n\}$, $p = p_0$, then $\bigwedge_{1\le i\le n} \phi_i$ isolates p when the formulas ϕ_i are chosen so that $\phi_i \in p - p_i$). Thus every model of T is atomic and so (i) follows from Lemma 9.8.

(i)$\Rightarrow$(ii): Suppose $S_n(\emptyset)$ is infinite. We show that T is not ω-categorical.

1 Claim. *There is non-isolated $r \in S_n(\emptyset)$.*

Proof. Suppose not. For every $p \in S_n(\emptyset)$, let $\phi_p \in p$ be a formula that isolates p. Then $q = \{\neg\phi_p | \ p \in S_n(\emptyset)\}$ can be realized in a model of T by compactness since for all $p \in S_n(\emptyset)$ every realization of p realizes $q - \{\neg\phi_p\}$. Let $\mathcal{B} \models T$ and $b \in \mathcal{B}^n$ be such that b realizes q. Then $r = t(b/\emptyset)$ is a complete consistent type but $r \notin S_n(\emptyset)$, a contradiction. □ Claim 1.

Let r be as in Claim 1. By omitting types theorem, T has a countable model $\mathcal{A}$ that omits r. Since T is complete and has infinite models, every model of T is infinite and so $\mathcal{A}$ has power ω.

On the other hand, since $r \in S(\emptyset)$, there is a model $\mathcal{B}$ of T that realizes r. As above $\mathcal{B}$ is infinite and so by Lemma 4.8 can be chosen to have cardinality ω. Clearly $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic. □

9.12 Example. *T_{acf_0} is not ω-categorical.*

Proof. Exercise. □

9.13 Exercise. *Suppose T is complete. Let $A \subseteq \mathcal{A} \models T$. We say that $p \in S_n(A;\mathcal{A})$ is isolated if there is $\phi(x,a) \in p$, $x = (v_1, ..., v_n)$, such that for all $\theta(x,b) \in p$, $Th(\mathcal{A}, A) \models \forall v_1 ... \forall v_n(\phi(x,\underline{a}) \rightarrow \theta(x,\underline{b}))$. Assume that $\mathcal{A}$ is primary over A i.e. there are $a_i \in \mathcal{A}$, $i < \alpha$, such that $\mathcal{A} = A \cup \{a_i | \ i < \alpha\}$ and for all $i < \alpha$, $t(a_i/A \cup \{a_j | \ j < i\})$ is isolated. Show that if $\mathcal{B} \models T$ and $f : \mathcal{A} \rightarrow \mathcal{B}$ is a partial elementary map with $dom(f) = A$, then there is an elementary embedding $g : \mathcal{A} \rightarrow \mathcal{B}$ such that $f \subseteq g$ (i.e. $\mathcal{A}$ is prime over A).*

9.14 Exercise.

(i) Show that there is a model $\mathcal{A}$ such that $\mathbf{Z} \subseteq \mathcal{A} \preceq (\mathbf{R}, <)$ and it is primary over $\mathbf{Z}$.

(ii) Let T_{rgr} be the theory of random graphs i.e. the theory of the existentially closed models of T_{gr} (see Exercise 5.18) and suppose $\mathcal{B} = (\mathcal{B}, E) \models T_{rgr}$ and there is countably infinite $X \subseteq \mathcal{B}$ such that for all $a, b \in X$, $(a,b) \notin E$. Show that there is no model $X \subseteq \mathcal{A} \preceq \mathcal{B}$ such that it is prime over X.

9.15 Exercise. *Prove the direction (i)⇒(ii) in Lemma 9.5 the following way: Let $\mathcal{A} \models T$. Show that there are $a_i \in \mathcal{A}$ such that for all $n < \omega$, $t((a_0, ..., a_n)/\emptyset)$ is isolated and $A = \{a_i | \ i < \omega\}$ passes the Tarski-Vaught test (Theorem 4.6).*

We finish this section by giving one more application to the Henkin construction technique from the proof of Theorem 9.2.

9.16 Craig's interpolation theorem. *Let L_1 and L_2 be vocabularies, $L_0 = L_1 \cap L_2$, ϕ a sentence in the vocabulary L_1 and ψ a sentence in the vocabulary L_2. If $\phi \models \psi$, then there is a sentence θ in the vocabulary L_0 such that $\phi \models \theta$ and $\theta \models \psi$.*

Proof. W.o.l.g. we may assume that L_1 and L_2 are finite. For a contradiction we assume that $\phi \models \psi$ but there is no sentence θ in the vocabulary L_0 such that $\phi \models \theta$ and $\theta \models \psi$. Let c_i, $i < \omega$, be new constants and $L_i^* = L_i \cup \{c_i | \ i < \omega\}$ for $i < 3$. If T_1 is an L_1^*-theory and T_2 an L_2^*-theory, we say that T_1 and T_2 are separable if there is an L_0^* sentence θ such that $T_1 \models \theta$ and $T_2 \cup \{\theta\}$ is inconsistent. Then we say that θ separates T_1 and T_2. Notice that from our counter assumption it follows that $\{\phi\}$ and $\{\neg\psi\}$ are not separable (if $\theta^*(c_0, ..., c_n)$, where $\theta^*(x_0, ..., x_n)$ is an L_0-formula, separates them, then $\theta = \exists x_0 ... \exists x_n \theta$ separates them as well and thus $\phi \models \theta$ and $\theta \models \psi$) and that if T_1 and T_2 are not separable, then they are consistent.

Let η_i, $i < \omega$, list sentences so that $\{\eta_i | \ i < \omega, \ i \text{ even}\}$ is the set of all L_1^*-sentences and $\{\eta_i | \ i < \omega, \ i \text{ odd}\}$ is the set of all L_2^*-sentences (thus every L_0^*-sentence appears in both of the sets). Furthermore we assume that if c_j appears in η_i, then $j < i$. By recursion on $i < \omega$, we construct L_1^*-theories T_i^1 and L_2^*-theories T_i^2 so that

(i) T_i^1 and T_i^2 are not separable and if $i < j$, then $T_i^k \subseteq T_j^k$ for $k \in \{1, 2\}$,

(ii) if i is even then $\eta_i \in T_{i+1}^1$ or $\neg\eta_i \in T_{i+1}^1$ and if i is odd then $\eta_i \in T_{i+1}^2$ or $\neg\eta_i \in T_{i+1}^2$,

(iii) if i is even and $\eta_i = \exists v_k \xi(v_k) \in T_{i+1}^1$ for some formula ξ, then $\xi(c_i) \in T_{i+1}^1$ and if i is odd and $\eta_i = \exists v_k \xi(v_k) \in T_{i+1}^1$ for some formula ξ, then $\xi(c_i) \in T_{i+1}^1$,

(iv) if c_j appears in some sentence in $T_i^1 \cup T_i^2$, then $j < i$.

We let $T_0^1 = \{\phi\}$ and $T_0^2 = \{\neg\psi\}$. Suppose that we have defined T_i^1 and T_i^2. We also assume that i is even, the other case is similar. We let $T_{i+1}^2 = T_i^2$. If $T' = T_i^1 \cup \{\eta_i\}$ and T_{i+1}^2 are separated by θ and $T'' = T_i^1 \cup \{\neg\eta_i\}$ and T_{i+1}^2 are separated by θ', then $\theta \vee \theta'$ separates T_i^1 and T_i^2 and thus we can choose T to be the one that is not separable from T_{i+1}^2. If $T = T_i^1 \cup \{\eta_i\}$ and $\eta_i = \exists v_i \xi(v_i)$, we let $T_{i+1}^1 = T \cup \{\xi(c_i)\}$ and otherwise we let $T_{i+1}^1 = T$. Clearly T_{i+1}^1 and T_{i+1}^2 are not separable (exercise, see the proof of Theorem 9.2 or above).

Let $T_1 = \cup_{i<\omega} T_i^1$ and $T_2 = \cup_{i<\omega} T_1^2$. Then they are consistent and we let $\mathcal{A} \models T_1$ and $\mathcal{B} \models T_2$. Notice that T_1 and T_2 contain exactly the same L_0^*-sentences i.e. $T_1 \cap T_2$ is a complete L_0^*-theory. Thus we can also choose $\mathcal{A}$ and $\mathcal{B}$ so that for all $i < \omega$, $c_i^{\mathcal{A}} = c_i^{\mathcal{B}}$. Let $X = \{c_i^{\mathcal{A}} | \ i < \omega\}$. Then as before $\mathcal{A} \restriction X \preceq \mathcal{A}$ and $\mathcal{B} \restriction X \preceq \mathcal{B}$. Thus letting $\mathcal{C}$ be $L_1^* \cup L_2^*$-model such that $\mathcal{C} \restriction L_1^* = \mathcal{A} \restriction X$ and $\mathcal{C} \restriction L_2^* = \mathcal{B} \restriction X$, $\mathcal{C} \models \phi$ and $\mathcal{C} \models \neg\psi$ contradicting the assumption that $\phi \models \psi$. □

Notice that from Craig's interpolation theorem it follows that $\Delta(L_{\omega\omega}) = L_{\omega\omega}$ (for $\Delta(L_{\omega\omega})$ see the slides of the course Strong logics by J. Väänänen).

Let $L^* = L \cup \{P\}$, where P is a new n-ary relation symbol and $\phi(x)$, $x = (x_1, ..., x_n)$, be an L^*-sentence. If P' is another new n-ary relation symbol, we write $\phi(P')$ for the formula we get from ϕ by replacing P by P' everywhere. So $\phi(P) = \phi$. We say that ϕ defines P implicitly if $\phi(P) \wedge \phi(P') \models \forall x_1 ... \forall x_n (P(x) \leftrightarrow P'(x))$.

9.17 Exercise. *Suppose $\phi(P)$ defines P implicitly. Show that there is an L-formula $\psi(x)$ such that it defines $P^{\mathcal{A}}$ in every L^*-model $\mathcal{A} \models \phi(P)$ (this is usually expressed by saying that ψ defines P explicitly). Hint: Let $c_1, ..., c_n$ be new constant symbols and notice that $\phi(P) \wedge P(c_1, ..., c_n) \models \phi(P') \rightarrow P'(c_1, ..., c_n)$.*

10. Indiscernible sequences

In Section 12 indiscernible sequences will play an important role. In this section we make some general observations about them.

10.1 Definition. *Suppose* $(I,<)$ *is a linear ordering, for all* $i\in I$, $a_i\in\mathcal{A}^n$ *and* $A\subseteq\mathcal{A}$. *We say that* $(a_i)_{i\in I}$ *is* m^**-indiscernible over* A *if for all* $m\le m^*$, $\phi(x^1,...,x^m,a)$, $x^k=(x^k_1,...,x^k_n)$ *and* $a\in A^{n^*}$, *the following holds: If* $i_1<i_2<...<i_m$ *and* $j_1<j_2<...<j_m$ *are from* I, *then* $\mathcal{A}\models\phi(a_{i_1},...,a_{i_m},a)$ *iff* $\mathcal{A}\models\phi(a_{j_1},...,a_{j_m},a)$. $(a_i)_{i\in I}$ *is indiscernible over* A *if it is* m^**-indiscernible over* A *for all* $m^*\in\mathbb{N}$.

If for all $i,j\in I$, $a_i=a_j$, $(a_i)_{i\in I}$ is called trivial (indiscernible sequence). When we talk about indiscernible sequences we mean non-trivial ones.

Notice that $(a_i)_{i\in I}$ is indiscernible over A can be defined also as follows (exercise): For all $n<\omega$ and $i_1,...,i_n,j_1,...,j_n\in I$, if $x=(x_1,...,x_n)$ and

$$t^x_{at}((i_1,...,i_n)/\emptyset;(I,<))=t^x_{at}((j_1,...,j_n)/\emptyset;(I,<))$$

(see Definition 5.12), then for all formulas $\phi(x,y)$, $y=(y_1,...,y_k)$ and $b\in A^k$,

$$\mathcal{A}\models\phi(a_{i_1},...,a_{i_n},b)\leftrightarrow\phi(a_{j_1},...,a_{j_n},b).$$

10.2 Definition. *Let* κ,λ *and* ξ *be cardinals and* $X\subseteq\kappa$.

(i) By $[X]^n$ *we mean the set* $\{(\alpha_1,...,\alpha_n)\in X^n|\ \alpha_1<\alpha_2<...<\alpha_n\}$.

(ii) We write $\kappa\to(\lambda)^n_\xi$ *if the following holds: For all functions* $f:[\kappa]^n\to\xi$ *there is* $X\subseteq\kappa$ *of power* λ *such that* $f\restriction[X]^n$ *is constant (such* X *is called homogeneous).*

10.3 Ramsay's theorem. $\omega\to(\omega)^n_k$ *for all* $n,k\in\mathbb{N}-\{0\}$.

Proof. By induction on n:

$n=1$: This is just the pigeon hole principle.

$n=m+1$: By the induction assumption we can find by recursion on $i<\omega$, infinite sets $X_i\subseteq\omega$, $b_i\in X_i$, functions $f_i:[X_i-\{b_i\}]^m\to k$ and $c_{i+1}\in k$ as follows:

$i=0$: $X_0=\omega$, $b_0=0$ and $f_0(a_1,...,a_m)=f(b_0,a_1,...,a_m)$.

$i=j+1$: We let $X_i\subseteq X_j$ and $c_i\in k$ be such that X_i is infinite and for all $(a_1,...,a_m)\in[X_i]^m$, $f_j(a_1,...,a_m)=c_i$. We let b_i be the least element of X_i and $f_i(a_1,...,a_m)=f(b_i,a_1,...,a_m)$.

By the case $i=1$, we can find infinite $I\subseteq\omega$ and $c\in k$ such that for all $i\in I$, $c_{i+1}=c$. Then $X=\{b_i|\ i\in I\}$ is as wanted (exercise). □

The following theorem is just one example of what kind of indiscernible sequences can be found by compactness.

10.4 Theorem. *Suppose* $(I,<)$ *is a linear ordering,* $a_i=(a^i_1,...,a^i_n)\in\mathcal{A}^n$, $i<\omega$, *are such that for* $i\ne j$, $a_i\ne a_j$. *Then there are* $\mathcal{A}\preceq\mathcal{B}$ *and* $e_i\in\mathcal{B}^n$ *such that* $(e_i)_{i\in I}$ *is indiscernible over* $\mathcal{A}$ *and for all* $i_1<...<i_m\in I$, $d\in\mathcal{A}^k$ *and formula* $\phi(z_1,...,z_m,y)$, *if* $\mathcal{B}\models\phi(e_{i_1},...,e_{i_m},d)$, *then for some* $j_1<...<j_m<\omega$, $\mathcal{A}\models\phi(a_{j_1},...,a_{j_m},d)$.

Proof. Let c^i_j, $i \in I$ and $1 \le j \le n$, be constants not in $L(\mathcal{A})$. Denote $c_i = (c^i_1, ..., c^i_n)$. Clearly it is enough to show that the following theory T is consistent:

$$T = Th(\mathcal{A}, \mathcal{A}) \cup$$

$$\{\psi(c_{i_1}, ..., c_{i_k}, \underline{b}) \leftrightarrow \psi(c_{j_1}, ..., c_{j_k}, \underline{b}) |\ \psi(z_1, ..., z_k, z)\ L\text{-formula},\ b \in \mathcal{A}^{lg(z)},$$

$$i_1 < i_2 < ... < i_k,\ j_1 < j_2 < ... < j_k\} \cup$$

$$\{\neg\phi(c_{i_1}, ..., c_{i_m}, \underline{d}) |\ \phi(z_1, ..., z_m, z)\ L\text{-formula such that}$$

$$\text{for all } j_1 < ... < j_m < \omega,\ \mathcal{A} \not\models \phi(a_{j_1}, ..., a_{j_m}, d),$$

$$i_1 < ... < i_m \in I,\ d \in \mathcal{A}^{lg(z)}\},$$

where $lg(z)$ denotes the length of z.

Let $\psi_s(z_1, ..., z_k, z)$, $s < s^* < \omega$ be L-formulas, $d \in \mathcal{A}^{lg(z)}$, $\phi(z_1, ..., z_m, z)$ be an L-formula such that for all $i_1 < ... < i_m < \omega$, $\mathcal{A} \not\models \phi(a_{i_1}, ..., a_{i_m}, d)$, for all $s < s^*$, $i^s_1 < ... < i^s_k \in I$ and $j^s_1 < ... < j^s_k \in I$ and $i_1 < ... < i_m \in I$. By compactness, it is enough to show that

$$T' = Th(\mathcal{A}, \mathcal{A}) \cup$$

$$\{\psi_s(c_{i^s_1}, ..., c_{i^s_k}, \underline{d}) \leftrightarrow \psi_s(c_{j^s_1}, ..., c_{j^s_k}, \underline{d}) |\ s \le s^*\} \cup$$

$$\{\neg\phi(c_{i_0}, ..., c_{i_m}, \underline{d})\}$$

is consistent.

For each $s \le s^*$ define $f_s : [\omega]^k \to 2$ so that $f_s(n_1, ..., n_k) = 1$ if $\mathcal{A} \models \psi_s(a_{n_1}, ..., a_{n_k}, b)$ and otherwise $f_s(n_1, ..., n_k) = 0$. By Ramsey's theorem there is infinite $X \subseteq \omega$ such that it is homogeneous for every function f_s, $s \le s^*$.

Let $\pi : \{i^s_l, j^s_l |\ s \le s^*.\ 1 \le l \le k\} \cup \{i_0, ..., i_m\} \to X$ be order preserving. Let $\mathcal{A}^*$ be a model we get from $(\mathcal{A}, \mathcal{A})$ by interpreting $c^{i^s_l}_p$ to $a^{\pi(i^s_l)}_p$ and $c^{j^s_l}_p$ to $a^{\pi(j^s_l)}_p$, where $1 \le l \le k$ and $1 \le p \le n$, and $c^{i_l}_p$ to $a^{\pi(i_l)}_p$ for $l \le m$ and $1 \le p \le n$. Clearly $\mathcal{A}^* \models T'$. □

Theorem 10.4 can be used to prove e.g. the following result which is at the hearth of many so called non-structure theorem:

10.5 Corollary. *Suppose $(I, <)$ is an infinite linear ordering, $a_i = (a^i_1, ..., a^i_n) \in \mathcal{A}^n$, $i < \omega$, and $d \in \mathcal{A}^m$ are such that for $i \ne j$, $\mathcal{A} \models \phi(a_i, a_j, d)$ iff $i < j$, where $\phi(x, y, z)$ is a formula. Then there are $\mathcal{A} \preceq \mathcal{B}$ and $e_i \in \mathcal{B}^n$ such that $(e_i)_{i \in I}$ is indiscernible over $\mathcal{A}$ and for all $i, j \in I$, $i \ne j$, $\mathcal{B} \models \phi(e_i, e_j, d)$ iff $i < j$.*

Proof. Exercise. □

10.6 Theorem. *Suppose $(I, <) \subseteq (J, <)$ are infinite linear orderings, $A \subseteq \mathcal{A}$ and $a_i \in \mathcal{A}^n$, $i \in I$, are such that $(a_i)_{i \in I}$ is indiscernible over A. Then there are $\mathcal{A} \preceq \mathcal{B}$ and $b_i \in \mathcal{B}^n$ such that $(b_i)_{i \in J}$ is indiscernible over A and for all $i \in I$, $b_i = a_i$.*

Proof. As the proof of the previous theorem (exercise, Ramsey's theorem is not needed). □

10.7 Lemma. *Suppose* $(J,<)$ *is a linear ordering and* $(a_i)_{i\in J}$ *is indiscernible over* $A\subseteq\mathcal{A}$, $a_i=(a^i_1,...,a^i_n)\in\mathcal{A}^n$. *Let* $I_i\subseteq J$, $i<3$, *be such that* $J=\cup_{i<3}I_i$ *and for all* $x_i\in I_i$, $i<3$, $x_0<x_1<x_2$. *Then* $(a_i)_{i\in I_1}$ *is indiscernible over* $A\cup\{a^j_k|\ j\in I_0\cup I_2,\ 1\le k\le n\}$.

Proof. Exercise. □

10.8 Exercise. *Suppose* $\mathcal{A}\models T_{acf_0}$. *For* $A\subseteq\mathcal{A}$, *by* $acl(A)$ *we mean the algebraic closure of* A *(i.e. the set of all roots from* $\mathcal{A}$ *of all non-zero polynomials* $P(X)$ *over the field generated by* A*). Then* $(a_i)_{i<\omega}$, $a_i\in\mathcal{A}$ *(*a_i *a singleton), is indiscernible over* $A\subseteq\mathcal{A}$ *(and non-trivial) iff for all* $i<\omega$, $a_i\notin acl(A\cup\{a_j|\ j<i\})$. *Hint for the direction from right to left: There is essentially only one such sequence. If* $(a'_i)_{i<\omega}$ *is another one, then by induction on* i *one can find an isomorphism* f *between the fields generated by* $A\cup\{a_i|\ i<\omega\}$ *and by* $A\cup\{a'_i|\ i<\omega\}$ *such that* $f\restriction A=id$ *and for all* $i<\omega$, $f(a_i)=a'_i$. *(See also the next section.)*

We say that $(a_i)_{i\in I}$, $a_i\in\mathcal{A}^m$, is set-indiscernible over $A\subseteq\mathcal{A}$ if for all $n<\omega$, $i_0,...,i_n,j_0,...,j_n\in I$, a formula $\phi(x^0,...,x^n,y)$ and $a\in A^k$ the following holds: if for all $l<p\le n$, $i_l\neq i_p$ and $j_l\neq j_p$, then $\mathcal{A}\models\phi(a_{i_0},...,a_{i_n},a)$ iff $\mathcal{A}\models\phi(a_{j_0},...,a_{j_n},a)$.

For κ-stable see Exercise 8.16.

10.9 Exercise. *Let* $(a_i)_{i\in\omega}$, $a_i\in\mathcal{A}^m$, *be indiscernible over* $A\subseteq\mathcal{A}$.

(i) Let $\pi\neq id$ *be a permutation of* $n+1$, $n<\omega$. *Show that there are* $m_i\in n$, $i\le k<\omega$, *such that* $\pi=(m_k,m_k+1)\circ(m_{k-1},m_{k-1}+1)\circ...\circ(m_0,m_0+1)$, *where the permutations* (m,m') *are the usual transpositions.*

(ii) Suppose $(a_i)_{i\in I}$ *is not set-indiscernible. Show that there are* $k<n<\omega$, *a formula* $\phi(x^0,...,x^{n+1},y)$ *and* $a\in A^p$ *such that* $\mathcal{A}\models\phi(a_0,...,a_{n+1},a)$ *but* $\mathcal{A}\models\neg\phi(a_0,...,a_{k-1},a_{k+1},a_k,a_{k+2},...,a_{n+1},a)$.

(iii) Show that if $Th(\mathcal{A})$ *is* ω*-stable (or* κ*-stable for some* κ*), then* $(a_i)_{i\in\omega}$ *is set-indiscernible over* A. *Hint: Exercise 8.13.*

11. Pregeometries

In this section we look at combinatorial geometries and their relation to model theory.

11.1 Definition. *Let* X *be a set and* $cl:\mathcal{P}(X)\to\mathcal{P}(X)$. *We say that* (X,cl) *is a pregeometry if for all* $A\subseteq B\subseteq X$ *and* $a,b\in X$, *the following holds:*

(g1) $A\subseteq cl(A)\subseteq cl(B)=cl(cl(B))$,

(g2) if $a\in cl(A)$, *then there is finite* $C\subseteq A$ *such that* $a\in cl(C)$,

(g3) if $b\in cl(A\cup\{a\})-cl(A)$, *then* $a\in cl(A\cup\{b\})$.

If in addition, $cl(\emptyset)=\emptyset$ *and* $cl(\{a\})=\{a\}$ *for all* $a\in X$, *we say that* (X,cl) *is a geometry.*

The property (g3) above is called Steinitz exchange principle.

11.2 Exercise.
(i) Show that $(\mathbf{R}^{87}, span)$ *is a pregeometry, where* $\mathbf{R}^{87}$ *is considered as a vector space over the reals* $\mathbf{R}$ *(of dimension 87) and for* $A \subseteq \mathbf{R}^{87}$, $span(A)$ *is the subspace generated by* A $(span(\emptyset) = span(\{0\}) = \{0\})$*.*

(ii) Show that $(\mathbf{R}^{87}, aff)$ *is a geometry, where* $aff(A)$ *is the affine subspace generated by* A *i.e.* $\{\Sigma_{i=0}^{n} r_i a_i |\ n < \omega,\ r_i \in \mathbf{R},\ a_i \in A$ *and* $\Sigma_{i=0}^{n} r_i = 1\}$ $(aff(\emptyset) = \emptyset)$*. Hint: One may want to start by showing that for any* $a \in A$, $aff(A) = f_a(span(f_a^{-1}(A)))$*, where* $f_a : \mathbf{R}^{87} \to \mathbf{R}^{87}$ *is such that* $f_a(x) = x + a$*.*

Notice that $(\mathbf{R}^2, aff)$ can be see also as the structure in Example 1.19 (the lines are $aff(\{a, b\})$ for $a, b \in \mathbf{R}^2$, $a \neq b$).

For the rest of this section, we let (X, cl) be an arbitrary pregeometry.

11.3 Exercise. *Show that for all* $A \subseteq X$, $cl(A) = \bigcup\{cl(C) |\ C \subseteq A,\ C \text{ finite}\}$*.*

Let $Y = X - cl(\emptyset)$. On Y we define a binary relation E so that $(a, b) \in E$ if $cl(\{a\}) = cl(\{b\})$.

11.4 Exercise. *Show that for all* $a \in Y$ *and* $A \subseteq Y$*, if* $cl(\{a\}) \cap A \neq \emptyset$*, then* $cl(\{a\}) \subseteq cl(A)$*. Conclude that* E *is an equivalence relation.*

Let $X^* = Y/E = \{a/E |\ a \in Y\}$, where $a/E = \{b \in Y |\ (a, b) \in E\}$. Let $cl^* : \mathcal{P}(X^*) \to \mathcal{P}(X^*)$ be such that $cl^*(A) = (cl(\cup A) \cap Y)/E$ (i.e. for $a \in Y$, $a/E \in cl^*(\{a_i/E |\ i \in I\})$ iff $a \in cl(\{a_i |\ i \in I\})$).

11.5 Exercise. *Show that* (X^*, cl^*) *is a geometry.*

The geometry $((\mathbf{R}^{87})^*, span^*)$ is also called a projective space and denoted by $\mathrm{P}_{86}(\mathbf{R})$. As with the affine closure, also $\mathrm{P}_2(\mathbf{R}) = ((\mathbf{R}^3)^*, span^*)$ can be seen as a structure of the form of Exercise 1.19. Exercise: What is the difference between $A_2(\mathbf{R})$ and $\mathrm{P}_2(\mathbf{R})$? (In geometry, a/E e.g. for $a = (x, y, z) \in \mathbf{R}^3 - \{0\}$, is usually denoted by $(x : y : z)$ or $[x, y, z]$.)

For all $A \subseteq X$, by $cl_A : \mathcal{P}(X) \to \mathcal{P}(X)$ we mean the closure operation $cl_A(B) = cl(A \cup B)$.

11.6 Exercise. *Show that for all* $A \subseteq X$, (X, cl_A) *is a pregeometry.*

11.7 Definition. *We say that* $(a_i)_{i \in I} \subseteq X$ *is independent (in* (X, cl)*) if for all* $i \in I$, $a_i \notin cl(\{a_j |\ j \in I - \{i\}\})$*.*

11.8 Exercise. *Suppose* $I = (I, <)$ *is linearly ordered. Show that* $(a_i)_{i \in I} \subseteq X$ *is independent if for all* $i \in I$, $a_i \notin cl(\{a_j |\ j \in I,\ j < i\})$*.*

11.9 Definition. *Suppose* $(a_i)_{i \in I} \subseteq A \subseteq X$*. We say that* $(a_i)_{i \in I}$ *is a basis of* A *(in* (X, cl)*) if it is independent (in* (X, cl)*) and* $A \subseteq cl(\{a_i |\ i \in I\})$*.*

11.10 Exercise. *Show that every set* $A \subseteq X$ *has a basis.*

11.11 Lemma. *Suppose $I \cap J = \emptyset$.*

(i) Suppose $A \subseteq B \subseteq X$, $(a_i)_{i\in I}$ is a basis of A (in (X,cl)) and $(a_i)_{i\in J}$ is a basis of B in (X,cl_A). Then $(a_i)_{i\in I\cup J}$ is a basis of B in (X,cl).

(ii) Suppose $(a_i)_{i\in I\cup J}$ is a basis of $A \subseteq X$. Then $(a_i)_{i\in J}$ is a basis of A in $(X,cl_{\{a_i|\ i\in I\}})$.

Proof. (i): Clearly $(a_i)_{i\in I\cup J} \subseteq B$. It is also independent: Pick some linear order $<$ on $I\cup J$ such that for all $i\in I$ and $j\in J$, $i<j$. By Exercise 11.8, it is enough to show that for all $i\in I\cup J$, $a_i \notin cl(\{a_j|\ j\in I\cup J,\ j<i\})$. If $i\in I$, this is clear. So suppose $i\in J$. Then $a_i \notin cl_A(\{a_j|\ j\in J,\ j<i\}) = cl(A\cup\{a_j|\ j\in J,\ j<i\}) \supseteq cl(\{a_j|\ j\in I\cup J,\ j<i\})$.

So we are left to prove that $B\subseteq cl(\{a_i|\ i\in I\cup J\})$. But $B\subseteq cl_A(\{a_i|\ i\in J\}) = cl(A\cup\{a_i|\ i\in J\}) \subseteq cl(cl(\{a_i|\ i\in I\})\cup\{a_i|\ i\in J\}) \subseteq cl(cl(\{a_i|\ i\in I\cup J\})) = cl(\{a_i|\ i\in I\cup J\})$.

(ii): Exercise. □

11.12 Theorem. *Suppose $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ are basis of $A\subseteq X$. Then $|I|=|J|$ (i.e. I and J have the same cardinality).*

Proof. If both I and J are infinite, then the claim is immediate: By symmetry, it is enough to show that $|I|\le|J|$. For this, for each $j\in J$, we can find finite $I_j\subseteq I$ such that $b_j\in cl(\{a_i|\ i\in I_j\})$. Let $I'=\cup_{j\in J}I_j$. Then $|I'|\le|J|$ and $A\subseteq cl(\{b_j|\ j\in J\})\subseteq cl(cl(\{a_i|\ i\in I'\})) = cl(\{a_i|\ i\in I'\})$ and thus $I'=I$.

So we may assume that $I=\{i_k|k<n\}$. If $n=0$, then $A\subseteq cl(\emptyset)$ and thus also $J=\emptyset$. So it is enough to prove the claim for $n=m+1$ under the assumption that the claim holds for m.

Choose finite $J'\subseteq J$ such that $a_{i_m}\in cl(\{b_j|\ j\in J'\})$ and choose J' so that in addition $|J'|$ minimal. By well-ordering J, we may assume that J is an ordinal and that $J'=\{0,...,p\}$. Similarly we may assume that $i_k=k$ for all $k<n$. Since $a_m\in cl(\{b_0,...,b_p\}) - cl(\{b_0,...,b_{p-1}\})$. $cl(\{b_0,...,b_{p-1},a_m\}) = cl(\{b_0,...,b_p\})$ and thus by Exercise 11.8 and a simple manipulation, $\{b_j|\ j\in J-\{p\}\}\cup\{a_m\}$ is a basis of A. Thus by Lemma 11.11 (ii), $\{a_0,...,a_{m-1}\}$ and $\{b_j|\ j\in J-\{p\}\}$ are basis of A in $(X,cl_{\{a_m\}})$. By the induction assumption, $|J-\{p\}|=m$ and thus $|J|=|I|$. □

11.13 Definition. *Suppose $A,B\subseteq X$. By the dimension of A, $dim(A)$, we mean the cardinality of a basis of A. Similarly by $dim(A/B)$ we mean the dimension of A in the pregeometry (X,cl_B).*

Notice that by Exercise 11.10 and Theorem 11.12, dimensions are well-defined. Notice also that $dim(\mathbf{R}^2)$ in $A_2(\mathbf{R}) = (\mathbf{R}^2, aff)$ as well as $dim((\mathbf{R}^3)^*)$ in $\mathrm{P}_2(\mathbf{R}) = ((\mathbf{R}^3)^*, span^*)$ is 3 (although they are called planes).

11.14 Exercise.

(i) Show that $dim(A\cup B) = dim(A) + dim(B/A)$.

(ii) Show that $(\mathbf{R}^{87}, span)$ is modular i.e. for all $A,B\subseteq\mathbf{R}^{87}$, $dim(A\cup B) = dim(A)+dim(B)-dim(span(A)\cap span(B))$.

(iii) Show that $(\mathbf{R}^{87}, aff)$ *is not modular but* $(\mathbf{R}^{87}, aff_{\{a\}})$ *is modular for any* $a \in \mathbf{R}^{87}$ *(i.e.* $(\mathbf{R}^{87}, aff)$ *is locally modular).*

11.15 Definition. *Let* $\mathcal{A}$ *be a model.*

(i) Suppose $A \subseteq \mathcal{A}$. *We say that* a *is algebraic over* A *if there are* $b \in A^n$ *and* $\phi(x,y)$ *such that* $\mathcal{A} \models \phi(a,b)$ *and* $\phi(\mathcal{A},b)\,(=\{c \in \mathcal{A}|\ \mathcal{A} \models \phi(c,b)\})$ *is finite. By the algebraic closure* $acl_{\mathcal{A}}(A)$ *of* A *we mean the set of elements of* $\mathcal{A}$ *that are algebraic over* A. *Usually we write just* $acl(A)$ *for* $acl_{\mathcal{A}}(A)$ *(see Exercise 11.16 (ii)). If* $A \subseteq \mathcal{A}^n$, *we write* $acl(A)$ *for the set* $acl(Pr(A))$, *where* $Pr(A)$ *is the set of all* $b \in \mathcal{A}$ *such that for some* $a = (a_1, ..., a_n) \in A$ *and* $1 \le i \le n$, $b = a_i$. *Instead of* $acl(Pr(\{a\}))$ *we write just* $acl(a)$.

(ii) We say that a definable infinite set $A \subseteq \mathcal{A}^n$ *is minimal if for all definable sets* $B \subseteq \mathcal{A}^n$ *either* $A \cap B$ *is finite or* $A - B$ *is finite.*

(iii) We say that a definable set $A \subseteq \mathcal{A}^n$ *is strongly minimal if for all* $\mathcal{B} \succeq \mathcal{A}$, $\phi(\mathcal{B}^n, b)$ *is minimal, where* $\phi(x,y)$ *and* $b \in \mathcal{A}^m$ *are such that* $A = \phi(\mathcal{A}^n, b)$.

(iv) We say that $\mathcal{A}$ *is (strongly) minimal if the universe of* $\mathcal{A}$ *is (strongly) minimal.*

Typical examples of strongly minimal sets are (irreducible) algebraic curves. E.g. an elliptic curve $C = \{(x,y,z)/E \in \mathrm{P}_2(\mathbf{C})|\ y^2z = x^3 + axz^2 + bz^3\}$, $4a^3 + 27b^2 \neq 0$ is a strongly minimal subset of $\mathrm{P}_2(\mathbf{C})$ when $\mathrm{P}_2(\mathbf{C})$ is equipped with a suitable structure (e.g. the structure from Exercise 1.19; then C is definable using e.g. lines $\{(x,y,z)/E|\ x=0\}$, $\{(x,y,z)/E|\ y=0\}$, $\{(x,y,z)/E|\ z=0\}$ and points $(1,0,1)/E$, $(0,1,1)/E$, $(a,0,1)/E$ and $(b,0,1)/E$ as parameters). The curve C is also a strongly minimal subset of $\mathbf{C}^{eq}$ (for $\mathbf{C}^{eq}$, see any book on stability theory or [Ho]).

Notice that if $\phi(\mathcal{A}^n/b)$ is finite, then for all $a = (a_1, ..., a_n) \in \phi(\mathcal{A}^n, b)$ and $1 \le i \le n$, $a_i \in acl(b)$ (exercise).

11.16 Exercise.

(i) Suppose $\mathcal{A} \models T_{acf_0}$ *and* $A \subseteq \mathcal{A}$. *Show that* $acl(A)$ *as defined above is the same as* $acl(A)$ *as defined in Example 10.8. Conclude that* $\mathcal{A}$ *is strongly minimal.*

(ii) Suppose $A \subseteq \mathcal{A} \prec \mathcal{B}$. *Then* $acl_{\mathcal{B}}(A) = acl_{\mathcal{A}}(A)$.

(iii) Show that Definition 11.15 (iii) does not depend on the choice of ϕ *and* b.

(iv) Give an example of a minimal structure $\mathcal{A}$ *such that it is not strongly minimal. Hint: Think of an equivalence relation whose equivalence classes are finite but of different size. Theorem 7.8 may be useful.*

(v) Find $\mathcal{A}$ *such that* $(\mathcal{A}, acl)$ *does not satisfy Steinitz exchange principle (g3) of Definition 11.1.*

11.17 Definition. *Suppose* $X \subseteq \mathcal{A}^n$ *is minimal and definable with parameters* a^*. *Then by* $acl^X_{a^*}$ *we denote the operation* $acl^X_{a^*}(A) = acl(A \cup a^*)^n \cap X$ *for* $A \subseteq X$. *If there is no risk for confusion, we write just* acl *for* $acl^X_{a^*}$.

11.18 Exercise. *Suppose* $X \subseteq \mathcal{A}^n$ *is minimal and definable with parameters* a^* *and* $A \subseteq X$.

(i) Show that $a \in acl^X_{a^*}(A)$ iff $a \in X$ and there are $\phi(x,y,z)$ and $b \in A^m$ such that $\phi(X,b,a^*)$ $(= \{c \in X \mid \mathcal{A} \models \phi(c,b,a^*)\})$ is finite and $\mathcal{A} \models \phi(a,b,a^*)$.

(ii) Suppose $a,b \in X - acl^X_{a^*}(A)$. Show that $t(a/Pr(A) \cup a^*) = t(b/Pr(A) \cup a^*)$.

Notice that in the proof of the following theorem, the strong minimality assumption is not needed in the proof of (g1) and (g2). Also for (g3), minimality is enough, see Exercise 11.20.

11.19 Theorem. *Suppose $X \subseteq \mathcal{A}^k$ is strongly minimal and definable with parameters a^*. Then (X, acl) $(= (X, acl^X_{a^*}))$ is a pregeometry.*

Proof. So let $A \subseteq B \subseteq X$ and $a,b \in X$. We show that (g1)-(g3) are satisfied. We assume that $a^* = \emptyset$. E.g. by naming a^* with new constants, this can be done without loss of generality (but it simplifies notations).

(g1): Formulas $v_1 = c_1 \wedge ... \wedge v_k = c_k$, $c = (c_1, ..., c_k) \in A$, show that $A \subseteq acl(A)$. $acl(A) \subseteq acl(B)$ follows immediately from the definition. For acl(B)=acl(acl(B)), pick $c \in acl(acl(B))$. It is enough to show that $c \in acl(B)$. Now, by Exercise 11.18 (i), there are $\phi(x,y)$, $y = (y_1, ..., y_n)$, and $d = (d_1, ..., d_n) \in acl(B)^n$ such that $\phi(X,d)$ is finite and $\mathcal{A} \models \phi(c,d)$. Since $d_i \in acl(B)$, $1 \le i \le n$, there are $\phi_i(y_i, z_i)$ and $e_i \in B^{n_i}$ such that $\phi_i(X, e_i)$ is finite and $\mathcal{A} \models \phi_i(d_i, e_i)$.

Let $m = |\phi(X,d)|$ and $\psi(y)$ such that $\mathcal{A} \models \psi(e)$ iff $|\phi(X,e)| \le m$ and $e \in X$ (keep in mind that X is definable without parameters). Now it is easy to see that

$$\theta(x, z_1, ..., z_n) = \exists y_1 ... \exists y_n (\psi(y) \wedge (\bigwedge_{1 \le i \le n} \phi_i(y_i, z_i)) \wedge \phi(x,y))$$

and $e_1 \cup ... \cup e_n$ witnesses that $c \in acl(B)$.

(g2): Immediate by the definition.

(g3): The proof we give here is not the most simple one (see Exercise 11.20). Instead it is the one that can be used to prove symmetries also in many other cases (e.g. symmetry of non-forking for stable theories). In order to avoid some cardinal arithmetical considerations in this proof, we assume that the language is countable. This guarantees that for all $A \subseteq X$, $|acl(A)| \le max\{|A|, \omega\}$ (exercise). In Exercise 11.20 this assumption is not used.

By Exercise 11.16 (ii), it is enough to show that (g3) holds in some elementary extension of $\mathcal{A}$. Thus by Theorem 8.7, we may assume that $\mathcal{A}$ is $(2^\omega)^+$-saturated and strongly $(2^\omega)^+$-homogeneous. Also if (g3) fails, there is an example in which A is finite. So without loss of generality, we may assume that A is finite.

For a contradiction, suppose $a \in acl(A \cup \{b\}) - acl(A)$ and $b \notin acl(A \cup \{a\})$. Let $\phi(x,y,A)$ be such that $\phi(X,b,A)$ is finite and $\mathcal{A} \models \phi(a,b,A)$.

Choose $a_i \in X$, $i < \omega$, such that for all $i < \omega$, $a_i \notin acl(A \cup \bigcup_{j<i} a_j)$ (this is possible since for all $i < \omega$, $|acl(A \cup \bigcup_{j<i} a_j)| \le \omega < |X|$). Now by Exercise 11.18 (ii), for all $j < i < \omega$, $t(a_j a_i / Pr(A)) = t(ab/Pr(A))$. Thus for all $j < i < \omega$, $\mathcal{A} \models \phi(a_j, a_i, A) \wedge \neg\phi(a_i, a_j, A)$. Now we could get a contradiction by modifying Exercises 10.9 and 11.18 (ii) or we can argue as follows: By Theorem 10.4 (and

$(2^\omega)^+$-saturation of $\mathcal{A}$), we can find $e_r \in X$, $r \in \mathbf{R}$, such that for all $r, s \in \mathbf{R}$, $\mathcal{A} \models \phi(e_r, e_s, A)$ iff $r < s$. Since $acl(\{e_q |\ q \in \mathbf{Q}\})$ is countable, we can find $r, s \in \mathbf{R}$ such that $r < s$ and $e_r, e_s \notin acl(A \cup \{e_q |\ q \in \mathbf{Q}\})$. By Exercise 11.18 (ii), $t(e_r/Pr(A \cup \{e_q |\ q \in \mathbf{Q}\}) = t(e_s/Pr(A \cup \{e_q |\ q \in \mathbf{Q}\})$. But letting q be a rational between r and s, $\mathcal{A} \models \phi(e_r, e_q, A) \wedge \neg\phi(e_s, e_q, A)$, a contradiction. □

11.20 Exercise. *Suppose $X \subseteq \mathcal{A}^k$ is minimal and definable with parameters a^*. Show that (X, acl) $(= (X, acl^X_{a^*}))$ satisfies (g3). Hint: To simplify the notations, suppose $a^* = \emptyset$ and $k = 1$. Let $\phi(x)$ define X and suppose that there are $A \subseteq X$ and $a, b \in X$ such that $a \in acl(A \cup \{b\}) - acl(A)$ but $b \notin acl(A \cup \{a\})$. Then there are $c \in A^m$, formulas $\phi^*(x, y, c)$, $\psi(y, c)$ and $\theta(x, c)$ and $N \in \omega$ such that $\mathcal{A} \models \psi(b, c) \wedge \theta(a, c)$ and ψ says that $y \in X$ and there are $\leq N$ many $x \in X$ such that $\phi^*(x, y, c)$ holds and θ says that $x \in X$ and there are $\leq N$ many $y \in X$ such that $\neg\phi^*(x, y, c)$ holds. Then find distinct $b_i \in X$, $i < N(N+1)+1$, that satisfy ψ and distinct $a_i \in X$, $i < N+1$, that satisfy θ. Finally find $j < N(N+1)+1$ such that for all $i < N+1$, $\mathcal{A} \models \phi^*(a_i, b_j, c)$, a contradiction.*

11.21 Definition. *We say that a complete theory T is strongly minimal if some $\mathcal{A} \models T$ is strongly minimal.*

So by Exercise 11.16 (i), T_{acf_0} is strongly minimal.

Now we can formulate Schanuel's conjecture that was mentioned in Remark 6.5. Let $\mathcal{A} \models T_{acf_0}$. The dimension of $A \subseteq \mathcal{A}$ in the pregeometry $(\mathcal{A}, acl)$ is called the trancendence degree and we denote it by $Tr(A)$. However, $\mathcal{A}$ can also be seen as a vector space over $\mathbf{Q}$ (see Exercise 6.7) by letting the addition of the vector space to be the same as the addition of the field and for all $a \in \mathbf{Q}$ and $x \in \mathcal{A}$, letting $f_a(x) = ax$. Then $(\mathcal{A}, span)$ is a pregeometry and by $dim(A)$ we denote the dimension of A in this pregeometry. Now letting $\mathcal{A} = \mathbf{C}$, Schanuel's conjecture says that for all $a_1, ..., a_n \in \mathbf{C}$,

$$Tr(\{a_1, ..., a_n, exp(a_1), ..., exp(a_n)\}) - dim(\{a_1, ..., a_n\}) \geq 0.$$

11.22 Exercise.

(i) Show that if T is strongly minimal, then every $\mathcal{A} \models T$ is strongly minimal. Conclude that if T is strongly minimal, then it is κ-stable for all $\kappa \geq |L_{\omega\omega}|$. Hint: Exercise 11.18 (ii).

(ii) Show that T_{qv} from Exercise 6.7 is strongly minimal and that for $A \subseteq \mathcal{A} \models T_{qv}$, $acl(A) = span(A)$.

11.23 Theorem. *Suppose T is strongly minimal and $\mathcal{A}, \mathcal{B} \models T$. Then $\mathcal{A} \cong \mathcal{B}$ iff $dim(\mathcal{A}) = dim(\mathcal{B})$, where the dimensions are calculated in the pregeometries $(\mathcal{A}, acl)$ and $(\mathcal{B}, acl)$, respectively.*

Proof. The claim from left to right is trivial. So we assume that for some cardinal κ, $\mathcal{A}$ has a basis $(a_i)_{i<\kappa}$ and $\mathcal{B}$ has basis $(b_i)_{i<\kappa}$ and we prove that $\mathcal{A} \cong \mathcal{B}$. Let $f : \{a_i |\ i < \kappa\} \to \{b_i |\ i < \kappa\}$ be such that $f(a_i) = b_i$ for all $i < \kappa$. Then as in the proof of Theorem 11.19 (notice that, by Lemma 8.1 (i) or Theorem 8.7, w.o.l.g. we may assume that there is $\mathcal{C}$ such that $\mathcal{A},\mathcal{B} \prec \mathcal{C} \models T$), f is a partial elementary map from $\mathcal{A}$ to $\mathcal{B}$. By starting from f and recursively going through all the elements of $\mathcal{A}\cup\mathcal{B}$ one can find the isomorphism once one has proved the following claim (and one noticed that the claim is symmetric):

11.23.1 Claim. *Suppose $g : \mathcal{A} \to \mathcal{B}$ is a partial elementary map such that $\mathcal{A} = acl(dom(g))$ and $\mathcal{B} = acl(rng(g))$ and $a \in \mathcal{A}$. Then there is a partial elementary map $h : \mathcal{A} \to \mathcal{B}$ such that $g \subseteq h$ and $a \in dom(h)$.*

Proof. Let $\phi(x, y)$ and $c \in dom(g)^k$ witness that $a \in acl(dom(g))$. Since $\phi(\mathcal{A}, c)$ is finite, we may assume that we have chosen these so that in addition, for all $a' \in \phi(\mathcal{A}, c)$, $t(a'/dom(g)) = t(a/dom(g))$. Now $\mathcal{A} \models \exists x\phi(x, c)$ and thus there is $b \in \mathcal{B}$ such that $\mathcal{B} \models \phi(b, g(c))$. We claim that $h = g \cup \{(a, b)\}$ is as wanted.

Let $\psi(x, z)$ and $d \in dom(g)^n$ be arbitrary such that $\mathcal{A} \models \psi(a, d)$. We need to show that $\mathcal{B} \models \psi(b, g(d))$. Suppose not. Then $\mathcal{B} \models \exists x(\phi(x, g(c)) \wedge \neg\psi(x, g(d)))$. Thus there is $a' \in \mathcal{A}$ such that $\mathcal{A} \models \phi(a', c) \wedge \neg\psi(a', d)$. This contradicts the choice of ϕ and c. □ Claim 11.23.1.

□

Remark: There is an easier proof for Theorem 11.23: One could assume that $\mathcal{C}$ (see the proof) is strongly κ-homogeneous for large enough κ and then extend f to an automorphism g of $\mathcal{C}$ and show that $g(\mathcal{A}) = \mathcal{B}$. We gave the more complicated proof since it contains an important observation that can be used in many situations.

11.24 Exercise. *Suppose T is strongly minimal.*

(i) Show that T is κ-categorical for all $\kappa > |L_{\omega\omega}|$.

(ii) Suppose, in addition, that L is countable. Show that T has, upto isomorphism, at most ω many countable models.

(iii) Suppose, in addition, that L is countable and $A \subseteq \mathcal{A} \models T$. Show that if $acl(A)$ is infinite, then $acl(A) \preceq \mathcal{A}$ (here we think $acl(A)$ as the submodel of $\mathcal{A}$ generated by $acl(A)$ and notice that $acl(A)$ is L-closed). Conclude that T has upto isomorphism either one or infinitely many countable models.

12. Ehrenfeucht-Mostowski models

We start by looking at ideas from the proof of Lemma 4.8.

12.1 Definition. *Given a vocabulary L, a skolemization L^S of L is the vocabulary $L \cup \{f_{\phi(v_i,x)} |\ \phi(v_i, x)$ L-formula, $x = (x_1, ..., x_n)\}$, where $f_{\phi(v_i,x)}$ are new n-ary function symbols (0-ary function symbols are constants). Skolem theory T^S is the set of all sentences*

$$\forall x_1 ... \forall x_n(\exists v_i\phi(v_i, x) \to \phi(f_{\phi(v_i,x)}(x), x)),$$

where $\phi(v_i, x)$ is an L-formula.

Misusing the terminology, we call both the new function symbols and their interpretations (in case they satisfy T^S) Skolem-functions.

The following lemma was part of the proof of Lemma 4.8.

12.2 Lemma.

(i) For all L-structures $\mathcal{A}$ there is L^S-structure $\mathcal{A}^S$ such that $\mathcal{A}^S \models T^S$ and $\mathcal{A}^S \restriction L = \mathcal{A}$.

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are L^S-structures and $\mathcal{B} \subseteq \mathcal{A} \models T^S$, then $\mathcal{B} \restriction L \preceq \mathcal{A} \restriction L$.

Proof. (i) is trivial and (ii) follows immediately from Tarski-Vaught. □

12.3 Definition. *Let $\mathcal{A}$ be an L^S-structure and $A \subseteq \mathcal{A}$. By $SH(A)$ (Skolem hull) we mean the set*

$$\{t^{\mathcal{A}}(a) |\ t(x)\ \ L^S\text{-term},\ x = (x_1, ..., x_n),\ a \in A^n\}.$$

Then $SH(A)$ is closed under all $f^{\mathcal{A}}$, $f \in L^S$ a function symbol, and contains all $c^{\mathcal{A}}$, $c \in L^S$ a constant symbol, and thus it can be equipped with the structure induced from $\mathcal{A}$. This substructure of $\mathcal{A}$ is also called $SH(A)$.

Notice that by Lemma 12.2 (ii), $SH(A) \restriction L \preceq \mathcal{A} \restriction L$ assuming $\mathcal{A} \models T^S$.

12.5 Definition. *Let κ be an infinite cardinal. For all ordinals α, a cardinal $\beth_\alpha(\kappa)$ is defined as follows: $\beth_0(\kappa) = \kappa$, $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$ and for limit α, $\beth_\alpha(\kappa) = \bigcup_{\beta<\alpha} \beth_\beta(\kappa)$. Also we write $\beth_\alpha = \beth_\alpha(\omega)$.*

Notice that for all infinite cardinals κ, $\beth_{\kappa^+}(\lambda) = \beth_{\kappa^+}$ for all $\lambda < \beth_{\kappa^+}$ and that for all $\alpha \geq \omega$, $\beth_\alpha((2^\kappa)^+) = \beth_\alpha(\kappa)$.

12.6 Fact (Erdös-Rado). *For all infinite cardinals κ and $n \in \mathbb{N}$,*

$$(\beth_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa.$$

Proof. See e.g. [Je]. □

Only for notational simplicity, in the following theorem we look at elements $a^\alpha_i \in \mathcal{A}_\alpha$ instead n-sequences $a^\alpha_i \in \mathcal{A}^n_\alpha$ for $n \in \mathbb{N}$.

12.7 Theorem. *Let $\kappa = |L_{\omega\omega}|$ and $\lambda = (2^\kappa)^+$. Suppose that for all $\alpha < \lambda$, we have $\mathcal{A}_\alpha$ and $a^\alpha_i \in \mathcal{A}_\alpha$, $i < (\beth_\alpha(\lambda))^+$, such that $a^\alpha_i \neq a^\alpha_j$ for $i \neq j$. For all $\alpha < \lambda$, let $\mathcal{A}^S_\alpha$ be as in Lemma 12.2 (i) for $\mathcal{A}_\alpha$. Then there is a collection Φ of L^S-formulas with the following properties:*

(i) for every L^S-formula $\phi(v_1, ..., v_n)$, $n \in \mathbb{N}$, either $\phi \in \Phi$ or $\neg\phi \in \Phi$,

(ii) for all linear orderings $(I, <)$, there are L^S-structure $\mathcal{B}$ and $b_i \in \mathcal{B}$, $i \in I$, such that

(a) for all $\phi(v_1, ..., v_n)$ and $i_1 < i_2 < ... < i_n$ from I, $\mathcal{B} \models \phi(b_{i_1}, ..., b_{i_n})$ iff $\phi(v_1, ..., v_n) \in \Phi$,

(b) for all $i_1 < i_2 < ... < i_n$ from I, there are $\alpha < \lambda$, $\gamma_1 < ... < \gamma_n < (\beth_\alpha(\lambda))^+$ and an isomorphism $\pi : SH(\{b_{i_1}, ..., b_{i_n}\}) \to SH(\{a^\alpha_{\gamma_1}, ..., a^\alpha_{\gamma_n}\})$ such that for all $1 \leq k \leq n$, $\pi(b_{i_k}) = a^\alpha_{\gamma_k}$.

Proof. By Lemma 10.7 it is enough to prove (ii) in the case $I=\omega$. Furthermore, by (i) and (a), it is enough to prove (b) in the case $i_k=k$ for all $1\le k\le n\in \mathbb{N}-\{0\}$. We do this.

By recursion on $n\in\mathbb{N}$ we construct n-types Φ_n over $\emptyset$ in vocabulary L^S, and for $\alpha<\lambda$, $\alpha^n\in\lambda-\alpha$ and $X^\alpha_n\subseteq(\beth_{\alpha^n}(\lambda))^+$ of power $\ge\beth_\alpha(\lambda)$ so that

(I) $\alpha^n<\beta^n$ for $\alpha<\beta$ and $\alpha^{n+1}=\beta^n$ for some $\beta\ge\alpha$ and then $X^\alpha_{n+1}\subseteq X^\beta_n$,

(II) for all $\phi(v_1,...,v_n)$ and $i_1<...<i_n$ from X^α_n, $\phi(v_1,...,v_n)\in\Phi_n$ iff $\mathcal{A}^S_{\alpha^n}\models\phi(a^{\alpha^n}_{i_1},...,a^{\alpha^n}_{i_n})$,

(III) if $(a_1,...,a_n)\in\mathcal{A}^n$ realizes Φ_n, then $(a_1,...,a_n)$ is n-indiscernible over $\emptyset$,

(IV) $\Phi_n\subseteq\Phi_{n+1}$.

$n=0$: Since the number of possible L^S-theories is $<\lambda$, there is $X\subseteq\lambda$ of power λ such that for all $\alpha,\beta\in X$, $Th(\mathcal{A}^S_\alpha)=Th(\mathcal{A}^S_\beta)$ and we let Φ_0 be the common theory. $\alpha^0=min(X-\{\beta^0|\ \beta<\alpha\})$ and $X^\alpha_0=(\beth_{\alpha^0}(\lambda))^+$. Clearly (I)-(IV) hold.

$n=m+1$: For all $\alpha<\lambda$, let $\alpha^n_*=(\alpha+n)^m$ and notice that

(*) $|X^{\alpha+n}_m|\ge(\beth_m(\beth_\alpha(\lambda)))^+$.

Define $f^\alpha_n:[X^{\alpha+n}_m]^n\to S_n(\emptyset)$ so that $f^\alpha_n(i_1,...,i_n)=t((a^{\alpha^n_*}_{i_1},...,a^{\alpha^n_*}_{i_n})/\emptyset;\mathcal{A}^S_{\alpha^n_*})$. Since

(**) $|S_n(\emptyset)|<\lambda$,

by Erdös-Rado and (*) above, there is homogeneous $X^{\alpha^n_*}\subseteq X^{\alpha+n}_m$ of power $\ge\beth_\alpha(\lambda)$. Let $p_{\alpha^n_*}$ be the constant value.

By (**), there is $X\subseteq\{\alpha^n_*|\ \alpha<\lambda\}$ of power λ and p such that for all $\gamma\in X$, $p_\gamma=p$. Now let $\alpha^n=min(X-\{\beta^n|\ \beta<\alpha\})$, $X^\alpha_n=X^{\alpha^n}$ and $\Phi_n=p$. Notice that by the assumptions (I) and (II) for m, $\Phi_m\subseteq\Phi_n$. So clearly (I)-(IV) hold.

Then we let $\Phi=\cup_{n<\omega}\Phi_n$. This is as wanted (exercise). □

12.8 Corollary. *Let $\kappa=|L_{\omega\omega}|$ and suppose that T is a theory and D is a collection of types over $\emptyset$. If for all $\lambda<\beth_{(2^\kappa)^+}$ there is $\mathcal{A}\models T$ of power $\ge\lambda$ such that it omits every $p\in D$, then for all $\theta\ge\kappa$ there is $\mathcal{A}\models T$ of power θ such that it omits every $p\in D$.*

Proof. For all $\alpha<(2^\kappa)^+$, since $(\beth_\alpha((2^\kappa)^+))^+<\beth_{(2^\kappa)^+}$, we can find $\mathcal{A}_\alpha\models T$ and $a^\alpha_i\in\mathcal{A}_\alpha$, $i<(\beth_\alpha((2^\kappa)^+))^+$ such that

(i) $a^\alpha_i\ne a^\alpha_j$ for $i\ne j$,

(ii) $\mathcal{A}_\alpha$ omits every $p\in D$.

Let $\Phi=\cup_{n<\omega}\Phi_n$ be as in Theorem 12.7. Let $\mathcal{B}$ and b_i, $i<\theta$, be also as in Theorem 12.7 (ii) for $(I,<)=(\theta,<)$. We claim that $\mathcal{C}=SH(\{b_i|\ i<\theta\})\restriction L$ is as wanted.

Clearly $|\mathcal{C}|=\theta$. Also $T\cup T^S\subseteq\Phi_0$ and thus $\mathcal{B}\models T\cup T^S$ and so $\mathcal{C}\preceq\mathcal{B}\restriction L$ and $\mathcal{C}\models T$. Finally, suppose $p\in D$. For a contradiction, assume that $c=(c_1,...,c_m)\in\mathcal{C}^m$ realizes p. Then there are $i_1<...<i_n$ in θ such that $c_i\in SH(\{b_{i_1},...,b_{i_n}\})=\mathcal{D}$ for all $1\le i\le m$. Since $\mathcal{D}\restriction L\preceq\mathcal{B}\restriction L$ and $\mathcal{D}\subseteq\mathcal{C}$, $\mathcal{D}\restriction L\preceq\mathcal{C}$, and so c realizes p in $\mathcal{D}$. Let α, $\gamma_1,...,\gamma_n$ and π be as in Theorem 12.7 (ii)(b). Then $d=(\pi(c_1),...,\pi(c_m))$ realizes p is $\mathcal{D}'=SH(\{a^\alpha_{\gamma_1},...,a^\alpha_{\gamma_n}\})\restriction L$. Since $\mathcal{D}'\preceq\mathcal{A}_\alpha$, d realizes p in $\mathcal{A}_\alpha$, a contradiction. □

The model $\mathcal{C}=SH(\{b_i|\ i\in I\})\restriction L$ from the proof of Corollary 12.8 is called an Ehrenfeucht-Mostowski model and is denoted by $EM(I,\Phi)$. Notice that I and

Φ determine $EM(I,\Phi)$ upto isomorphism (and not more). In the literature, by Φ one usually means our Φ restricted to quantifier free formulas. (Exercise: Why?)

It is also important to notice that although the easiest way to show that the model $EM(I,\Phi)$ exists (i.e. that the set Φ constructed in the proof of Theorem 12.7 satisfies (ii) from the theorem) is to use compactness, this can also be done without it and thus the construction works in many other context than the one above, perhaps with a bit more carefully chosen interpretations for Skolem-functions.

Recall that Remark 9.3 shows that in Corollary 12.8, one can not replace $\beth_{(2^\kappa)^+}$ by κ^+.

12.9 Exercise. *Show that in Corollary 12.8, one can not replace $\beth_{(2^\kappa)^+}$ by any cardinal $< \beth_{\kappa^+}$. (Hint: Look at models in which there are a countable set, codes for subsets of the set, codes for subsets of subsets of the set etc.)*

By $Ho(\kappa)$ (Hanf number for omitting types) we denote the least cardinal λ such that the following holds for all vocabularies L of size $\leq \kappa$: If T is an L-theory, D is a collection of L-types over $\emptyset$ of size $\leq \kappa$ and there is $\mathcal{A} \models T$ of power $\geq \lambda$ such that it omits every $p \in D$, then for all $\theta \geq \kappa$ there is $\mathcal{A} \models T$ of power θ such that it omits every $p \in D$. $Ho_1(\kappa)$ is defined similarly except that D is assumed to be a singleton.

12.10 Exercise.

(i) Show that for all infinite κ, $Ho(\kappa) = Ho_1(\kappa)$. Hint: For the non-trivial inequality $Ho(\kappa) \leq Ho_1(\kappa)$, suppose $\mathcal{K}$ is the class of all models of T that omit every $p \in D$. We want to code $\mathcal{K}$ as a class of the type the definition of $Ho_1(\kappa)$ talks about. To do this, extend the vocabulary by adding new unary predicates P, Q, constants c_i, $i < \kappa$, and function symbols F_p, $p \in D$. Look at a theory T' such that $\mathcal{A} \models T'$ implies the following: $P^{\mathcal{A}}$ and $Q^{\mathcal{A}}$ form a partition of the universe, for all $i < \kappa$, $c_i^{\mathcal{A}} \in Q^{\mathcal{A}}$, for all $i < j < \kappa$, $c_i^{\mathcal{A}} \neq c_j^{\mathcal{A}}$ and $\mathcal{A} \restriction P^{\mathcal{A}}, L \models T$ (see, Exercise 1.15). In addition put to T' requirements for the functions F_p so that if $\mathcal{A} \models T'$ omits $\{Q(v_0)\} \cup \{\neg v_0 = c_i |\ i < \kappa\}$, then $\mathcal{A} \restriction P^{\mathcal{A}}, L$ omits every $p \in D$.

(ii) Show that $Ho(\kappa) < \beth_{(2^\kappa)^+}$.

Recall that we say that a complete theory T is κ-stable if for all $n < \omega$, $\mathcal{A} \models T$ and $A \subseteq \mathcal{A}$ of power $\leq \kappa$, $|S_n(A;\mathcal{A})| \leq \kappa$.

12.11 Exercise. *Suppose κ is an uncountable cardinal, L is countable, T a complete theory, $\mathcal{A} \models T$, $a_i \in \mathcal{A}$ for all $i < \beth_{(2^\omega)^+}$, for all $i < j < \beth_{(2^\omega)^+}$, $a_i \neq a_j$ and Φ is as in Theorem 12.7 for $\mathcal{A}_\alpha = \mathcal{A}$ and $a_i^\alpha = a_i$.*

(i) Show that for all $n < \omega$ and countable $A \subseteq \kappa$, $|\{t^x_{at}(a/A;(\kappa,<))|\ a \in \kappa^n\}| \leq \omega$, where $x = (v_1, ..., v_n)$.

(ii) Show that for all $n < \omega$ and countable $A \subseteq EM(\kappa,\Phi)$,

$$|\{t(a/A;EM(\kappa,\Phi))|\ a \in EM(\kappa,\Phi)^n\}| \leq \omega.$$

(iii) Show that if T is κ-categorical then it is ω-stable.

13. $L_{\kappa\omega}$ and omitting types

In this section we look at extensions $L_{\kappa\omega}$ of the first-order logic. The main motivation to look extension (also other than $L_{\kappa\omega}$) is that many interesting classes of structures are not axiomatizable in the first-order logic. The following is a good example although it is not axiomatizable even in $L_{\kappa\omega}$. Look at the class $\mathbf{H}$ of all complex Hilbert spaces $H = (H, +, f_a, < \, , \, >)_{a \in \mathbf{C}}$ (there are many ways to handle the inner product $< \, , \, >$ so that the structure H indeed is a structure in our sense). If one looks the class of all models of the complete first-order theory of H, the resulting class behaves very badly, the structures are extremely complicated e.g. stationary sets (see [Je]) can be coded in them. However, the original class $\mathbf{H}$ is very nice, at least in comparison. E.g. if $\kappa^\omega = \kappa$, then there is upto isomorphism only one model in $\mathbf{H}$ of power κ.

13.1 Definition. *$L_{\kappa\omega}$-formulas are defined as follows:*

(i) atomic formulas ϕ are $L_{\kappa\omega}$-formulas and v_i is free in ϕ is it appears in ϕ,

(ii) if ψ is $L_{\kappa\omega}$-formula, then so are $\neg\psi$ and $\exists v_k \psi$ and v_i is free in $\neg\psi$ if it is free in ψ and it is free in $\exists v_k \psi$ if it is free in ψ and $k \neq i$,

(iii) if $|I| < \kappa$, for all $i \in I$, ψ_i is $L_{\kappa\omega}$-formula and there is $n \in \mathbb{N}$ such that for all $i \in I$, if v_k is free in ψ_i, then $k < n$, then $\bigwedge_{i \in I} \psi_i$ is a formula and v_k is free in $\bigwedge_{i \in I} \psi_i$ if it is free in some ψ_i.

An $L_{\kappa\omega}$-formula ϕ is an $L_{\kappa\omega}$-sentence if no v_k is free in ϕ.

Notice that for all $L_{\kappa\omega}$-formulas ϕ, only finitely many v_k are free in ϕ. The notation $\phi(x)$ is used as in the case of first-order logic and also $\mathcal{A} \models \phi(a)$ is defined as in the case of first-order logic ($\mathcal{A} \models (\bigwedge_{i \in I} \psi_i)(a)$ if $\mathcal{A} \models \psi_i(a)$ for all $i \in I$). Symbols $\bigvee$, $\rightarrow$, $\leftrightarrow$ and $\forall$ are used as in the case of first-order logic ($\bigvee_{i \in I} \psi_i = \neg \bigwedge_{i \in I} \neg\psi_i$). Notice that $L_{\omega\omega}$ is still the first-order logic (i.e. the two definitions for $L_{\omega\omega}$ coincide) and we say that ϕ is $L_{\infty\omega}$-formula if it is $L_{\kappa\omega}$-formula for some κ.

13.2 Definition.

(i) Let κ be a cardinal or ∞. We say that $\mathcal{A}$ and $\mathcal{B}$ are $L_{\kappa\omega}$ equivalent ($\mathcal{A} \equiv_{\kappa\omega} \mathcal{B}$) if for all $L_{\kappa\omega}$-sentences ϕ, $\mathcal{A} \models \phi$ iff $\mathcal{B} \models \phi$.

(ii) A variant $EF_k^*(\mathcal{A}, \mathcal{B})$, $k \leq \omega$, of the game $EF_k(\mathcal{A}, \mathcal{B})$ is defined exactly as the game $EF_k(\mathcal{A}, \mathcal{B})$ except that at each round $m < k$, II is required to choose the function f_m so that (in addition) it is a partial isomorphism.

13.3 Exercise. *Show that for all models $\mathcal{A}$ and $\mathcal{B}$, $II \uparrow EF_\omega(\mathcal{A}, \mathcal{B})$ iff $II \uparrow EF_\omega^*(\mathcal{A}, \mathcal{B})$.*

13.4 Theorem. *The following are equivalent:*

(i) $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$,

(ii) $II \uparrow EF_\omega(\mathcal{A}, \mathcal{B})$.

Proof. (ii)$\Rightarrow$(i): Exactly as in the first-order case (just forget the quantifier ranks).

(i)$\Rightarrow$(ii): Clearly it is enough to prove the following claim:

1 Claim. *If $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$, then for all $a \in \mathcal{A}$, there is $b \in \mathcal{B}$ such that $(\mathcal{A}, a) \equiv_{\infty\omega} (\mathcal{B}, b)$.*

Proof. Suppose not. Then for all $b \in \mathcal{B}$ there is an $L_{\infty\omega}$-formula $\phi_b(v_1)$ such that $\mathcal{A} \models \phi_b(a)$ but $\mathcal{B} \not\models \phi_b(b)$. Then $\mathcal{A} \models \exists v_1 \bigwedge_{b\in\mathcal{B}} \phi_b(v_1)$ but $\mathcal{B} \not\models \exists v_1 \bigwedge_{b\in\mathcal{B}} \phi_b(v_1)$, a contradiction. □ Claim 1.
□

13.5 Definition. *$L_{\kappa\omega}$-formulas in negation normal form are defined as follows: $L_{\kappa\omega}$-formula ϕ is in negation normal form if it is atomic or negated atomic formula, or of the form $\exists v_i\psi$ or $\forall v_i\psi$, where ψ is in negation normal form or of the form $\bigwedge_{i\in I} \psi_i$ or $\bigvee_{i\in I} \psi_i$, where each ψ_i is in negation normal form.*

13.6 Lemma. *For all $L_{\kappa\omega}$-formulas $\phi(x)$, $x = (x_1, ..., x_n)$, there is an $L_{\kappa\omega}$-formula $\psi(x)$ in negation normal form such that for all $\mathcal{A}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{A} \models \psi(a)$.*

Proof. Clearly it is enough to prove the following claim (exercise):

1 Claim. *For all $L_{\kappa\omega}$-formulas $\phi(x)$ in negation normal form, $x = (x_1, ..., x_n)$, there is an $L_{\kappa\omega}$-formula $\psi(x)$ in negation normal form such that for all $\mathcal{A}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \neg\phi(a)$ iff $\mathcal{A} \models \psi(a)$.*

Proof. Easy induction on ϕ. E.g. if $\phi = \bigwedge_{i\in I} \theta_i$, then by the induction assumption there are $L_{\kappa\omega}$-formulas θ'_i in negation normal form such that for all $\mathcal{A}$ and $a \in \mathcal{A}^n$, $\mathcal{A} \models \neg\theta_i(a)$ iff $\mathcal{A} \models \theta'_i(a)$ and we can choose $\psi = \bigvee_{i\in I} \theta'_i$. □ Claim 1.
□

13.7 Lemma. *Suppose T is a theory and D is a collection of types. Let κ be such that $|L_{\omega\omega}|, |D| < \kappa$. Then there is an $L_{\kappa\omega}$-sentence ϕ such that for all $\mathcal{A}$, $\mathcal{A} \models \phi$ iff $\mathcal{A} \models T$ and $\mathcal{A}$ omits every $p \in D$.*

Proof. Exercise. □

In the following definition, we assume that v_0 does not appear in ϕ and when we write $\phi(x)$, $x = (x_1, ..., x_n)$, we assume that x is chosen so that $v_i \in \{x_1, ..., x_n\}$ iff v_i is free in ϕ. And in item (iv) our notation is even more sloppy than usually.

13.8 Definition. *Suppose ϕ is an $L_{\kappa\omega}$-formula in negation normal form. In items (ii)-(iv) below ϕ is assumed to be a sentence.*
(i) A fragment F_ϕ of ϕ is defined as follows:
(a) if ϕ is atomic or negated atomic formula, then $F_\phi = \{\phi\}$,
(b) if $\phi = \bigwedge_{i\in I} \psi_i$ or $\phi = \bigvee_{i\in I} \psi_i$, then $F_\phi = \{\phi\} \cup \bigcup_{i\in I} F_{\psi_i}$,
(c) if $\phi = \exists v_k\psi$ or $\phi = \forall v_k\psi$, then $F_\phi = \{\phi\} \cup F_\psi$.
(ii) $L^\phi = L \cup \{R_{\psi(x_1,...,x_n)} \mid \psi \in F_\phi\}$, where R_ψ are new $n+1$-ary relation symbols.
(iii) T^ϕ consists of the following formulas:
(a) if $\psi(x_1, ..., x_n) \in F_\phi$ is atomic or negated atomic formula then $\forall v_0\forall x_1...\forall x_n(R_\psi(v_0, x_1, ..., x_n) \leftrightarrow \psi(x_1, ..., x_n)) \in T^\phi$,
(b) if $\psi(x_1, ..., x_n) = \bigwedge_{i\in I} \psi_i \in F_\phi$, then for all $i \in I$,

$\forall v_0 \forall x_1 ... \forall x_n (R_\psi(v_0, x_1 ... x_n) \to R_{\psi_i}(v_0, x_1, ..., x_n)) \in T^\phi$,
(c) if $\psi(x_1, ..., x_n) = \bigvee_{i \in I} \psi_i \in F_\phi$, then for all $i \in I$,
$\forall v_0 \forall x_1 ... \forall x_n (R_{\psi_i}(v_0, x_1 ... x_n) \to R_\psi(v_0, x_1, ..., x_n)) \in T^\phi$,
(d) if $\psi(x_1, ..., x_n) = \exists x \theta(x, x_1, ..., x_n) \in F_\phi$, then
$\forall v_0 \forall x_1 ... \forall x_n (R_\psi(v_0, x_1, ..., x_n) \leftrightarrow \exists x R_\theta(v_0, x, x_1, ..., x_n)) \in T^\phi$,
(e) if $\psi(x_1, ..., x_n) = \forall x \theta(x, x_1, ..., x_n) \in F_\phi$, then
$\forall v_0 \forall x_1 ... \forall x_n (R_\psi(v_0, x_1, ..., x_n) \leftrightarrow \forall x R_\theta(v_0, x, x_1, ..., x_n)) \in T^\phi$,
(f) $\forall v_0 R_\phi(v_0)$.

(iv) D^ϕ *consists of the following types:*
(a) if $\psi(x_1, ..., x_n) = \bigwedge_{i \in I} \psi_i \in F_\phi$, then
$\{\neg R_\psi(v_0, v_1 ... v_n)\} \cup \{R_{\psi_i}(v_0, v_1, ..., v_n) |\ i \in I\} \in D^\phi$,
(b) if $\psi(x_1, ..., x_n) = \bigvee_{i \in I} \psi_i \in F_\phi$, then
$\{R_\psi(v_0, v_1 ... v_n)\} \cup \{\neg R_{\psi_i}(v_0, v_1, ..., v_n) |\ i \in I\} \in D^\phi$.

13.9 Lemma. *Suppose* ϕ *is an* $L_{\kappa\omega}$*-sentence in negation normal form.*
(i) For all L*-structures* $\mathcal{A}$*, if* $\mathcal{A} \models \phi$*, then there is an* L^ϕ*-structure* $\mathcal{B} \models T^\phi$ *so that* $\mathcal{B}$ *omits every* $p \in D^\phi$ *and* $\mathcal{B} \restriction L = \mathcal{A}$*. Furthermore such* $\mathcal{B}$ *is unique.*
(ii) If $\mathcal{A} \models T^\phi$ *is an* L^ϕ*-structure and* $\mathcal{A}$ *omits every* $p \in D^\phi$*, then* $\mathcal{A} \restriction L \models \phi$*.*

Proof. Just check the definitions (exercise). □

13.10 Theorem. *Suppose* ϕ *is an* $L_{\kappa^+\omega}$*-sentence such that for all* $\lambda < \beth_{(2^\kappa)^+}$*, there is* $\mathcal{A} \models \phi$ *of power* $\geq \lambda$*. Then for all* $\lambda \geq \kappa$*, there is* $\mathcal{A} \models \phi$ *of power* λ*.*

Proof. Clearly in ϕ at most κ symbols from the vocabulary can appear and thus we may assume that $|L_{\omega\omega}| \leq \kappa$. But then the claim is immediate by Lemmas 13.6 and 13.9 and Corollary 12.8. □

In Theorem 13.10 in the case $\kappa = \omega$, $\beth_{(2^\kappa)^+}$ can be replaced by $\beth_{\omega_1}$. This is because in this case in the construction of a model in the Skolem-language with the indiscernibles one can make use of Henkin construction.

13.11 Remark. *Suppose* F *is a collection of* $L_{\kappa^+\omega}$*-formulas of power* $\leq \kappa$ *and* $A \subseteq \mathcal{A}$*. Then there is a substructure* $\mathcal{B}$ *of* $\mathcal{A}$ *of power* $|A| + \kappa$ *such that* $A \subseteq \mathcal{B}$ *and for all* $\phi(x_1, ..., x_n) \in F$ *and* $a \in \mathcal{B}^n$*,* $\mathcal{B} \models \phi(a)$ *iff* $\mathcal{A} \models \phi(a)$*.*

Proof. Clearly we may assume that if $\wedge_{i \in I} \psi_i \in F$, then $\psi_i \in F$ for all $i \in I$ and that if $\neg\psi \in F$ or $\exists v_k \psi \in F$, then $\psi \in F$. Then we can proceed as in the first-order case (exercise). □

13.12 Exercise. *Suppose* $\kappa > \omega$ *is such that* $L_{\kappa\omega}$ *is* κ*-compact i.e. if* T *is a collection of* $L_{\kappa\omega}$*-sentences of size* κ *and every* $T' \subseteq T$ *of size* $< \kappa$ *has a model, then* T *has a model. Show that* κ *is a regular limit cardinal (and thus the existence of such cardinal is not provable in ZFC).*

13.13 Exercise. *Let* $L = \{\times, 1\}$*, where* $\times$ *is a binary function symbol and* 1 *is a constant.*

(i) Show that there is an $L_{\omega_1\omega}$-sentence ϕ such that for all groups $\mathcal{A}$, $\mathcal{A} \models \phi$ iff $\mathcal{A}$ is locally finite i.e. every finitely generated subgroup is finite.

(ii) Show that there is an $L_{\omega_1\omega}$-sentence ϕ such that for all groups $\mathcal{A}$, $\mathcal{A} \models \phi$ iff $\mathcal{A}$ is simple i.e. $\{1\}$ and $\mathcal{A}$ are the only normal subgroups of $\mathcal{A}$.

13.14 Exercise. *Let $L = \{R\}$, where R is a binary relation. Show that there is no $L_{\infty\omega}$-sentence ϕ such that for all models $\mathcal{A}$, $\mathcal{A} \models \phi$ iff there are no $a_i \in \mathcal{A}$, $i < \omega$, such that for all $i < j < \omega$, $\mathcal{A} \models R(a_j, a_i)$. Hint: Use Lemma 13.9 and the proof of Corollary 12.8.*

14. Scott-rank

In this section we look at isomorphisms between countable structures. The starting point is the observation that by Theorem 13.4, for countable $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong \mathcal{B}$ iff $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$.

Throughout this section, L and all structures are assumed to be countable.

Now suppose $\mathcal{A}$ is a model. For all $\alpha < \omega_1$, $n < \omega$ and $a = (a_1, ..., a_n) \in \mathcal{A}^n$, we define an $L_{\omega_1\omega}$-formula $\phi^{\mathcal{A}}_{a,\alpha}(v_1, ..., v_n)$ by recursion on α as follows:
1. $\alpha = 0$: We let $\phi^{\mathcal{A}}_{a,0}$ be the conjunction of all atomic or negated atomic formulas $\psi(v_1, ..., v_n)$ such that $\mathcal{A} \models \psi(a)$. If $n = 0$ and there are no atomic sentences, we let $\phi^{\mathcal{A}}_{\emptyset,0}$ be $\top$.
2. $\alpha = \beta + 1$: For all $b \in \mathcal{A}$, by ab we denote the sequence $(a_1, ..., a_n, b)$. Then

$$\phi^{\mathcal{A}}_{a,\alpha} = \phi^{\mathcal{A}}_{a,\beta} \wedge (\bigwedge_{b\in\mathcal{A}} \exists v_{n+1} \phi^{\mathcal{A}}_{ab,\beta}) \wedge (\forall v_{n+1} \bigvee_{a\in\mathcal{A}} \phi^{\mathcal{A}}_{ab,\beta}).$$

3. α is a limit: $\phi^{\mathcal{A}}_{a,\alpha} = \bigwedge_{\beta<\alpha} \phi^{\mathcal{A}}_{a,\beta}$.
If $a = \emptyset$, then we denote $\phi^{\mathcal{A}}_{a,\alpha}$ by just $\phi^{\mathcal{A}}_{\alpha}$. Notice that $\mathcal{A} \models \phi^{\mathcal{A}}_{a,\alpha}(a)$ and that for all $\alpha < \beta < \omega_1$ and $a \in \mathcal{A}^n$, $\models \forall v_1 ... \forall v_n (\phi^{\mathcal{A}}_{a,\beta} \rightarrow \phi^{\mathcal{A}}_{a,\alpha})$.

14.1 Lemma. *Suppose $|\mathcal{A}| = \omega$. There is $\alpha < \omega_1$ such that for all $n < \omega$ and $a, b \in \mathcal{A}^n$, if $\mathcal{A} \models \phi^{\mathcal{A}}_{a,\alpha}(b)$, then $\mathcal{A} \models \phi^{\mathcal{A}}_{a,\alpha+1}(b)$.*

Proof. For any pair $(a, b) \in (\mathcal{A}^n)^2$, let $\alpha(a, b)$ be the least $\alpha < \omega_1$ such that $\mathcal{A} \not\models \phi^{\mathcal{A}}_{a,\alpha}(b)$ if there is such α and otherwise let $\alpha(a, b) = 0$. Now let α be any ordinal $< \omega_1$ such that $\alpha > \alpha(a, b)$ for any $(a, b) \in (\mathcal{A}^n)^2$, $n < \omega$. Clearly α is as wanted (exercise). □

The least α as in Lemma 14.1 is called Scott-rank of (countable) $\mathcal{A}$ and we denote it by $Sr(\mathcal{A})$. The Scott-sentence $\phi^{\mathcal{A}}$ of $\mathcal{A}$ is the formula

$$\phi^{\mathcal{A}}_{\alpha+1} \wedge (\bigwedge_{0<n<\omega} \bigwedge_{a\in\mathcal{A}^n} \forall v_1 ... \forall v_n (\phi^{\mathcal{A}}_{a,\alpha} \rightarrow \phi^{\mathcal{A}}_{a,\alpha+1})),$$

where $\alpha = Sr(\mathcal{A})$. Notice that $\mathcal{A} \models \phi^{\mathcal{A}}$ and that $\phi^{\mathcal{A}}$ is an $L_{\omega_1\omega}$-sentence.

14.2 Theorem. *Let L be countable. For all countable structures $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent:*
(i) $\mathcal{A} \cong \mathcal{B}$,
(ii) $\mathcal{B} \models \phi^{\mathcal{A}}$.

Proof. We prove that (ii) implies (i), the other direction is clear. Let $\alpha = Sr(\mathcal{A})$. We notice first that since $\mathcal{B} \models \phi^{\mathcal{A}}_{\alpha+1}$, for all $a \in \mathcal{A}$, there is $b \in \mathcal{B}$ such that $\mathcal{B} \models \phi^{\mathcal{A}}_{a,\alpha}(b)$.

We also notice that $a = (a_1, ..., a_n) \in \mathcal{A}^n$, if $b = (b_1, ..., b_n) \in \mathcal{B}^n$ and $\mathcal{B} \models \phi^{\mathcal{A}}_{a,\alpha}(b)$, then $\mathcal{B} \models \phi^{\mathcal{A}}_{a,0}(b)$ and thus $a_i \mapsto b_i$, $1 \leq i \leq n$, is a partial isomorphism.

Thus it is enough to prove the following claim (exercise).

14.2.1 Claim. *Suppose $a \in \mathcal{A}^n$ and $b \in \mathcal{B}^n$ are such that $\mathcal{B} \models \phi^{\mathcal{A}}_{a,\alpha}(b)$.*
(i) For all $c \in \mathcal{A}$ there is $d \in \mathcal{B}$ such that $\mathcal{B} \models \phi^{\mathcal{A}}_{ac,\alpha}(b, d)$.
(ii) For all $d \in \mathcal{B}$ there is $c \in \mathcal{A}$ such that $\mathcal{B} \models \phi^{\mathcal{A}}_{ac,\alpha}(b, d)$.

Proof. By symmetry it is enough to prove (ii). Since $\mathcal{B} \models \phi^{\mathcal{A}} \wedge \phi^{\mathcal{A}}_{a,\alpha}(b)$, $\mathcal{B} \models \phi^{\mathcal{A}}_{a,\alpha+1}(b)$. In particular, $\mathcal{B} \models \forall v_{n+1} \bigvee_{c \in \mathcal{A}} \phi^{\mathcal{A}}_{ac,\alpha}(b)$. Thus there is $c \in \mathcal{A}$ such that $\mathcal{B} \models \phi^{\mathcal{A}}_{ac,\alpha}(b, d)$. □ Claim 14.2.1.
□

14.3 Definition.
(i) We say that a (possibly empty) partial ordering $T = (T, <)$ is a tree if for all $t \in T$, the set $\{u \in T |\ u < t\}$ is well-ordered by $<$. We say that $B \subseteq T$ is a branch if it is linearly ordered by $<$ and downwards closed. $u \in T$ is a successor of a branch B if $t < u$ for all $t \in B$ and it is an immediate successor if in addition there is no $w \in T$ such that $B < w < u$. $u \in T$ is an immediate successor of $t \in T$ if it is an immediate successor of the branch $\{w \in T |\ w \leq t\}$. T is a κ, λ-tree if every branch has $< \kappa$ many immediate successors and every branch has size $< \lambda$.
(ii) For an ordinal α, by T_α we mean the tree of all strictly decreasing $f : n \to \alpha$, $n \in \omega - \{0\}$, ordered by the subset relation.
(iii) Let T be a tree and $\mathcal{A}$ and $\mathcal{B}$ be structures. The game $EF^d_T(\mathcal{A}, \mathcal{B})$ is defined exactly as $EF^*_\omega(\mathcal{A}, \mathcal{B})$ except that now the number of rounds may be any ordinal and at each round $m \in On$, the player I is required to choose in addition to $c_m \in \mathcal{A} \cup \mathcal{B}$ also $t_m \in T$ and he is required to do this so that for all $k < m$, $t_k < t_m$. The game ends when I cannot choose t_m anymore (i.e. $\{u \in T |$ for some $k < m$, $u \leq t_k\}$ is a maximal branch).

Often one requires that trees have the least element (=root) but here that is inconvenient. Also when we talk about κ, λ-trees we assume that κ and λ are infinite cardinals. Notice that $\emptyset$ is a branch, T_α is a $|\alpha|^+, \omega$-tree and that $T_0 = \emptyset$. Notice also that $EF^*_\omega(\mathcal{A}, \mathcal{B})$ and $EF^d_\omega(\mathcal{A}, \mathcal{B})$ are equivalent i.e. a player has a winning strategy in one iff he/she has a winning strategy in the other and that these games are determined i.e. one of the players has a winning strategy, compare this to Fact 15.12 (if I does not have a winning strategy then at every round, II can find a move so that after that move, I still does not have a winning strategy in the rest of the game, exercise). Finaly notice that if $\mathcal{A}$ and $\mathcal{B}$ are countable, then $\mathcal{A} \cong \mathcal{B}$ iff II has a winning strategy in $EF^d_\omega(\mathcal{A},\mathcal{B})$.

14.4 Exercise. *Show that for all ordinals α and structures $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent.*
(i) $\mathcal{B} \models \phi^{\mathcal{A}}_{\emptyset,\alpha}$,
(ii) $II \uparrow EF^d_{T_\alpha}(\mathcal{A},\mathcal{B})$.

We can define quantifier rank for $L_{\infty\omega}$-formulas: If ϕ is atomic, $qr(\phi) = 0$, if $\phi = \neg\psi$, then $qr(\phi) = qr(\psi)$, if $\phi = \exists v_i \psi$, then $qr(\phi) = qr(\psi) + 1$ and finally if $\phi = \bigwedge_{i \in I} \psi_i$, then $qr(\phi) = \cup\{qr(\psi_i) | \ i \in I\}$.

14.5 Exercise. *Let $\mathcal{A}$ and $\mathcal{B}$ be models and α an ordinal. The following are equivalent.*
(i) For all $L_{\infty\omega}$-sentences ϕ of quantifier rank $\leq \alpha$, $\mathcal{A} \models \phi$ iff $\mathcal{B} \models \phi$.
(ii) $II \uparrow EF^d_{T_\alpha}(\mathcal{A},\mathcal{B})$.
Conclude that for all countable $\mathcal{A}$ there is an ω_1,ω-tree T such that for all countable $\mathcal{B}$, $\mathcal{A} \cong \mathcal{B}$ iff $II \uparrow EF^d_T(\mathcal{A},\mathcal{B})$.

15. More on Ehrenfeucht-Fraïssé games

In this section we take a quick look on what happens when one cannot use ordinals to rank games in the style of the previous section. The analysis below is not limitted to Ehrenfeucht-Fraïsseé games but works for all so called closed games.

15.1 Definition. *Let T_0 and T_1 be trees. The comparison game $CG(T_0,T_1)$ is defined as follows: At each round $m \in On$, first I chooses $t_m \in T_0$ and then II chooses $u_m \in T_1$. Both players must choose these so that the element they pick is strictly greater than any of the elements they have chosen earlier. The one who can not move, looses. We write $T_0 \leq T_1$ if $II \uparrow CG(T_0,T_1)$ and $T_1 << T_0$ if $I \uparrow CG(T_0,T_1)$.*

15.2 Exercise.
(i) Show that $T_0 << T_1$ implies $T_0 \leq T_1$ and $T_1 \not\leq T_0$.
(ii) Show that $T_0 \leq T_1$ iff there is $f: T_0 \to T_1$ such that for all $t, u \in T_0$, if $u < t$, then $f(u) < f(t)$.
(iii) Show that for all κ^+,ω-trees T there is $\alpha < \kappa^+$ such that $T \leq T_\alpha$ and $T_\alpha \leq T$.
(iv) Show that for all ordinals α and β, $\alpha \leq \beta$ iff $T_\alpha \leq T_\beta$ and $\alpha < \beta$ iff $T_\alpha << T_\beta$.

15.3 Definition. *Let T be a tree. By $\sigma(T)$ we mean the tree of all branched of T ordered by the subset relation.*

15.4 Exercise. *Show that $\sigma(T_\alpha) \leq T_{\alpha+1}$ and $T_{\alpha+1} \leq \sigma(T_\alpha)$.*

15.5 Lemma. *Let T be a tree.*

(i) $T << \sigma(T)$.

(ii) If $T << T'$, then $\sigma(T) \leq T'$. In particular, there is no T' such that $T << T' << \sigma(T)$.

(iii) If T is a κ, λ-tree, then so is $\sigma(T)$.

Proof. (i): At each round $m \in On$ in $CG(\sigma(T), T)$, I simply chooses t_m to be the set of all $w \in T$ such that for some $k < m$, $w \leq u_k$, where u_k is the choice of II at round k (so $t_0 = \emptyset$).

(ii) and (iii): Exercise. □

15.6 Definition. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are models of size $\geq \kappa$. We write $T_\kappa(\mathcal{A}, \mathcal{B})$ for the tree of all winning strategies of II in games EF^d_α, $\alpha < \kappa$, ordered by the subset relation.*

15.7 Exercise.

(i) Suppose $\mathcal{A}$ and $\mathcal{B}$ be models of size $\leq \lambda \geq \kappa$. Then II does not have a winning strategy in $EF^d_\kappa(\mathcal{A}, \mathcal{B})$ iff $T_\kappa(\mathcal{A}, \mathcal{B})$ is a $(2^\lambda)^+, \kappa$-tree.

(ii) If $|\mathcal{A}| = |\mathcal{B}| = \kappa$, then $\mathcal{A} \cong \mathcal{B}$ iff $II \uparrow EF^d_\kappa(\mathcal{A}, \mathcal{B})$.

15.8 Theorem. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are models of size $\lambda \geq \kappa$, II does not have a winning strategy in $EF^d_\kappa(\mathcal{A}, \mathcal{B})$ and $T = \sigma(T_\kappa(\mathcal{A}, \mathcal{B}))$. Then II does not have a winning strategy in $EF^d_T(\mathcal{A}, \mathcal{B})$.*

Proof. It is enough to show that for any tree T, if $II \uparrow EF^d_T(\mathcal{A}, \mathcal{B})$, then $T \leq T_\kappa(\mathcal{A}, \mathcal{B})$. Let π be the winning strategy and $t \in T$. Let $B = \{u \in T |\ u < t\}$ and α the order-type of B. Then B and π canonically define a winning strategy π_t for II to $EF^d_\alpha(\mathcal{A}, \mathcal{B})$. But then clearly $t \mapsto \pi_t$ is an order-preserving function from T to $T_\kappa(\mathcal{A}, \mathcal{B})$. □

15.9 Corollary. *Suppose $|L| \leq \kappa$. For all models $\mathcal{A}$ of power κ, there is a $(2^\kappa)^+, \kappa$-tree T such that for all models $\mathcal{B}$ of power κ, $\mathcal{A} \cong \mathcal{B}$ iff $II \uparrow EF^d_T(\mathcal{A}, \mathcal{B})$.*

Proof. Let $\mathcal{B}_\alpha$, $\alpha < 2^\kappa$, list upto isomorphism all models of power κ that are not isomorphic to $\mathcal{A}$ and for all $\alpha < 2^\kappa$, let $T_\alpha = \sigma(T_\kappa(\mathcal{A}, \mathcal{B}))$. Then if T is the disjoint union of these trees, T is as wanted. □

Corollary 15.9 can not be improved:

15.10 Fact. *For every (e.g.) unstable theory Σ in a countable language and uncountable κ such that $\kappa^{<\kappa} = \kappa$, there is $\mathcal{A} \models \Sigma$ of power κ such that for all κ^+, κ-trees T there is $\mathcal{B} \models \Sigma$ of power κ for which $II \uparrow EF^d_T(\mathcal{A}, \mathcal{B})$ but $\mathcal{A} \not\cong \mathcal{B}$.*

There is another way to prove Corollary 15.9: Instead of looking when II has a winning strategy one can look when I does not have a winning strategy. In many cases of many (closed) games (but not always) this is the right way to go.

15.11 Exercise. *Suppose $|\mathcal{A}| = |\mathcal{B}| = \kappa$ and $\mathcal{A} \not\cong \mathcal{B}$. Show that there is a $(\kappa^{<\kappa})^+, \kappa$-tree T such that $I \uparrow EF^d_T(\mathcal{A}, \mathcal{B})$.*

15.12 Fact. *There are (e.g.) dense linear orders* $\mathcal{A}$ *and* $\mathcal{B}$ *of power* ω_1 *s.t.* $II \uparrow EF^d_\alpha(\mathcal{A},\mathcal{B})$ *iff* $\alpha \le \omega+1$ *and* $I \uparrow EF^d_\alpha(\mathcal{A},\mathcal{B})$ *iff* $\alpha \ge \omega_1$.

Appendix: On recursive definitions

A.1 Definition. *Suppose* X *is a set.*

(i) Suppose n *is a natural number,* $f : X^n \to X$ *is a function and* $C \subseteq X$. *We say that* C *is closed under* f *if for all* $x \in C^n$, $f(x) \in C$.

(ii) Suppose $Y \subseteq X$ *and for all* $i \in I$, n_i *is a natural number and* $f_i : X^{n_i} \to X$ *is a function. Then by* $C(Y, f_i)_{i\in I}$ *we mean the* $\subseteq$*-least subset* C *of* X *such that it contains* Y *and is closed under every* f_i, $i \in I$ *(if such* C *exists).*

A.2 Lemma. *Let* X, Y, I *and* n_i *and* f_i, $i \in I$, *be as in Definition A.1 (ii). Then* $C(Y, f_i)_{i\in I}$ *exists.*

Proof. Just let $C(Y, f_i)_{i\in I}$ be the intersection of all sets $C \subseteq X$ which contain Y and are closed under every f_i (notice that X is such a set). □

A.3 Lemma. *Let* X, Y, I *and* n_i *and* f_i, $i \in I$, *be as in Definition A.1 (ii). Suppose that* P *is a property, every element of* Y *has it and for all* $k \in I$ *and* $x = (x_0, ...x_{n_{k-1}}) \in (C(Y, f_i)_{i\in I})^{n_k}$ *the following holds: If every* x_j, $j < n_k$, *has the property, then also* $f_k(x)$ *has the property. Then every element of* $C(Y, f_i)_{i\in I}$ *has the property* P.

Proof. Let C be the set of all elements of $C(Y, f_i)_{i\in I}$ that have the property P. Then C contains Y and is closed under every f_i. Thus $C(Y, f_i)_{i\in I} \subseteq C$. □

A.4 Definition. *Let* X, Y, I *and* n_i *and* f_i, $i \in I$, *be as in Definition A.1 (ii). For all ordinals* $\alpha \le \omega$, *we define* $C_\alpha(Y, f_i)_{i\in I}$ *as follows:*

(i) $C_0(Y, f_i)_{i\in I} = Y$,

(ii) $C_{\alpha+1}(Y, f_i)_{i\in I} = C_\alpha(Y, f_i)_{i\in I} \cup \{f_i(x) |\ i \in I,\ x \in (C_\alpha(Y, f_i)_{i\in I})^{\alpha_i}\}$,

(iii) if α *is limit, then* $C_\alpha(Y, f_i)_{i\in I} = \bigcup_{\beta<\alpha} C_\beta(Y, f_i)_{i\in I}$.

A.5 Exercise. *Show that for all ordinals* $\alpha < \beta \le \omega$, $Y \subseteq C_\alpha(Y, f_i)_{i\in I} \subseteq C_\beta(Y, f_i)_{i\in I} \subseteq C(Y, f_i)_{i\in I}$.

A.6 Lemma. *Let* X, Y, I *and* n_i *and* f_i, $i \in I$, *be as in Definition A.1 (ii). Then* $C(Y, f_i)_{i\in I} = C_\omega(Y, f_i)_{i\in I}$.

Proof. By Exercise A.5, it is enough to show that $C_\omega(Y, f_i)_{i\in I}$ is closed under every f_k, $k \in I$. For this let $x \in (C_\omega(Y, f_i)_{i\in I})^{n_k}$. Then there is $\gamma < \omega$ such that $x \in (C_\gamma(Y, f_i)_{i\in I})^{n_k}$ But then $f_i(x) \in C_{\gamma+1}(Y, f_i)_{i\in I} \subseteq C_{\omega_1}(Y, f_i)_{i\in I}$. □

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