Real Analysis I

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1 L^p-spaces

1.1 Measure space

Definition 1.2. Let X be an arbitrary set and $\mathcal{P}(X) = \{A\colon A \subset X\}$ its power set. A family $\Gamma \subset \mathcal{P}(X)$ is called a *σ-algebra* ("sigma algebra") on X if

(1) $\emptyset \in \Gamma$;

(2) $A \in \Gamma \;\Rightarrow\; X \setminus A \in \Gamma$; (denote $A^c = X \setminus A$)

(3) $A_i \in \Gamma,\ i \in \mathbb{N} \;\Rightarrow\; \bigcup_{i=1}^{\infty} A_i \in \Gamma$.

Definition 1.3. Let Γ be a σ-algebra on X. A function $\mu\colon \Gamma \to [0, +\infty]$ is a (positive) *measure* on X (or on the σ-algebra Γ) if

(a) $\mu(\emptyset) = 0$;

(b) $A_i \in \Gamma,\ i \in \mathbb{N}$, disjont $\Rightarrow\; \mu\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i\in\mathbb{N}} \mu(A_i)$. "countably additivity"

The triple (X, Γ, μ) is a *measure space* (and Γ is the family of μ-measurable sets).

Example 1.4.

1. $X = \mathbb{R}^n$, $\Gamma = \operatorname{Leb}\mathbb{R}^n =$ the set of all Lebesgue measurable sets and $\mu = m_n =$ Lebesgue measure.

2. $X = \mathbb{R}^n$, $\Gamma = \operatorname{Bor}\mathbb{R}^n =$ the Borel σ-algebra and $\mu = m_n|\operatorname{Bor}\mathbb{R}^n =$ the restriction of the Lebesgue measure to the Borel σ-algebra. (Recall: $\operatorname{Bor}\mathbb{R}^n =$ the smallest σ-algebra on $\mathbb{R}^n$ that contains all closed subsets of $\mathbb{R}^n$.)

3. Let $X \neq \emptyset$ be an arbitrary set. Fix $x \in X$ and define, for all $A \subset X$,

$$\mu(A) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

   Then $\mu\colon \mathcal{P}(X) \to [0, +\infty]$ is a measure (so-called *Dirac measure* at $x \in X$). Usually it is denoted by $\mu = \delta_x$.

1.5 Complete measures

Let (X, Γ, μ) be a measure space and $F \in \Gamma$. Suppose that P $\big(= P(x)\big)$ is some property whose validity depends on the point $x \in X$.

We say: *The property P holds μ-a.e. in F* (a.e. = almost everywhere) if there exists $E \in \Gamma$ such that $\mu(E) = 0$, $E \subset F$ and P holds in $F \setminus E$.

Sometimes we would like to be more specific: "P holds in F with the execption of a set of measure 0". The problem is the case:

$$\underbrace{\{x \in F\colon P(x) \text{ does not hold}\}}_{=A} \notin \Gamma$$

although $A \subset E$ and $\mu(E) = 0$. Note: In the case of a general measure space it is possible that

$$A \subset E,\ \mu(E) = 0,\ \textit{but } A \notin \Gamma$$

(i.e. A *is not* μ-measurable).

Example 1.6. Consider a measure space $(\mathbb{R}^n, \operatorname{Bor}\mathbb{R}^n, \mu)$, $\mu = m_n|\operatorname{Bor}\mathbb{R}^n$. Then there exists $B \in \operatorname{Bor}\mathbb{R}^n$, $\mu(B) = 0$, and $A \subset B$ s.t. $A \notin \operatorname{Bor}\mathbb{R}^n$.

Let's prove the case $n \geq 2$: ($n = 1$ later). Let $A \subset \mathbb{R}$ be a non-Lebesgue measurable set and $f\colon \mathbb{R} \to \mathbb{R}^n$,

$$f(x) = (x, 0, \dots, 0)\,.$$

Then f is continuous and

$$m_n^*(fA) \leq m_n^*\big(\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0\ \forall i = 2, \dots, n\}\big) = 0,$$

hence $fA \in \operatorname{Leb}\mathbb{R}^n$.

Claim: $fA \notin \operatorname{Bor}\mathbb{R}^n$.

Assume on the contrary: $fA \in \operatorname{Bor}\mathbb{R}^n$. Then the pre-image $f^{-1}(fA) = A$ is a Borel set since f is continuous [see (1.8)]. This leads to a contradiction since A is not Lebesgue measurable.

Let $G \in \operatorname{Bor}\mathbb{R}^n$. We say that a mapping $g\colon G \to \mathbb{R}^m$ is a *Borel mapping* (or just Borel) if

$$U \subset R^m \text{ open} \;\;\Rightarrow\;\; g^{-1}U \in \operatorname{Bor}\mathbb{R}^n.$$

In particular, every continuous mapping $g\colon G \to \mathbb{R}^m$, $G \in \operatorname{Bor}\mathbb{R}^n$, is Borel because then $g^{-1}U$ is open in G for every open $U \subset \mathbb{R}^m$. In other words, $g^{-1}U = G \cap V$, where $V \subset \mathbb{R}^n$ is open, and so $g^{-1}U \in \operatorname{Bor}\mathbb{R}^n$.

Lemma 1.7. *Let $G \subset \mathbb{R}^n$ be a Borel set and $g\colon G \to \mathbb{R}^m$ a Borel mapping. Then*

$$A \in \operatorname{Bor}\mathbb{R}^m \;\;\Rightarrow\;\; g^{-1}A \in \operatorname{Bor}\mathbb{R}^n\,. \tag{1.8}$$

Proof. Let $\Gamma = \{V \subset \mathbb{R}^m : g^{-1}V \in \operatorname{Bor}\mathbb{R}^n\}$. Then Γ is a σ-algebra because:

(1) $g^{-1}\emptyset = \emptyset \in \operatorname{Bor}\mathbb{R}^n \;\Rightarrow\; \emptyset \in \Gamma$;

(2) $V \in \Gamma \;\Rightarrow\; g^{-1}V^c = \underbrace{G}_{\in \operatorname{Bor}\mathbb{R}^n} \setminus \underbrace{g^{-1}V}_{\in \operatorname{Bor}\mathbb{R}^n} \in \operatorname{Bor}\mathbb{R}^n \Rightarrow\; V^c \in \Gamma$;

(3) $V_i \in \Gamma,\ i \in \mathbb{N} \;\Rightarrow\; g^{-1}\big(\bigcup_{i\in\mathbb{N}} V_i\big) = \bigcup_{i\in\mathbb{N}} \underbrace{g^{-1}V_i}_{\in \operatorname{Bor}\mathbb{R}^n} \in \operatorname{Bor}\mathbb{R}^n. \Rightarrow\; \bigcup_{i\in\mathbb{N}} V_i \in \Gamma$.

Moreover, Γ contains all open subsets of $\mathbb{R}^m$ since:

$$U \subset \mathbb{R}^m \text{ open} \;\Rightarrow\; g^{-1}U \in \operatorname{Bor}\mathbb{R}^n \;\;\Rightarrow\;\; U \in \Gamma\,.$$

Hence $\Gamma \supset \operatorname{Bor}\mathbb{R}^m$ (= the smallest σ-algebra on $\mathbb{R}^n$ that contains open sets). □

The situation as in Example 1.6 (i.e. $A \subset E$, $\mu(E) = 0$, $A \notin \Gamma$) does not occur if μ is so called complete measure.

Definition 1.9. Let (X, Γ, μ) be a measure space. The measure μ is *complete* if

$$E \in \Gamma,\ \mu(E) = 0,\ F \subset E \;\;\Rightarrow\;\; F \in \Gamma\,.$$

Remark 1.10. μ monotonic $\;\Rightarrow\; \mu(F) = 0$.
Completeness = "subsets of a set of measure 0 are measurable and of measure 0".

Example 1.11. 1. $(\mathbb{R}^n, \operatorname{Leb}\mathbb{R}^n, m_n)$, the Lebesgue measure m_n is complete.

2. $(\mathbb{R}^n, \operatorname{Bor}\mathbb{R}^n, \mu)$, $\mu = m_n | \operatorname{Bor}\mathbb{R}^n$ is not complete.

This is a minor problem and usually does not cause problems since a (non compete) measure μ can be completed:

Theorem 1.12. *Let (X, Γ, μ) be a measure space. Define $\bar{\Gamma} \subset \mathcal{P}(X)$ by setting*

$$\bar{\Gamma} = \{A \cup F \colon A \in \Gamma \text{ and } F \subset E \text{ for some } E \in \Gamma,\ \mu(E) = 0\}.$$

and define $\bar{\mu} \colon \bar{\Gamma} \to [0, +\infty]$,

$$\bar{\mu}(A \cup F) = \mu(A),$$

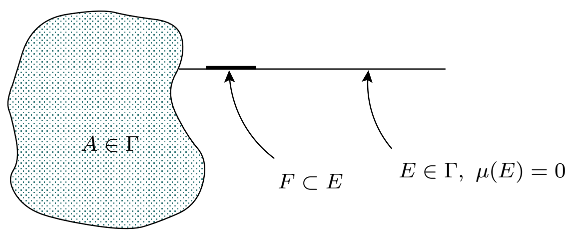
where A and F are as above. Then

(1) $\bar{\Gamma}$ is a σ-algebra on X;

(2) $\bar{\mu}$ is a complete measure;

(3) $\mu = \bar{\mu}|\Gamma$.

$\bar{\mu}$ is called the completion *of μ (and $(X, \bar{\Gamma}, \bar{\mu})$ is the completion of (X, Γ, μ)).*



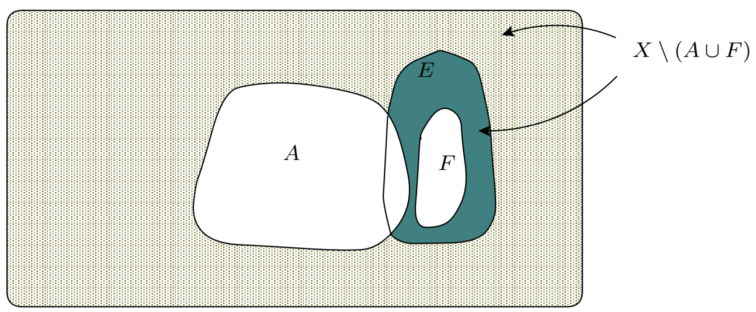
Proof. Let's proof some claims (the rest are left as an exercise).

(1) (i): $\emptyset \in \bar{\Gamma}$.

(ii): Let $B \in \bar{\Gamma}$, $B = A \cup F$, where $A \in \Gamma$, $F \subset E \in \Gamma$ and $\mu(E) = 0$. Claim: $X \setminus B \in \bar{\Gamma}$. Proof: Since

$$X \setminus B = X \setminus (A \cup F) = \underbrace{\big(X \setminus (A \cup E)\big)}_{\in \Gamma} \cup \underbrace{\big(E \setminus (A \cup F)\big)}_{\subset E}$$

is of desired form, we have $X \setminus B \in \bar{\Gamma}$.



(iii): If $B_i \in \bar{\Gamma}$, $i \in \mathbb{N}$, then clearly $\bigcup_{i \in \mathbb{N}} B_i \in \bar{\Gamma}$ (Exerc.).

(2) (i): $\bar{\mu}$ is well-defined: Let $B = A_1 \cup F_1 = A_2 \cup F_2$, where $A_i \in \Gamma,\ F_i \subset E_i,\ \mu(E_i) = 0,\ i = 1, 2$. Then

$$A_1 \subset A_1 \cup F_1 = A_2 \cup F_2 \subset A_2 \cup E_2$$

$$\Rightarrow\ \mu(A_1) \le \mu(A_2) + \underbrace{\mu(E_2)}_{=0} = \mu(A_2)\,.$$

Similarly,

$$\mu(A_2) \le \mu(A_1)\,,$$

and therefore $\mu(A_1) = \mu(A_2) = \bar{\mu}(B)$.

(ii): $\bar{\mu}$ is a measure. (Exerc.)

(iii): $\bar{\mu}$ is complete. (Exerc.)

(3) (Exerc.) □

Example 1.13. $(\mathbb{R}^n, \Gamma, \mu)$, $\Gamma = \operatorname{Bor}\mathbb{R}^n$, $\mu = m_n|\operatorname{Bor}\mathbb{R}^n$.
Claim: $\bar{\Gamma} = \operatorname{Leb}\mathbb{R}^n$, $\bar{\mu} = m_n$. (Exerc.)

Example 1.14. Let $f_j\colon \mathbb{R}^n \to \mathbb{R}$, $j \in \mathbb{N}$, be *Borel functions*, i.e.

$$U \subset \mathbb{R} \text{ open } \ \Rightarrow\ f_j^{-1}U \in \operatorname{Bor}\mathbb{R}^n\,,$$

$\mu = m|\operatorname{Bor}\mathbb{R}^n$ and suppose that $f_j \to f$ μ-a.e., i.e.

$$\{x \in \mathbb{R}^n \colon f_j(x) \not\to f(x)\} \subset E \in \operatorname{Bor}\mathbb{R}^n\,,\ \mu(E) = 0\,.$$

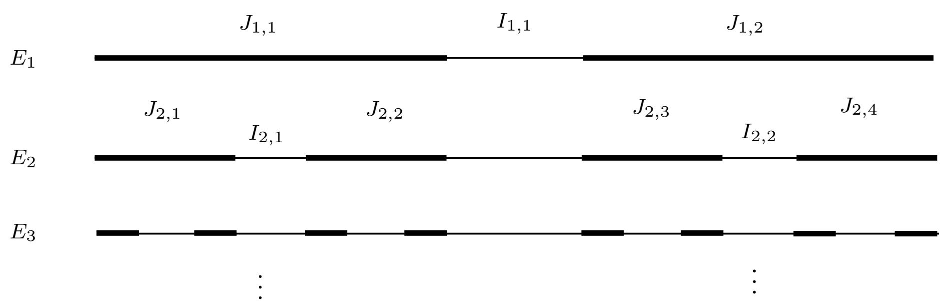
Then we can *not* conclude that f is a Borel function. Instead, we can conclude that f is Lebesgue measurable (see [Ho, L. 2.23 and 2.27]). (Reason: m complete.)

Remark 1.15. If a measure μ is complete, it makes sense to talk about measurability of functions that are defined μ-a.e.

1.16 Cantor set in $\mathbb{R}$

Let $I = [0, 1]$ and let $p = (p_1, p_2, \ldots)$ be a sequence of real numbers $0 < p_i < 1$. We erase (exactly) from the middle of I the open interval $I_{1,1}$ whose length is p_1.

$$I = J_{1,1} \cup I_{1,1} \cup J_{1,2}\,, \quad \text{where } J_{1,1} \text{ and } J_{1,2} \text{ are closed intervals of length } \frac{1-p_1}{2}\,.$$



Next we erase from the middle of each $J_{1,k}$ the open interval $I_{2,k}$ whose length is $= p_2\ell(J_{1,k}) = \frac{p_2(1-p_1)}{2}$.

What remains is the set

$$\begin{aligned} I \setminus (I_{1,1} \cup I_{2,1} \cup I_{2,2}) &= J_{2,1} \cup J_{2,2} \cup J_{2,3} \cup J_{2,4} \\ \text{length of } J_{2,k} &= \frac{1-p_2}{2} \cdot \frac{1-p_1}{2} \quad (< \frac{1}{2^2}) . \\ \text{total length of } J'_{2,k}\text{s} &= (1-p_1)(1-p_2) . \end{aligned}$$

We continue the process . . .:

Finally, what remains is the set:

$$E = I \setminus \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{2^{j-1}} I_{j,k} = \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{2^j} J_{j,k} \quad \text{or equivalently}$$

$$E = \bigcap_{j=1}^{\infty} E_j \,, \quad \text{where } E_j = \bigcup_{k=1}^{2^j} J_{j,k} \text{ is compact} .$$

In this course, we say that $E = E(p) =$ is the *Cantor set determined by the sequence p.*

$$\begin{aligned} E_1 \supset E_2 \supset \cdots , \quad m(E_1) < \infty &\overset{\text{[Ho, L. 1.60]}}{\Longrightarrow} \\ m(E) = \lim_{j\to\infty} m(E_j) &= \lim_{j\to\infty} (1-p_1)(1-p_2)\cdots(1-p_j) \\ &= \prod_{j=1}^{\infty}(1-p_j) . \end{aligned}$$

If $p_j = 1/3\ \forall j$, the corresponding Cantor set E is called the *Cantor $\frac{1}{3}$-set*, in which case

$$m(E) = \lim_{j\to\infty} \left(\frac{2}{3}\right)^j = 0.$$

Remark 1.17. The numbers p_j can be chosen such that $m(E)$ takes any given value on the interval $[0, 1[$. (Exerc.) [Hint: By taking log from the (infinite) product that gives $m(E)$ we obtain an infinite series. Then choose the numbers p_j so that this series is a geometric series (whose sum we can, of couse, compute). Or more simpler: Let $a = m(E) \in]0, 1[$. Choose $0 < p_1 < 1$ s.t. $a < 1 - p_1 < a + 1,\ p_2 \in]0, 1[$ s.t. $a < (1 - p_1)(1 - p_2) < a + 1/2$ etc.]

Theorem 1.18. *The Cantor set $E = E(p)$ satisfies:*

(a) E is compact and it does not contain any open interval.

(b) If $E(p)$ and $E(q)$ are Cantor sets determined by sequences p and q, then there exists a homeomorphism
$f : E(p) \to E(q)$.

(c) E is uncountable.

Remark 1.19. (i) E is closed and does not contain any (non-empty) open sets $\Rightarrow$ E is *nowhere dense*. [Def. $A \subset \mathbb{R}^n$ is nowhere dense if $\operatorname{int} \bar{A} = \emptyset$.]

(ii) There exists a strictly increasing continuous bijection $f\colon \mathbb{R} \to \mathbb{R}$ s.t. $f(E_p) = E_q$.

(iii) Recall: $f\colon A \to B$ is a homeomorphism if f is a continuous bijection whose inverse map f^{-1} is also continuous.

Proof. (a):

$$\left.\begin{array}{c} E_j \text{ closed } \Rightarrow \ E \text{ closed.} \\ E \subset I,\ I \text{ compact} \end{array}\right\} \Rightarrow \ E \text{ compact}$$

The construction $\Rightarrow$ E does not contain open intervals.

(b): If $x \in E$, the clearly $\exists$ a unique sequence of closed intervals $I \supset J_{1,k_1} \supset J_{2,k_2} \supset \cdots$ s.t.

$$\bigcap_{j=1}^{\infty} J_{j,k_j} = \{x\}\,.$$

Conversely: If $I \supset J_{1,k_1} \supset J_{2,k_2} \supset \cdots$ is a sequence of closed intervals, the intersection

$$\bigcap_{j=1}^{\infty} J_{j,k_j} \quad \text{is a singleton, i.e. contains exactly one point,}$$

because $m\big(J_{j,k_j}\big) \xrightarrow{j\to\infty} 0$.

Define $f\colon E(p) \to E(q)$ as follows:

$$\text{if } \ \{x\} = \bigcap_{j=1}^{\infty} J_{j,k_j}(p)\,, \quad \text{then } \ \{f(x)\} = \bigcap_{j=1}^{\infty} J_{j,k_j}(q)\,.$$

<u>Note:</u> the indexes j, k_j corresponding to x and $f(x)$ are the *same*.

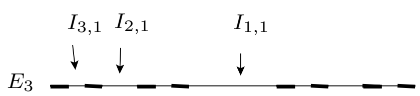
The construction $\Rightarrow$ f is bijective.

Let's prove next that f is continuous:

Denote

$$\delta_j(p) = \min\{m\big(I_{i,k}(p)\big) \colon i \le j\}\,, \quad \text{and so}$$

$$\delta_j(p) \xrightarrow{j\to\infty} 0\,.$$



If $x, y \in E(p)$ and $|x-y| < \delta_j(p)$, then necessarily x and y belong to the same interval $J_{j,k}(p)$, and therefore

$$\begin{aligned} & f(x), f(y) \in J_{j,k}(q) \quad \text{(same indexes)} \\ \Rightarrow\ & |f(x) - f(y)| \le m\big(J_{j,k}(q)\big) < \frac{1}{2^j} \quad \text{(the number of disjoint intervals } J_{j,k} \text{ is } 2^j) \\ \Rightarrow\ & f \quad \text{continuous.} \end{aligned}$$

Concluding similarly we get that f^{-1} is continuos (or: f continuous bijection and $E(p)$ compact $\Rightarrow$ f homeomorphism).

(c): If $m(E) > 0$, then E must be uncountable. Let then $E = E(p)$ be such that $m(E) = 0$. Choose a sequence q s.t. $m\big(E(q)\big) > 0$.

$$\left.\begin{array}{c}\text{(b)} \quad \Rightarrow \ \exists \ \text{homeomorphism } f\colon E(p) \to E(q) \\ E(q) \ \ \text{uncountable}\end{array}\right\} \quad \Rightarrow \ E(p) \ \ \text{uncountable.}$$

□

Example 1.20. Let us now prove the claim in Example 1.6 in the remaining case $n = 1$. In other words, there exists a Lebesgue measurable subset of $\mathbb{R}$ that is not Borel.

<u>Proof:</u> Choose Cantor sets E and E' s.t. $m(E) > 0$ and $m(E') = 0$. Then there exists a non-Lebesgue measurable set $F \subset E$.

$$\begin{array}{c}\text{Thm. 1.18} \quad \Rightarrow \exists \text{ homeomorphism } f\colon E \to E' \\ fF \subset E' \ \Rightarrow \ m^*(fF) = 0 \ \Rightarrow \ fF \in \operatorname{Leb}\mathbb{R}.\end{array}$$

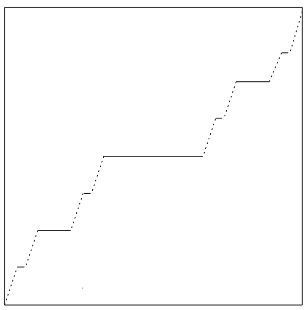
Suppose that fF is a Borel set.

$$\left.\begin{array}{c} E \in \operatorname{Bor}\mathbb{R} \\ f\colon E \to \mathbb{R} \text{ continuous} \\ fF \in \operatorname{Bor}\mathbb{R}\end{array}\right\} \overset{\text{L. 1.7}}{\Longrightarrow} f^{-1}(fF) = F \in \operatorname{Bor}\mathbb{R}\,.$$

<u>Contradiction</u> since $F \notin \operatorname{Leb}\mathbb{R}$. Hence fF is Lebesgue measurable but not Borel. □

Example 1.21. Let E be the Cantor 1/3-set. Define $f\colon I \to I,\ I = [0,1]$, first by setting

$$\begin{aligned} f(x) &= \frac{1}{2}, \quad \forall x \in I_{1,1}, \\ f(x) &= \frac{1}{4}, \quad \forall x \in I_{2,1}, \\ f(x) &= 1 - \frac{1}{4}, \quad \forall x \in I_{2,2}, \\ &\vdots \\ f(x) &= \frac{1 + 2(k-1)}{2^j}, \quad \forall x \in I_{j,k}, \\ &\vdots \end{aligned}$$



Now

$$f\colon \underbrace{\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{2^{j-1}} I_{j,k}}_{=A} \to I$$

is increasing and $\forall y \in]0,1[$:

$$\lim_{\substack{x \to y+ \\ x\in A}} f(x) = \lim_{\substack{x \to y- \\ x\in A}} f(x)\,.$$

Define $f(y)$ as the above limit at points $y \in E\setminus\{0,1\}$ and as the one-sided limit at points $y \in \{0,1\}$. We obtain $f\colon I \to I$ that satisfies:

(a) f is a continuous and increasing surjection;

(b) $f'(x) = 0$ a.e. $x \in I$ (since $f'(x) = 0\ \forall x \in I_{j,k}$ and $m(E) = 0$);

(c) $fE = I$ (because f is constant on each open interval $I_{j,k} \subset I \setminus E$ and their end points belong to E).

The function f is called the *Cantor* 1/3-*function* (the Cantor ternary function, "Devil's Staircase"). We will return to this function later.

1.22 The space L^1

Let (X, Γ, μ) be a complete measure space, i.e. μ is complete. The theory of (Lebesgue) measurable functions and the theory of Lebesgue integration that were developed in the course "Mitta ja integraali" work (almost verbatim) in this more general context. Furthermore, no problems occur with the concept of "almost everywhere" if the measure μ is complete. Also the convergence theorems (Monotone convergence theorem, Dominated convergence theorem, Fatou's lemma) and their proofs are the same as in the case of Lebegue integration. Instead Fubini's theorems require different proofs in the case of a general product measure $\mu \times \nu$ (in $X \times Y$) than in the case of Lebesgue measure (see e.g. [Ru, s. 136-142]).

Remark 1.23. The notions like "an open set, a continuous function, etc." require a topological space X, similarly Bor X (= the smallest σ-algebra on X containing all closed subsets of X).

Let $A \subset X$ be μ-measurable (abbr. measurable), i.e. $A \in \Gamma$. Note: It is, in fact, misleading to talk about μ-measurability because it is a property that depends on the σ-algebra Γ, not on the measure $\mu\colon \Gamma \to [0, +\infty]$. A function $f\colon A \to \dot{\mathbb{R}}$ is said to be *measurable* (or *μ-measurable, Γ-measuable*) if

$$f^{-1}(-\infty) \in \Gamma,\ f^{-1}(+\infty) \in \Gamma, \text{ and } f^{-1}U \in \Gamma \quad \forall U \subset \mathbb{R} \text{ open.}$$

Let us define

$$L^1(A) = \{f\colon A \to \dot{\mathbb{R}} \mid f \text{ measurable and } \int_A |f|d\mu < \infty\}\,.$$

We also denote $L^1(A, \mu)$ and $L^1 = L^1(\mu) = L^1(X)$.

Remark 1.24. If $A \subset X$ is measurable and $\Gamma_A = \{B \cap A\colon B \in \Gamma\}$, then $(A, \Gamma_A, \mu|\Gamma_A)$ is also a complete measure space. Therefore it suffices (usually) to study the "whole" measure space (X, Γ, μ).

Recall: a measurable function $f\colon X \to \dot{\mathbb{R}}$ is integrable (over X) if

$$\int_X f^+ d\mu < \infty \quad \text{and} \quad \int_X f^- d\mu < \infty\,.$$

In that case

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu \quad \text{and} \quad \int_X |f| d\mu = \int_X f^+ d\mu + \int_X f^- d\mu\,.$$

Thus: $f \in L^1 \iff f\colon X \to \dot{\mathbb{R}}$ is integrable.

Remark 1.25. If $f\colon X \to \dot{\mathbb{R}}$ is integrable, then $f(x) \in \mathbb{R}$ μ-a.e. We define $f^*\colon X \to \mathbb{R}$,

$$f^*(x) = \begin{cases} f(x), & \text{if } f(x) \in \mathbb{R}\,, \\ 0, & \text{if } f(x) \in \{-\infty, +\infty\}\,. \end{cases}$$

Then f^* is integrable and

$$\int_X f^* d\mu = \int_X f d\mu\,.$$

Threfore we may often assume that the functions in question are real valued.

If $f, g \in L^1,\ a, b \in \mathbb{R}$, then $af + bg \in L^1$. Hence L^1 is a vector space with real coefficients. We denote

$$\|f\|_1 = \|f\|_{1,X} = \int_X |f| d\mu = \int |f| d\mu\,.$$

Theorem 1.26. $\|\cdot\|$ *satisfies:*

(1) $\|f\|_1 \geq 0$,

(2) $\|\lambda f\|_1 = |\lambda| \|f\|_1,\ \forall \lambda \in \mathbb{R}$,

(3) $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$,

(4) $\|f\|_1 = 0 \iff f = 0$ *a.e.*

Proof. Clear. ☐

Theorem 1.26 $\Rightarrow$ $f \mapsto \|f\|_1$ is a *semi norm*. It is not a norm because: $\|f\|_1 = 0 \not\Rightarrow f = 0$.

Example.

$$X = \mathbb{R},\ \mu = m,\ f = \chi_{\mathbb{Q}},\ \Rightarrow\ \|f\|_1 = 0, \quad \text{but} \quad f \neq 0\,.$$

Definition 1.27. Functions $f, g \in L^1$ ae *equivalent*, denoted by $f \sim g$, if $f = g$ a.e.

We denote

$$[f] = \tilde{f} = \{g \in L^1 \colon g \sim f\} = \text{the equivalence class of } f\,,$$
$$\tilde{L}^1 = \{\tilde{f} \colon f \in L^1\}\,.$$

$\tilde{L}^1$ is a vector space:

$$[af + bg] = a[f] + b[g]\,.$$

We define

$$\|\tilde{f}\|_1 = \|f\|_1 \quad \text{(well defined, does not depend on the representative } f)\,.$$

$\tilde{L}^1$ is a normed space since in addition to the cases (1)–(3) in Theorem 1.26 we also have:

(4') $\|\tilde{f}\|_1 = 0 \iff \tilde{f} = 0$,

where $0 = \tilde{0} = \{f \in L^1 \colon f = 0 \text{ a.e.}\}$. In what follows we will get rid of the notation $\tilde{L}^1$ and we say: *a normed space* L^1. We also talk about (L^1-)functions rather than equivalence classes, that is, we identify functions that agree almost everywhere.

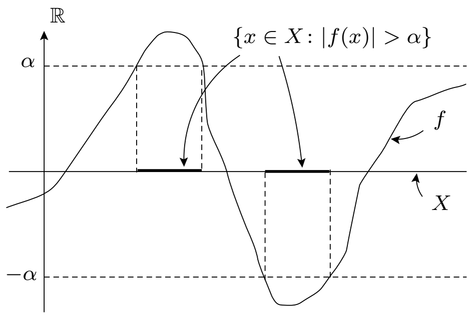
1.28 The space L^∞

Let (X, Γ, μ) be a complete measure space and $f \colon X \to \dot{\mathbb{R}}$ measurable.

We write

$$\|f\|_\infty = \inf \underbrace{\{\alpha \geq 0 \colon \mu\big(\{x \in X \colon |f(x)| > \alpha\}\big) = 0\}}_{=S}.$$

If $S = \emptyset$, we set $\|f\|_\infty = \infty$.



If $\|f\|_\infty < \infty$, that is $S \neq \emptyset$, then we have:

$$\{x \in X \colon |f(x)| > \|f\|_\infty\} \subset \bigcup_{j \in \mathbb{N}} \{x \in X \colon |f(x)| > \|f\|_\infty + 1/j\}$$

$$\Rightarrow \mu\big(\{x \in X \colon |f(x)| > \|f\|_\infty\}\big) \leq \sum_{j \in \mathbb{N}} \underbrace{\mu\big(\{x \in X \colon |f(x)| > \|f\|_\infty + 1/j\}\big)}_{=0} = 0$$

$$\Rightarrow \|f\|_\infty \in S \quad \text{and hence } |f| \leq \|f\|_\infty \text{ a.e.}$$

Therefore we often denote $\|f\|_\infty = \operatorname{ess\,sup}|f|$ ("essential supremum"). Denote

$$L^\infty(X) = L^\infty = L^\infty(\mu) = \{f \colon X \to \dot{\mathbb{R}} \mid f \text{ measurable and } \|f\|_\infty < \infty\}.$$

Again we identify functions $f, g \in L^\infty$ if $f = g$ a.e. Equivalence classes are denoted by f etc.: L^∞ = the set of equivalence classes but nevertheless we talk about functions.

Theorem 1.29. *The space L^∞ is a normed space equipped with the norm $\|\cdot\|_\infty$.*

Proof. Clearly:

(i) L^∞ is a vector space (see (iv)).

(ii) $\|f\|_\infty \geq 0$ and $\|f\|_\infty = 0 \iff f = 0$ a.e. (notice the equivalence class).

(iii) $\|\lambda f\|_\infty = |\lambda| \|f\|_\infty \; \forall \lambda \in \mathbb{R}$.

Moreover:

(iv) the triangle inequality holds:

$$\left.\begin{array}{l} |f| \le \|f\|_\infty \text{ a.e.} \\ |g| \le \|g\|_\infty \text{ a.e.} \end{array}\right\} \Rightarrow |f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty \text{ a.e.}$$

Hence

$$\mu\big(\{x \in X : |f(x) + g(x)| > \|f\|_\infty + \|g\|_\infty\}\big) = 0,$$

and so $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ and $f+g \in L^\infty$.

□

Example 1.30. Let $X = \mathbb{R}$, $\mu = m$, and $f : \mathbb{R} \to \mathbb{R}$ *continuous*.
Claim: $f \in L^\infty \iff f$ is bounded.
Proof: $\boxed{\Leftarrow}$ clear.
$\boxed{\Rightarrow}$:
Assume on the contrary: f is not bounded, and hence $\forall M > 0\ \exists\, x_0 \in \mathbb{R}$ s.t. $|f(x_0)| > M$.
Since f is continuous, we have $|f(x)| > M\ \forall x \in]x_0 - \delta, x_0 + \delta[= J$ for some $\delta > 0$.
$m(J) > 0 \Rightarrow \|f\|_\infty \ge M$.
$M > 0$ arbitrary $\Rightarrow \|f\|_\infty = \infty$. Contradiction □

1.31 The space L^p, $1 \le p < \infty$

Let (X, Γ, μ) be a measure space, μ complete, and $1 \le p < \infty$. Let us define

$$L^p(X) = L^p = L^p(\mu) = \{f : X \to \dot{\mathbb{R}} \mid f \text{ measurable and } \int_X |f|^p d\mu < \infty\}.$$

Denote

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p}.$$

We identify functions that agree a.e. as earlier. The exponent p has a *great influence on* which functions belong to L^p.

Example 1.32. Let $X =]0,1[$ and $\mu = m|]0,1[$ be the (restriction of) Lebesgue measure. If f is measurable and bounded, then $f \in L^p\ \forall p \ge 1$. (Reason: $|f|^p$ measurable and bounded, $\mu(X) < \infty \Rightarrow |f|^p$ integrable, hence $f \in L^p$).

Let

$$f(x) = \frac{1}{\sqrt{x}}.$$

Then

$$\int_X |f|^p d\mu = \lim_{a\to 0+} \int_a^1 x^{-p/2} dx = \begin{cases} \lim\limits_{a\to 0+} \dfrac{1}{1-\frac{p}{2}} \Big/_a^1 x^{1-p/2} = \dfrac{1}{1-\frac{p}{2}} < \infty, & \text{if } p < 2, \\ \lim\limits_{a\to 0+} \dfrac{1}{1-\frac{p}{2}} \Big/_a^1 x^{1-p/2} = \infty, & \text{if } p > 2, \\ \lim\limits_{a\to 0+} \Big/_a^1 \log x = \infty, & \text{if } p = 2. \end{cases}$$

Hence

$$f \in L^p \iff 1 \le p < 2.$$

Theorem 1.33. *If $\mu(X) < \infty$ and $1 \le q \le p$, then $L^p(\mu) \subset L^q(\mu)$.*

Proof. Exercise. □

We will prove next: L^p is a normed space. We need some "tools".

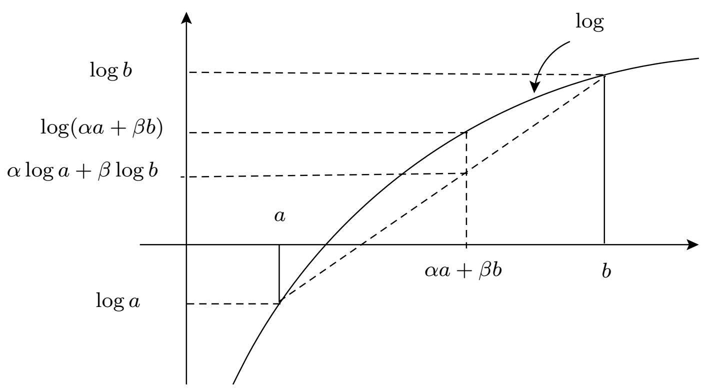
Lemma 1.34 (Young's inequality)**.** *If $a, b \ge 0$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$, then*

$$a^\alpha b^\beta \le \alpha a + \beta b\,.$$

Proof. The case $a = 0$ or $b = 0$ is trivial, therefore we may assume that $a, b > 0$.
The function $x \mapsto \log x$, $x > 0$, is concave, and therefore

$$\log\bigl(a^\alpha b^\beta\bigr) = \alpha \log a + \beta \log b \le \log(\alpha a + \beta b)\,.$$

Since log is increasing, we obtain the claim.



□

Next we will prove a very *important* inequality.

Theorem 1.35 (Hölder's inequality)**.** *If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, and $g \in L^q$, then*

$$fg \in L^1 \quad \textit{and} \quad \|fg\|_1 \le \|f\|_p \|g\|_q\,, \quad \textit{i.e.}$$

$$\int_X |fg| d\mu \le \left(\int_X |f|^p d\mu\right)^{1/p} \left(\int_X |g|^q d\mu\right)^{1/q}.$$

Proof. If $\|f\|_p = 0$, then $f = 0$ a.e.,and consequently $\|fg\|_1 = 0$ and the claim follows. Similarly, if $\|g\|_q = 0$. Hence we may assume that

$$\|f\|_p,\ \|g\|_q > 0\,.$$

Furthermore, we may assume that $f(x), g(x) \in \mathbb{R}\ \forall x$ (see Remark 1.25). By applying Young's inequality with

$$a = \frac{|f(x)|^p}{\|f\|_p^p}\,,\ b = \frac{|g(x)|^q}{\|g\|_q^q}\,,\ \alpha = \frac{1}{p}\,,\ \beta = \frac{1}{q}\,,$$

we obtain ($a^\alpha b^\beta \leq \alpha a + \beta b$)

$$\frac{|f(x)|}{\|f\|_p}\frac{|g(x)|}{\|g\|_q} \leq \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|^q}{\|g\|_q^q}\,.$$

By integrating over X:n (notice that the functions are measurable) we get

$$\frac{\|fg\|_1}{\|f\|_p\|g\|_q} \leq \frac{1}{p}\frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q}\frac{\|g\|_q^q}{\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1\,.$$

□

Remark 1.36. The constants $p, q > 1$, for which $\frac{1}{p} + \frac{1}{q} = 1$, are called Hölder conjugates (of each other). Often we denote $q = p' = \frac{p}{p-1}$. The exponent 2 is the only constant that is conjugate with itself.

Corollary 1.37 (Schwarz inequality)**.**

$$\|fg\|_1 \leq \|f\|_2\|g\|_2\,.$$

Example 1.38. Let $X = \{1, 2, \ldots, n\}$, $\mu\colon \mathcal{P}(X) \to [0, \infty[$, $\mu(A) = \operatorname{card} A$ = the number of elements in A.If $\frac{1}{p} + \frac{1}{q} = 1$ and $a_1, b_1, a_2, b_2, \ldots, a_n, b_n \in \mathbb{R}$, then

$$\sum_{i=1}^{n}|a_ib_i| \leq \left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|b_i|^q\right)^{1/q}\,.$$

Proof: Choose

$$f = \sum_i a_i\chi_{\{i\}}\,, \quad g = \sum_i b_i\chi_{\{i\}}\,.$$

In general:

$$X = \{x_i\colon i = 1, 2, \ldots\}\,,\ \Gamma = \mathcal{P}(X)\,,\ \text{and}\ \mu(A) = \operatorname{card} A\,.$$

If $f\colon X \to \mathbb{R}$, then

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p} = \left(\sum_{i=1}^{\infty}|f(x_i)|^p\right)^{1/p}\,.$$

(Note that every function f is measurable.) Denote

$$L^p(X) = \ell^p(X)\,.$$

If $f \in \ell^p(X)$, $g \in \ell^q(X)$, $\frac{1}{p} + \frac{1}{q} = 1$, and $a_i = f(x_i)$, $b_i = g(x_i)$, then the Hölder inequality takes the form

$$\sum_{i=1}^{\infty}|a_ib_i| \leq \left(\sum_{i=1}^{\infty}|a_i|^p\right)^{1/p}\left(\sum_{i=1}^{\infty}|b_i|^q\right)^{1/q}\,.$$

Theorem 1.39 (Minkowski's inequality)**.** *If $f, g \in L^p$, then $f + g \in L^p$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p\,.$$

Proof. The case $p = 1$ is already proven. Let $p > 1$ and $q = \frac{p}{p-1}$ so that $\frac{1}{p} + \frac{1}{q} = 1$ (Hölder conjugates).
If $a, b \geq 0$, then

$$(a+b)^p \leq \big(2\max(a,b)\big)^p = 2^p\big(\max(a,b)\big)^p \leq 2^p(a^p + b^p)\,.$$

We may assume that $f(x), g(x) \in \mathbb{R}\ \forall x$ (Remark 1.25). Then

$$|f+g|^p \leq (|f| + |g|)^p \leq 2^p\big(|f|^p + |g|^p\big) \quad \Rightarrow \quad f + g \in L^p\,.$$

On the other hand,

$$|f+g|^p = |f+g||f+g|^{p-1} \leq |f||f+g|^{p-1} + |g||f+g|^{p-1}$$

and

$$\big(|f+g|^{p-1}\big)^q = |f+g|^p \overset{f+g\in L^p}{\Longrightarrow} |f+g|^{p-1} \in L^q\,.$$

By Hölder's inequality we get

$$\begin{aligned}
\|f+g\|_p^p = \int |f+g|^p &\leq \int \underbrace{|f|}_{\in L^p}\underbrace{|f+g|^{p-1}}_{\in L^q} + \int \underbrace{|g|}_{\in L^p}\underbrace{|f+g|^{p-1}}_{\in L^q} \\
&\leq \|f\|_p \left(\int \big(|f+g|^{p-1}\big)^q\right)^{1/q} + \|g\|_p \left(\int \big(|f+g|^{p-1}\big)^q\right)^{1/q} \\
&= \|f\|_p \left(\int |f+g|^p\right)^{1/q} + \|g\|_p \left(\int |f+g|^p\right)^{1/q} \\
&= \big(\|f\|_p + \|g\|_p\big)\|f+g\|_p^{p/q}
\end{aligned}$$

$$\overset{p/q=p-1}{\Longrightarrow} \|f+g\|_p \leq \|f\|_p + \|g\|_p\,.$$

□

We have proved:

Theorem 1.40. *L^p is a normed space when equipped with the norm $\|\cdot\|_p$.*

Example 1.41. 1. If $\mu(X) = \infty$, it may happen that $L^p \not\subset L^q,\ q < p$: Let $X = \mathbb{R}$ and $\mu = m$.

$$f(x) = \frac{1}{1+|x|} \quad \begin{aligned} &f \in L^p,\ \text{if } p > 1\,,\\ &f \notin L^1. \end{aligned}$$

2. In general $L^p \not\subset L^q,\ p \neq q$. Above is the case $q < p$. Earlier: $X =]0,1[\,,\ \mu = m$ and

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{x}} \\ f &\in L^p, \quad \text{if } 1 \leq p < 2 \\ f &\notin L^q, \quad \text{if } q \geq 2\,. \end{aligned}$$

1.42 Completeness of L^p spaces

In this section we will prove that normed spaces L^p, $1 \le p \le \infty$, are *Banach spaces*, that is complete normed spaces.

Some terminology: Let (Y, d) be a metric space. We say that a sequence (x_j), $x_j \in Y$, is a *Cauchy sequence* in Y if for every $\varepsilon > 0$ there exists $i_\varepsilon \in \mathbb{N}$ s.t. $d(x_i, x_j) < \varepsilon$ for every $i, j \ge i_\varepsilon$. A metric space (Y, d) is *complete* if every Cauchy sequence in Y converges towards a point in Y. Recall that a sequence (x_j) converges to $x \in Y$ if $d(x_j, x) \to 0$ as $i \to \infty$.

Let $(V, \|\cdot\|)$ be a normed space. It is also a metric space equipped with the natural metric

$$d(x, y) = \|x - y\| .$$

A normed space $(V, \|\cdot\|)$ is a Banach space if, for every Cauchy sequence (x_j) in Y, there exists $x \in V$ s.t.

$$\|x_j - x\| \xrightarrow{j\to\infty} 0 .$$

Let (X, Γ, μ) be a complete measure space.
We say that (f_j) converges to f in L^p, denoted by $f_j \to f$ in L^p, if $f_j, f \in L^p$ and $\|f_j - f\|_p \to 0$ as $j \to \infty$.

Theorem 1.43. *If (f_j) is a Cauchy sequence in L^p, $1 \le p < \infty$, then there exists a subsequence (f_{j_k}) that converges (pointwise) almost everywhere.*

Proof. For every $k \in \mathbb{N}$ we choose j_k s.t.

(1) $\|f_i - f_j\|_p < \frac{1}{2^k}$ whenever $i, j \ge j_k$,

(2) $j_1 < j_2 < \cdots$.

Note that by Remark 1.25 we may assume that all functions (above) are real valued. We define

$$g_k = |f_{j_1}| + |f_{j_2} - f_{j_1}| + \cdots + |f_{j_{k+1}} - f_{j_k}| .$$

$$(g_k) \text{ increasing sequence } \Rightarrow \exists\, g = \lim_{k\to\infty} g_k .$$

By Minkowski's inequality,

$$\begin{aligned}
\|g_k\|_p &= \Big\| \, |f_{j_1}| + \sum_{\nu=1}^{k} |f_{j_{\nu+1}} - f_{j_\nu}| \, \Big\|_p \\
&\overset{\text{Minkowski}}{\le} \|f_{j_1}\|_p + \sum_{\nu=1}^{k} \|f_{j_{\nu+1}} - f_{j_\nu}\|_p \\
&\le \|f_{j_1}\|_p + \sum_{\nu=1}^{k} \frac{1}{2^\nu} \le \|f_{j_1}\|_p + 1 \quad \forall k
\end{aligned}$$

The Monotone convergence theorem (MCT) then implies that

$$\int g^p \overset{\text{MCT}}{=} \lim_{k\to\infty} \int g_k^p = \lim_{k\to\infty} \|g_k\|_p^p \le \big(\|f_{j_1}\|_p + 1\big)^p < \infty$$
$$\Rightarrow g(x) < \infty \quad \text{a.e.}$$

Hence the series

$$f_{j_1}(x) + \sum_{\nu=1}^{\infty} \big(f_{j_{\nu+1}}(x) - f_{j_\nu}(x)\big)$$

converges a.e. Denote its sum by $f(x)$,

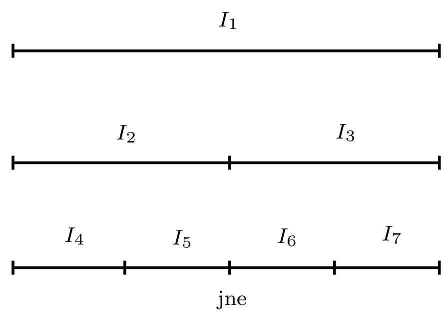
$$f(x) = f_{j_1}(x) + \sum_{\nu=1}^{\infty} \big(f_{j_{\nu+1}}(x) - f_{j_\nu}(x)\big).$$

We obtain

$$f_{j_{k+1}} = f_{j_1} + \sum_{\nu=1}^{k} \big(f_{j_{\nu+1}} - f_{j_\nu}\big) \to f \text{ a.e.}$$

□

Remark 1.44. The condition $f_j \to f$ in L^p does *not* (in general) imply that (the whole sequence) $f_j \to f$ a.e.
Example: Let I_k be the closed subinterval of $I = [0, 1]$ as in the picture.



Let $f_k = \chi_{I_k} \colon I \to \mathbb{R}$. Then $f_k \in L^p$, $\forall p \in [1, \infty)$ and

$$\|f_k - 0\|_p = \left(\int_I \chi_{I_k} dm\right)^{1/p} = m(I_k)^{1/p} \xrightarrow{k\to\infty} 0\,.$$

Hence $f_k \to 0$ in L^p.
Claim: $f_k(x) \not\to 0$ as $k \to \infty$ *for any* $x \in I$.
Proof: Let $x \in I$ and $k_0 \in \mathbb{N}$ be arbitrary.

$$\bigcup_{k>k_0} I_k = I \;\Rightarrow\; \exists\, k_1 > k_0 \text{ s.t. } x \in I_{k_1} \text{ and } f_{k_1}(x) = 1\,. \quad \square$$

Theorem 1.45. *L^p is a Banach avaruus for every $1 \le p \le \infty$.*

Remark 1.46. The case $1 \le p < \infty$ is so-called *Riesz-Fischer theorem* (1906).

Proof. (a) $1 \le p < \infty$: Let (f_j) be a Cauchy sequence in L^p. Theorem 1.43 implies that there exists a subsequence (f_{j_k}) s.t. $f_{j_k} \to f$ a.e.
<u>Claim</u>: $f \in L^p$ and $f_{j_k} \to f$ in L^p.
<u>Proof:</u> Let $\varepsilon > 0$. Then there exists $j_0 \in \mathbb{N}$ s.t.

$$\|f_i - f_j\|_p < \varepsilon \quad \text{as } i, j \ge j_0 \,.$$

If $j \ge j_0$, we have

$$\begin{aligned} \int |f_j - f|^p d\mu &= \int \lim_{k\to\infty} |f_j - f_{j_k}|^p d\mu \overset{\text{Fatou}}{\le} \liminf_{k\to\infty} \int |f_j - f_{j_k}|^p d\mu \\ &= \liminf_{k\to\infty} \|f_j - f_{j_k}\|_p^p \le \varepsilon^p \end{aligned}$$

$$\Rightarrow \begin{cases} f_j - f \in L^p \\ \|f_j - f\|_p \xrightarrow{j\to\infty} 0 \end{cases} \Rightarrow \begin{cases} f = f_j - (f_j - f) \in L^p \\ f_j \to f \text{ in } L^p. \end{cases} \qquad \square$$

(b) $p = \infty$: Let (f_j) be a Cauchy sequence in L^∞. Denote

$$\begin{aligned} A_j &= \{x \colon |f_j(x)| > \|f_j\|_\infty\} \\ A_{j,k} &= \{x \colon |f_j(x) - f_k(x)| > \|f_j - f_k\|_\infty\} \,. \end{aligned}$$

Then $\mu(A_j) = 0 = \mu(A_{j,k})$ (this follows directly from the definition of $\|\cdot\|_\infty$). Denote

$$A = \bigcup_j A_j \cup \bigcup_{j,k} A_{j,k} \,, \quad \text{hence} \quad \mu(A) = 0 \,.$$

If $x \in A^c$, then

$$|f_j(x) - f_k(x)| \le \|f_j - f_k\|_\infty \,. \tag{1.47}$$

Thus $\big(f_j(x)\big)$ is a Cauchy sequence in $\mathbb{R}$, and therefore the sequence $\big(f_j(x)\big)$ converges. Define

$$f(x) = \lim_{j\to\infty} f_j(x) \,.$$

The Cauchy criterion for the uniform convergence and (1.47) imply that

$$f_j \to f \quad \text{uniformly in } A^c.$$

For $x \in A$, we set $f(x) = 0$. By [Ho, L. 2.29] f is measurable.
<u>Claim</u>: $f \in L^\infty$ and $\|f_j - f\|_\infty \to 0$.
<u>Proof:</u> Let $j_0 \in \mathbb{N}$ s.t.

$$\|f_j - f_k\|_\infty < 1 \,, \quad \text{if } j, k \ge j_0 \,.$$

Since $\|\cdot\|_\infty$ is a norm, we have

$$\|f_j\|_\infty \le \|f_{j_0}\|_\infty + \|f_j - f_{j_0}\|_\infty \le \|f_{j_0}\|_\infty + 1 =: M$$

for $j \geq j_0$. If $x \in A^c$, we have

$$\begin{aligned} |f_j(x)| &\leq \|f_j\|_\infty\,, \quad \text{and so} \\ |f(x)| &= \lim_{j\to\infty} |f_j(x)| \leq M \\ &\Rightarrow \|f\|_\infty \leq M\,, \quad \text{because } \mu(A) = 0\,. \\ &\Rightarrow f \in L^\infty\,. \end{aligned}$$

(In fact, $|f(x)| \leq M \ \forall x$.)
Since $f_j \to f$ uniformly in A^c and $\mu(A) = 0$, we have $\|f_j - f\|_\infty \to 0$.

□

Remark 1.48. The theory of L^p spaces generalizes to mappings $f\colon X \to \mathbb{R}^m$, $f = (f_1, \ldots, f_m)$. Then the norm is

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p} ,$$

where $|f(x)| = \big(f_1(x)^2 + \cdots + f_m(x)^2\big)^{1/2}$ is the Euclidean norm. Similarly, for functions $f\colon X \to \mathbb{C}$.

2 Approximation in L^p spaces

2.1 Absolute continuity of measures

Let (X, Γ, μ) be a measure space and let $\sigma\colon \Gamma \to [0, +\infty]$ be another measure.

Definition 2.2. A measure σ is *absolutely continuous* with respect to μ (denoted by $\sigma \ll \mu$) if $\sigma(E) = 0$ for every $E \in \Gamma$ with $\mu(E) = 0$.

Example 2.3.

1. Let $f\colon X \to [0, +\infty]$ be Γ-measurable. Define

   $$\sigma(A) = \int_A f d\mu\,, \quad A \in \Gamma\,.$$

   By properties of an integral ([Ho, Lause 3.32]) σ is a measure and

   $$\mu(A) = 0 \;\Rightarrow\; \sigma(A) = 0\,.$$

   Hence $\sigma \ll \mu$. (See the next Remark for the converse direction.)

2. Let $X = \mathbb{R}$ and $\sigma\colon \operatorname{Leb}\mathbb{R} \to [0, +\infty]$ be the counting measure. Then the Lebesgue measure $m(\{0\}) = 0$ but $\sigma(\{0\}) = 1$, and therefore $\sigma \not\ll m$.

3. Let $X = \mathbb{R}$, $x \in \mathbb{R}$, and $\delta_x\colon \operatorname{Leb}\mathbb{R} \to [0, +\infty]$ be the Dirac measure at x (or, in fact, the restriction of the Dirac measure to $\operatorname{Leb}\mathbb{R}$). Then $m(\{x\}) = 0$ but $\delta(\{x\}) = 1$, and therefore $\delta \not\ll m$.

Remark 2.4. Let (X, Γ, μ) be a measure space and $\varphi\colon \Gamma \to \dot{\mathbb{R}}$. We say that

(a) φ is *countably additive* if

   (i) $\varphi(\emptyset) = 0$,

(ii) if $A_1, A_2, \ldots \in \Gamma$ are pairwise disjoint, then $\sum_i \varphi(A_i)$ is defined and

$$\sum_i \varphi(A_i) = \varphi\big(\cup_i A_i\big).$$

(b) φ is *absolutely continuous* with respect to μ if $\mu(A) = 0 \Rightarrow \varphi(A) = 0$. Then we denote $\varphi \ll \mu$.

(c) φ is *σ-finite* if

$$X = \cup_j A_j,\ A_j \in \Gamma,\ |\varphi(A_j)| < \infty.$$

Unfortunately, we shall not prove the following *Radon-Nikodym theorem* in this course: If (X, Γ, μ) is σ-finite measure space and if $\varphi\colon \Gamma \to \dot{\mathbb{R}}$ is countably additive, σ-finite, and $\varphi \ll \mu$, then there exists a measurable function $f\colon X \to \mathbb{R}$ s.t.

$$\varphi(E) = \int_E f d\mu, \quad \forall E \in \Gamma.$$

If, in addition, g is another function satisfying the equation above, then $f = g$ a.e.

Theorem 2.5. *Let (X, Γ, μ) be a measure space $\sigma\colon \Gamma \to [0, +\infty)$ a measure s.t. $\sigma(X) < \infty$. Then*

$$\sigma \ll \mu \iff$$

$$\forall \varepsilon > 0\ \exists\ \delta > 0 \textit{ s.t. } \mu(A) < \delta \ \Rightarrow\ \sigma(A) < \varepsilon\,. \tag{2.6}$$

Proof. $\boxed{\Rightarrow}$ Suppose that $\sigma \ll \mu$.
Assume on the contrary that there exists $\varepsilon > 0$ and a sequence $E_1, E_2, \ldots \in \Gamma$ s.t.

$$\sigma(E_i) \geq \varepsilon \quad \text{and} \quad \mu(E_i) < 2^{-i}\,.$$

Denote

$$A_k = \bigcup_{i \geq k} E_i\,, \quad A = \bigcap_{k=1}^{\infty} A_k\,.$$

$$A_k \supset E_k \ \Rightarrow\ \sigma(A_k) \geq \sigma(E_k) \geq \varepsilon \quad \forall k\,.$$

$$\left.\begin{array}{c} A_1 \supset A_2 \supset \cdots \\ \sigma(A_1) \leq \sigma(X) < \infty \end{array}\right\} \Rightarrow \sigma(A) = \lim_{k\to\infty} \sigma(A_k) \geq \varepsilon\,.$$

$$\mu(A) \leq \mu(A_k) \leq \sum_{i=k}^{\infty} \frac{1}{2^i} = \frac{1}{2^{k-1}} \quad \forall k$$

$$\Rightarrow\ \mu(A) = 0 \overset{\sigma\ll\mu}{\Longrightarrow} \sigma(A) = 0\,.\ \underline{\text{Contradiction}}$$

$\boxed{\Leftarrow}$ If the condition (2.6) holds, then $\sigma \ll \mu$ trivially ($\mu(A) = 0 \ \Rightarrow\ \sigma(A) < \varepsilon\ \forall \varepsilon > 0 \Rightarrow \sigma(A) = 0$). $\square$

Corollary 2.7. Let $f \in L^1$. Then $\forall \varepsilon > 0\ \exists\ \delta > 0$ s.t.

$$\mu(E) < \delta \ \Rightarrow\ \int_E |f| d\mu < \varepsilon\,.$$

Proof. We can apply Theorem 2.5 to the measure

$$\sigma(E) = \int_E |f| d\mu \quad E \in \Gamma$$

since $\sigma \ll \mu$ and $\sigma(X) < \infty$ (because $f \in L^1$). □

2.8 Egorov's Theorem and Lusin's Theorem

Let (X, Γ, μ) be a measure space. In "Mitta and integraali"-course we proved ([Ho, Lause 3.14]):

Theorem 2.9. *If $f\colon X \to [0, \infty]$ is measurable, there exists an increasing sequence $0 \le f_1 \le f_2 \le \cdots$ of simple functions s.t.*

$$f = \lim_{j \to \infty} f_j \,.$$

Remark 2.10.

1. $g\colon X \to [0, \infty)$ is simple if

$$g = \sum_{i=1}^{k} a_i \chi_{A_i}\,,\ a_i \ge 0,\ A_i \in \Gamma \text{ disjoint.}$$

2. The proof of Theorem 2.9 is the same as the one in the case of $(\mathbb{R}^n, \mathrm{Leb}\,\mathbb{R}^n, m)$.
3. If the function f in Theorem 2.9 is bounded, then $f_j \to f$ uniformly in X, i.e. $\forall \varepsilon > 0\ \exists\, i_\varepsilon \in \mathbb{N}$ s.t. $|f_j(x) - f(x)| < \varepsilon$ *for every* $x \in X$ (the index i_ε is *independent* of x).
4. $f\colon X \to [0, \infty]$ is measurable $\iff$ $f = \lim_{j \to \infty} f_j$, where (f_j) is an increasing sequence of simple functions $f_j\colon X \to [0, \infty)$.

In general: The convergence is uniform in a large portion of X as the following theorem reveals.

Theorem 2.11 (Egorov's Theorem)**.** *Let μ be complete, $\mu(X) < \infty$, and let functions $f_k\colon X \to \mathbb{R}$, $k = 1, 2, \ldots,$ be measurable s.t. $f_k \to f$ a.e., where $f\colon X \to \mathbb{R}$. Then*

1. *$\forall \varepsilon > 0\ \exists$ a measurable $F \subset X$ s.t. $\mu(X \setminus F) < \varepsilon$ and $f_k|F \to f|F$ uniformly;*
2. *if $X \subset \mathbb{R}^n$ and $\mu = m =$ is the Lebesgue measure, the set F can be chosen to be compact.*

Lemma 2.12. *Let $A \subset \mathbb{R}^n$ be measurable and $\varepsilon > 0$. Then*

1. *$\exists$ open $G \supset A$ s.t. $m(G \setminus A) < \varepsilon$;*
2. *$\exists$ closed $F \subset A$ s.t. $m(A \setminus F) < \varepsilon$.*
3. *If, in addition, $m(A) < \infty$, then $\exists$ compact $F \subset A$ s.t. $m(A \setminus F) < \varepsilon$.*

Proof. (Extra) exercise. □

Proof of Egorov's Theorem. 1. μ complete $\Rightarrow$ f measurable ([Ho, Lause 2.29]). Denote

$$E_{k,l} = \bigcap_{m=l}^{\infty} \underbrace{\{x \in X \colon |f_m(x) - f(x)| < \frac{1}{k}\}}_{\text{measurable}}, \quad k, l \in \mathbb{N}\,,$$

$$H = \{x \in X \colon \lim_{m \to \infty} f_m(x) = f(x)\}\,.$$

If $x \in H$ and $k \in \mathbb{N}$, then $\exists\, l_k \in \mathbb{N}$ s.t. $|f_m(x) - f(x)| < \frac{1}{k}\ \forall m \geq l_k \ \Rightarrow\ x \in E_{k,l_k}$. Hence

$$H \subset \bigcup_{l=1}^{\infty} E_{k,l} \quad \forall k\,.$$

The sets H and $E_{k,l}$ are measurable and $\mu(H) = \mu(X)$ since $f_m \to f$ a.e.

$$E_{k,l} \subset E_{k,l+1} \ \Rightarrow$$

$$\mu(X) \geq \lim_{l\to\infty} \mu(E_{k,l}) = \mu\big(\bigcup_{l=1}^{\infty} E_{k,l}\big) \geq \mu(H) = \mu(X) \tag{2.13}$$

$$\mu(X) < \infty,\ (2.13)\ \Rightarrow$$

$$\lim_{l\to\infty} \mu(X \setminus E_{k,l}) = \mu(X) - \lim_{l\to\infty} \mu(E_{k,l}) = 0 \quad \forall k\,.$$

Let $\varepsilon > 0$. Then $\forall k\ \exists\, l_k \in \mathbb{N}$ s.t.

$$\mu(X \setminus E_{k,l_k}) < \frac{\varepsilon}{2^k}\,.$$

Claim: The

$$F = \bigcap_{k=1}^{\infty} E_{k,l_k}$$

satisfies the desired conditions.
Proof. F measurable and

$$\mu(X \setminus F) = \mu\big(\bigcup_{k=1}^{\infty} (X \setminus E_{k,l_k})\big) \leq \sum_{k=1}^{\infty} \mu(X \setminus E_{k,l_k}) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon\,.$$

Furthermore $F \subset E_{k,l_k}\ \forall k$ and

$$|f_m(x) - f(x)| < \frac{1}{k}\,, \quad \text{if } x \in E_{k,l_k} \text{ and } m \geq l_k\,.$$

$$\Rightarrow |f_m(x) - f(x)| < \frac{1}{k}\,, \quad \text{if } x \in F \text{ and } m \geq l_k\,.$$

$$\Rightarrow f_m|F \to f|F \quad \text{uniformly (the index } l_k \text{ is independent of } x \in F).$$

2. Suppose now that $\mu = m =$ is the Lebesgue measure. Lemma 2.12 $\Rightarrow$ $\exists$ compact $F_0 \subset F$ s.t. $\mu(F \setminus F_0) < \varepsilon$. Thus

$$\mu(X \setminus F_0) \leq \mu(X \setminus F) + \mu(F \setminus F_0) < 2\varepsilon.$$

□

A measurable function is continuous in a large portion of its domain:

Theorem 2.14 (Lusin's Theorem). *Let $A \subset \mathbb{R}^n$ be measurable, $m(A) < \infty$, and let $f\colon A \to \mathbb{R}$ be measurable. Then*

$$\forall \varepsilon > 0\ \exists \text{ compact } F \subset A \text{ s.t. } m(A \setminus F) < \varepsilon \text{ and } f|F \text{ is continuous.}$$

Proof. Let $\varepsilon > 0$. $\boxed{\text{(a):}}$ *Suppose that f is simple*

$$f = \sum_{i=1}^{k} a_i \chi_{A_i} \,.$$

Lemma 2.12 $\Rightarrow$ $\exists$ compact sets $F_i \subset A_i$ s.t. $m(A_i \setminus F_i) < \varepsilon/k$. Then $F = F_1 \cup \cdots \cup F_k$ is compact and F_i's are disjoint. If $x \in F$, then $\exists\, r > 0$ s.t. $B(x,r) \cap F_i \neq \emptyset$ *for exactly one i.*

$$\text{(Reason:} \quad x \in F_i \Rightarrow \ \operatorname{dist}(x, F_j) = \inf_{y \in F_j} |x-y| \overset{F_j \text{ compact}}{=} \min_{y \in F_j} |x-y| > 0 \ \forall j \neq i \,.)$$

Hence

$$f(y) = f(x) = a_i \ \forall y \in B(x,r) \cap F \,,$$

and therefore $f|F$ is locally constant and hence $f|F$ is continuous. Also

$$m(A \setminus F) = m\Big(\bigcup_{i=1}^{k} (A_i \setminus F_i)\Big) \overset{\text{disjoint}}{=} \sum_{i=1}^{k} m(A_i \setminus F_i) < \varepsilon \,.$$

$\boxed{\text{(b):}}$ *Suppose then that $f \geq 0$ is measurable.* L. 2.9 $\Rightarrow$ $\exists$ simple functions f_j s.t. $f_j \nearrow f$. The part (a) of the proof $\Rightarrow$ $\exists$ compact sets $F_j \subset A$ s.t.

$$f_j|F_j \ \text{ continuous and } \ m(A \setminus F_j) < \frac{\varepsilon}{2^j} \,.$$

Let $F_0 = \bigcap_j F_j$, then

$$m(A \setminus F_0) = m\Big(\bigcup_j (A \setminus F_j)\Big) \leq \sum_j m(A \setminus F_j) < \sum_j \frac{\varepsilon}{2^j} = \varepsilon \,.$$

Egorov's Theorem $\Rightarrow$

$$\exists \text{ compact } F \subset F_0 \text{ s.t. } f_j|F \to f|F \text{ uniformly and } m(F_0 \setminus F) < \varepsilon \,.$$

Now

$$\left.\begin{array}{c} f_j|F \text{ compact} \\ f_j|F \to f|F \text{ uniformly} \end{array}\right\} \quad \Rightarrow \ f|F \text{ continuous,}$$

$$\text{moreover } m(A \setminus F) = m(A \setminus F_0) + m(F_0 \setminus F) < 2\varepsilon \,.$$

$\boxed{\text{(c):}}$ *Suppose finally that f is measurable* and write $f = f^+ - f^-$. The part (b) of the proof $\Rightarrow$ $\exists$ compact sets $F_1, F_2 \subset A$ s.t. $f^+|F_1,\ f^-|F_2$ are continuous and $m(A \setminus F_i) < \varepsilon/2,\ i = 1, 2$. The set $F = F_1 \cap F_2$ fulfils the requirements. □

Remark 2.15. 1. The assumption that f is real valued is essential in Egorov's and Lusin's theorems, that is the values $\pm\infty$ are not allowed.

2. The assumption $\mu(X) < \infty$ is essential in Egorov's theorem: Example: $X = \mathbb{R},\ f_j = \chi_{[j,\infty[}$. Then $f_j(x) \to 0\ \forall x \in \mathbb{R}$. Denote $f = 0$. If $F \subset \mathbb{R}$ s.t. $f_j|F \to f|F$ uniformly, then $\exists j_0$ s.t.

$$|f_j(x) - f(x)| = |f_j(x)| < \frac{1}{2} \,, \text{ if } j \geq j_0, x \in F$$
$$\Rightarrow \ F \cap [j,\infty[= \emptyset \ \forall j \geq j_0$$
$$\Rightarrow \ [j,\infty[\subset \mathbb{R} \setminus F \ \Rightarrow \ m(\mathbb{R} \setminus F) = \infty \,.$$

3. Lusin's theorem holds also in the case $m(A) = \infty$ if we just require that the set F is closed. (Exerc.)

2.16 Convolution in $\mathbb{R}^n$

In this section we consider the measure space $(\mathbb{R}^n, \mathrm{Leb}\,\mathbb{R}^n, m)$.

Let $f, g \in L^1(\mathbb{R}^n)$. The mapping $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(y) = x - y$, where $x \in \mathbb{R}^n$ is a constant, satisfies:

$$\varphi(A) \text{ measurable} \iff A \text{ measurable.}$$

Hence

$$\begin{aligned} & y \mapsto f(x-y) \text{ is measurable} \\ \Rightarrow\quad & y \mapsto f(x-y)g(y) \text{ is measurable.} \end{aligned}$$

Therefore the integral

$$h(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)$$

is defined if $f, g \geq 0$.

Questions: When $h(x) < \infty$? Can h be defined without assumptions $f, g \geq 0$?

Theorem 2.17. *Suppose that $f, g \in L^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |f(x-y)||g(y)|dm(y) < \infty \text{ a.e. } x \in \mathbb{R}^n\,. \tag{2.18}$$

For these x we denote

$$h(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)\,. \tag{2.19}$$

Then $h \in L^1(\mathbb{R}^n)$ and

$$\|h\|_1 \leq \|f\|_1 \|g\|_1\,. \tag{2.20}$$

The function h is called the convolution *of f and g and it is denoted by $h = f * g$.*

Proof. We are going to apply Fubini's theorem to the function $F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(x,y) \mapsto f(x-y)g(y)$, and therefore we must prove that it is measurable. We start with:

Claim: $\exists$ Borel functions $f_0, g_0\colon \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\begin{aligned} f_0 &= f \text{ a.e.} \\ g_0 &= g \text{ a.e.} \end{aligned}$$

(i.e. $f_0^{-1}U$, $g_0^{-1}U \in \mathrm{Bor}\,\mathbb{R}^n$ $\forall$ open $U \subset \mathbb{R}$.)

Proof: f^+ measurable $\Rightarrow$ $\exists$ a sequence of simple functions $0 \leq f_1 \leq f_2 \leq \cdots$ s.t.

$$\begin{aligned} & f_j \nearrow f^+\,, \\ & f_j = \sum_{i=1}^{k} a_i \chi_{A_i}\,. \end{aligned}$$

Choose[1] Borel sets $B_i \subset A_i$ s.t. $m(A_i \setminus B_i) = 0$. Then

$$\varphi_j = \sum_{i=1}^{k} a_i \chi_{B_i} \text{ is a Borel function, } 0 \le \varphi_j \le f_j \text{ and } \varphi_j = f_j \text{ a.e.;}$$
$$\varphi^+ = \liminf_{j\to\infty} \varphi_j \text{ Borel function and } \varphi^+ = f^+ \text{ a.e.}$$

(Note that (φ_j) need not be increasing $\Rightarrow \lim_{j\to\infty}\varphi_j$ need not exist.)

Similarly $\exists$ a Borel function $\varphi^- = f^-$ a.e.
Now $f_0 = \varphi^+ - \varphi^-$ is a Borel function and $f_0 = f$ a.e.
Similarly for g.

The values of integrals (2.18) and (2.19) do not change if f and g are replaced by f_0 and g_0. $\Rightarrow$
We mays assume: f, g are Borel functions.
Claim: $F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $F(x,y) = f(x-y)g(y)$, is a Borel function.
Proof: The mappings $u\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $u(x,y) = x-y$, and $v\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $v(x,y) = y$, are continuous. Now $F(x,y) = f\big(u(x,y)\big)g\big(v(x,y)\big)$, eli

$$F = (f \circ u)(g \circ v).$$

Let $V \subset \mathbb{R}$ be open. Since f is a Borel function, we have $f^{-1}V \in \operatorname{Bor}\mathbb{R}^n$. Furthermore,

$$(f\circ u)^{-1}V = u^{-1}(\underbrace{f^{-1}V}_{\in \operatorname{Bor}\mathbb{R}^n}) \in \operatorname{Bor}\mathbb{R}^{2n}$$

since $u\colon \mathbb{R}^{2n} \to \mathbb{R}^n$ is continuous (ks. L. 1.7). Hence $f\circ u$ is Borel. Similarly, we see that $g \circ v$ is Borel, and therefore is a Borel function as a product of two Borel functions. By Exercise 1/4, the "original" F agrees with the "new" F a.e. in $\mathbb{R}^{2n}$, and therefore the original F is measurable.

We may thus apply Fubini's theorems. Fubini 1. $\Rightarrow$

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,y)|dy\right)dx &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,y)|dx\right)dy\\ &= \int_{\mathbb{R}^n}\left(|g(y)|\int_{\mathbb{R}^n}|f(x-y)|dx\right)dy\\ &= \|f\|_1\|g\|_1 < \infty\,,\end{aligned}$$

because

$$\int_{\mathbb{R}^n}|f(x-y)|dx = \|f\|_1. \tag{2.21}$$

Hence (2.18) holds.

Fubini 2. $\Rightarrow$ $h \in L^1(\mathbb{R}^n)$ and

$$\begin{aligned}\|h\|_1 = \int_{\mathbb{R}^n}|h(x)|dx &= \int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} f(x-y)g(y)dy\Big|dx\\ &\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,y)|dy\right)dx = \|f\|_1\|g\|_1\,.\end{aligned}$$

$\square$

[1]Lemma 2.12 $\Rightarrow$ $\exists$ $\mathcal{F}_\sigma$-set $B_i \subset A_i$ s.t. $m(A_i \setminus B_i) = 0$.

Remark 2.22. The equation (2.21) holds by the translatoin invariance of the Lebesgue measure: If

$$f = \sum_{j=1}^{k} a_j \chi_{A_j}$$

is simple and $\varphi(x) = x - y$ is a translation, then $\varphi^{-1}(x) = x + y$,

$$f \circ \varphi = \sum_{j=1}^{k} a_j \chi_{\varphi^{-1} A_j} \quad \text{and} \quad m(\varphi^{-1} A_j) = m(A_j)$$
$$\Rightarrow \int f \circ \varphi = \sum_{j=1}^{k} a_j m(\varphi^{-1} A_j) = \sum_{j=1}^{k} a_j m(A_j) = \int f\,.$$

In other words, (2.21) holds for simple functions. After this, the general case follows from the definition of integral.

Question: Why we used Borel sets/functions and not just measurable sets/functions?
Reason: $g\colon \mathbb{R}^n \to \mathbb{R}^m$ measurable and $E \in \operatorname{Leb} \mathbb{R}^m \not\Rightarrow\ g^{-1}E \in \operatorname{Leb} \mathbb{R}^n$.

2.23 Approximation by C^∞-functions

First some notation:
Let $A \subset \mathbb{R}^n$, $f\colon A \to \mathbb{R}$. We denote

$$\begin{aligned} \operatorname{spt} f &= A \cap \overline{\{x \in A \colon f(x) \neq 0\}} \quad (\textit{support of } f), \\ f \in C(A) &= C^0(A) \iff f \text{ continuous.} \end{aligned}$$

Let $U \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$, $f : U \to \mathbb{R}$.

$$\begin{aligned} f \in C^k(U) &\iff f \text{ has continuous partial derivatives of order } k \\ (&\iff f \text{ is } k \text{ times continuously differentiable}), \\ f \in C^\infty(U) &\iff f \in C^k(U)\ \forall k, \\ f \in C_0^k(U) &\iff f \in C^k(U) \text{ and } \operatorname{spt} f \subset U \text{ compact}, \\ f \in C_0^\infty(U) &\iff f \in C^\infty(U) \text{ and } \operatorname{spt} f \subset U \text{ compact}. \end{aligned}$$

Denote also $f \in C^k$ etc.

Theorem 2.24. *If $f \in L^1(\mathbb{R}^n)$ and $g \in C_0(\mathbb{R}^n)$, then $f * g \in C(\mathbb{R}^n)$.*

Proof. Let's make a simple change of variables $x - y \mapsto y$. The value of the integral does not change (cf. Remark 2.22), and therefore

$$f * g(x) = \int_{\mathbb{R}^n} f(y) g(x - y) dy$$

that is defined for all x, because

$$\int_{\mathbb{R}^n} |f(y)g(x-y)|dy \le M \int_{\mathbb{R}^n} |f(y)|dy = M\|f\|_1 < \infty \ \forall x\,.$$

Here $M = \max|g|$ (the maximum exists since g is continuous and compactly supported).

$$\left.\begin{array}{r} f*g(x+h) - f*g(x) = \displaystyle\int_{\mathbb{R}^n} f(y)\big(g(x-y+h) - g(x-y)\big)dy \\ g \text{ uniformly continuous in } \mathbb{R}^n \end{array}\right\} \Rightarrow \ \forall \varepsilon > 0 \quad \exists\, \delta > 0 \ \text{ s.e.}$$

$$|f*g(x+h) - f*g(x)| \le \int_{\mathbb{R}^n} |f(y)| \underbrace{|g(x-y+h) - g(x-y)|}_{<\varepsilon} dy$$

$$< \varepsilon \underbrace{\|f\|_1}_{<\infty}, \text{ for } |h| < \delta \text{ and } x \in \mathbb{R}^n$$

$$\Rightarrow \ f*g \text{ continuous at } x.$$

□

Remark 2.25. The proof above implies that $f*g$ is uniformly continouos in $\mathbb{R}^n$.

Theorem 2.26. *If $f \in L^1(\mathbb{R}^n)$ and $g \in C_0^k(\mathbb{R}^n)$, then $f*g \in C^k(\mathbb{R}^n)$.*

Proof. For $(z,t) \in \mathbb{R}^n \times (\mathbb{R} \setminus \{0\})$, we set

$$\varphi(z,t) = \frac{g(z+te_i) - g(z)}{t} - D_i g(z)\,, \quad \text{where } e_1,\ldots,e_n \text{ is the standard basis of } \mathbb{R}^n.$$

The mean value theorem $\Rightarrow$

$$\varphi(z,t) = D_i g(z + \vartheta t e_i) - D_i g(z)\,, \quad \text{for some } 0 < \vartheta < 1\,. \tag{2.27}$$

$g \in C_0^k(\mathbb{R}^n) \ \Rightarrow \ D_i g$ uniformly continuous in $\mathbb{R}^n \overset{(2.27)}{\Longrightarrow} \varphi(z,t) \to 0$ uniformly in $\mathbb{R}^n$ as $t \to 0$, hence

$$\sigma(t) = \sup_{z\in\mathbb{R}^n} |\varphi(z,t)| \xrightarrow{t\to 0} 0\,.$$

Let $x \in \mathbb{R}^n$. Then

$$\begin{aligned} &\lim_{t\to 0}\left|\frac{f*g(x+te_i) - f*g(x)}{t} - f*D_i g(x)\right| \\ &= \lim_{t\to 0}\left|\int_{\mathbb{R}^n} f(y)\left(\frac{g(x+te_i-y) - g(x-y)}{t} - D_i g(x-y)\right) dy\right| \\ &\le \lim_{t\to 0}\int_{\mathbb{R}^n} |f(y)| \underbrace{|\varphi(x-y,t)|}_{\le\sigma(t)} dy \\ &\le \lim_{t\to 0}\sigma(t)\|f\|_1 = 0\,, \end{aligned}$$

and therefore

$$D_i\big(f*g\big)(x) = f*D_i g(x)\,.$$

$D_i g \in C_0(\mathbb{R}^n) \overset{2.24}{\Longrightarrow} D_i(f * g) \in C(\mathbb{R}^n)$. Repeating the above we get

$$D\big(f * g\big) = f * Dg\,,$$

where D is any partial derivative of order $p \le k$. □

Remark 2.28. Theorems 2.24 and 2.26 hold also for functions $f \in L^p(\mathbb{R}^n),\ 1 \le p \le \infty$. Reason: in the proofs we need not integrate over the whole $\mathbb{R}^n$, it suffices to integrate over a sufficiently large compact set K since g is compactly supperted. Indeed, by Hölder's inequality

$$\int_K |f| \le m(K)^{\frac{p-1}{p}} \Big(\int_K |f|^p\Big)^{1/p} \le m(K)^{\frac{p-1}{p}} \|f\|_p$$

if $p > 1$ (we interprete above $\frac{p-1}{p} = 1$ if $p = \infty$).

Our next goal is to use convolution in approximating L^p functions ($1 \le p < \infty$) by C_0^∞ functions.
For this purpose:

Theorem 2.29. *If $f \in L^p(\mathbb{R}^n),\ 1 \le p < \infty$, then*

$$\lim_{h \to 0} \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx = 0\,.$$

Proof. Let $\varepsilon > 0$. We need to show: $\exists\ \eta > 0$ s.t.

$$|h| < \eta \ \Rightarrow \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx < \varepsilon\,.$$

Denote

$$I_h(A) = \int_A |f(x+h) - f(x)|^p dx\,, \quad A \in \operatorname{Leb} \mathbb{R}^n,$$
$$B_k = B(0,k) = \{x \in \mathbb{R}^n \colon |x| < k\}\,, \quad k > 1\,.$$

Let $h \in B(0,1)$. (Then $|x| \ge k \ \Rightarrow\ |x+h| \ge k-1$.)
Now

$$\int_{\mathbb{R}^n \setminus B_k} |f(x+h) - f(x)|^p dx \le \int_{\mathbb{R}^n \setminus B_k} 2^p\big(|f(x+h)|^p + |f(x)|^p\big) dx$$
$$\le 2^p \left(\int_{\mathbb{R}^n \setminus B_{k-1}} |f|^p + \int_{\mathbb{R}^n \setminus B_k} |f|^p \right) \xrightarrow{\text{DCT}} 0, \quad \text{kun } k \to \infty\,.$$

$\Rightarrow\ \exists\ k$ s.t.

$$I_h(\mathbb{R}^n \setminus B_k) < \varepsilon/4\,. \tag{2.30}$$

Similarly,

$$I_h(A) \le 2^p \left(\int_{A+h} |f|^p + \int_A |f|^p \right)\,, \quad \text{if } A \in \operatorname{Leb} \mathbb{R}^n \text{ and } A + h = \{a + h \colon a \in A\}.$$

The integral is absolutely continuous with respect to the Lebesgue measure, and therefore Theorem 2.5 implies $\exists\, \delta > 0$ s.t.

$$I_h(A) < \varepsilon/4\,, \quad \text{if } m(A) < \delta\,. \tag{2.31}$$

Lusin's theorem $\Rightarrow$ $\exists$ a compact $F \subset B_{k+1}$ s.t.

$$m(B_{k+1} \setminus F) < \delta \quad \text{and} \quad f|F \text{ continuous.}$$

F compact $\Rightarrow$

$$f|F \text{ uniformly continuous.}$$

$\Rightarrow$ $\exists\, \eta \in]0,1[$ s.t.

$$|f(x+h) - f(x)|^p < \frac{\varepsilon}{4m(F)}, \quad \text{if } |h| < \eta \text{ and } x, x+h \in F\,. \tag{2.32}$$

Let $h \in B(0,\eta)$ be arbitrary. Denote

$$\begin{aligned} A_1 &= \{x \in F \colon x + h \in F\}, \\ A_2 &= \{x \colon x + h \in B_{k+1} \setminus F\}, \\ A_3 &= B_{k+1} \setminus F\,. \end{aligned}$$

(We observe: $A_2 = A_3 - h$, and so $m(A_2) = m(A_3) = m(B_{k+1} \setminus F) < \delta$.)
Then

$$x \in B_k \;\Rightarrow\; x + h \in B_{k+1}\,,$$

and therefore

$$B_k \cap F \subset \{x \in F \colon x + h \in B_{k+1}\} \subset \underbrace{\{x \in F \colon x + h \in F\}}_{=A_1} \cup \underbrace{\{x \in F \colon x + h \in B_{k+1} \setminus F\}}_{\subset A_2}$$

$$\subset A_1 \cup A_2$$

$$\Rightarrow\; B_k = \underbrace{(B_k \setminus F)}_{\subset A_3} \cup \underbrace{(B_k \cap F)}_{\subset A_1 \cup A_2} \subset A_1 \cup A_2 \cup A_3$$

Now $\mathbb{R}^n = A_1 \cup A_2 \cup A_3 \cup (\mathbb{R}^n \setminus B_k)$ and

$$I_h(\mathbb{R}^n) \le I_h(A_1) + I_h(A_2) + I_h(A_3) + I_h(\mathbb{R}^n \setminus B_k)\,.$$

Let us estimate the terms on the right hand side:

$$\left.\begin{aligned} &(2.32) \;\Rightarrow\; I_h(A_1) = \int_{A_1} \underbrace{|f(x+h) - f(x)|^p}_{< \frac{\varepsilon}{4m(F)}}\, dx \underset{A_1 \subset F}{<} \varepsilon/4 \\ &(2.31) \;\Rightarrow\; I_h(A_2) < \varepsilon/4 \\ &(2.31) \;\Rightarrow\; I_h(A_3) < \varepsilon/4 \\ &(2.30) : \quad I_h(\mathbb{R}^n \setminus B_k) < \varepsilon/4 \end{aligned}\right\} \Rightarrow$$

$$I_h(\mathbb{R}^n) = \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx < \varepsilon\,.$$

$\square$

Define $\eta\colon \mathbb{R} \to [0,\infty[$,

$$\eta(t) = \begin{cases} e^{\frac{1}{t^2-1}}, & \text{if } |t| < 1, \\ 0, & \text{if } |t| \geq 1. \end{cases}$$

Let $|t| < 1$. Then

$$\begin{aligned} &\eta^{(k)}(t) = \frac{e^{\frac{1}{t^2-1}} \cdot P_{3k}(t)}{(t^2-1)^{2k}}; \quad P_{3k} = \text{ polynomial of order } 3k, \\ &\Rightarrow \eta^{(k)}(t) \to 0, \quad \text{as } t \to 1 \text{ or } t \to -1\,. \\ &\Rightarrow \eta \in C_0^\infty(\mathbb{R})\,. \end{aligned}$$

We want a function $\varphi_k\colon \mathbb{R}^n \to [0,\infty[$, $k \in \mathbb{N}$, s.t.

(a) $\varphi_k \in C_0^\infty(\mathbb{R}^n)$,

(b) $\operatorname{spt} \varphi_k \subset \bar{B}_{1/k} = \bar{B}(0,1/k)$,

(c) $\displaystyle\int_{\mathbb{R}^n} \varphi_k = 1$.

We may choose

$$\varphi_k(x) = a_k \eta(k|x|)\,, \tag{2.33}$$

where the constant a_k is chosen such that (c) holds.

We notice: If $f \in L^p(\mathbb{R}^n)$ and $g \in C_0(\mathbb{R}^n)$ (i.e. g continuous and $\operatorname{spt} g$ compact), then

$$\begin{gathered} y \mapsto f(x-y)g(y) \quad \text{is integrable } \forall x \text{ since} \\ \int_{\mathbb{R}^n} |f(x-y)| \underbrace{|g(y)|}_{\leq M < \infty} dy \leq M \int_{\operatorname{spt} g} |f(x-y)| dy = M \int_A |f(y)| dy < \infty\,, \end{gathered}$$

where $A = x - \operatorname{spt} g = \{x - z \colon z \in \operatorname{spt} g\}$, and it holds:

$$f \in L^p(A),\ m(A) < \infty \ \Rightarrow \ f \in L^1(A)\,.$$

Hence the convolution $f * g(x)$ is defined $\forall x \in \mathbb{R}^n$.

We apply this to the following:

$$\begin{aligned} &g_k \colon \mathbb{R}^n \to [0,\infty[\ \text{ continuous}, \\ &\operatorname{spt} g_k \subset \bar{B}_{1/k}, \\ &\int_{\mathbb{R}^n} g_k = 1\,. \end{aligned}$$

Theorem 2.34. *Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and g_k as above. Then*

$$\lim_{k\to\infty} \|f - f * g_k\|_p = 0\,.$$

Proof. For the case $p = 1$ we have

$$\begin{aligned} f(x) - f * g_k(x) &= f(x) \int_{\mathbb{R}^n} g_k(y)dy - \int_{\mathbb{R}^n} f(x-y)g_k(y)dy \\ &= \int_{\mathbb{R}^n} \left(f(x) - f(x-y)\right) g_k(y)dy \\ \Rightarrow\ |f(x) - f * g_k(x)| &\le \int_{\mathbb{R}^n} |f(x) - f(x-y)| g_k(y)dy\,. \end{aligned}$$

If $p > 1$,

$$\begin{aligned} |f(x) - f * g_k(x)|^p &\le \Big(\int_{\mathbb{R}^n} |f(x) - f(x-y)| g_k(y)dy\Big)^p \\ &= \Big(\int_{\mathbb{R}^n} |f(x) - f(x-y)| \underbrace{g_k(y)^{1/p} g_k(y)^{1/q}}_{=g_k(y)} dy\Big)^p \quad (\text{where } q = \frac{p}{p-1}) \\ &\underset{\text{Hölder}}{\le} \int_{\mathbb{R}^n} |f(x) - f(x-y)|^p g_k(y)dy \underbrace{\Big(\int_{\mathbb{R}^n} \Big(g_k(y)^{1/q}\Big)^q dy\Big)^{p/q}}_{=1} \\ &= \int_{\mathbb{R}^n} |f(x) - f(x-y)|^p g_k(y)dy\,. \end{aligned}$$

Hence $\forall p \ge 1$:

$$\begin{aligned} \|f - f * g_k\|_p^p &= \int_{\mathbb{R}^n} |f(x) - f * g_k(x)|^p dx \\ &\le \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x) - f(x-y)|^p g_k(y)dy\Big) dx \\ &\overset{\text{Fubini 1.}}{=} \int_{\mathbb{R}^n} g_k(y) \Big(\int_{\mathbb{R}^n} |f(x) - f(x-y)|^p dx\Big) dy\,. \end{aligned}$$

Since $\operatorname{spt} g_k \subset \bar{B}_{1/k}$, we may assume that $|y| \le 1/k$ in the inner integration.

$$\left.\begin{aligned} \text{Thm. 2.29} \ \Rightarrow \ \int_{\mathbb{R}^n} |f(x) - f(x-y)|^p dx \to 0, \\ \text{as } k \to \infty \text{ and } |y| \le 1/k \\ \int_{\mathbb{R}^n} g_k = 1 \end{aligned}\right\} \Rightarrow \text{claim.} \quad \square$$

$\square$

Clearly the spaces $C_0(\mathbb{R}^n)$ and $C_0^k(\mathbb{R}^n)$, $k = 1, 2, \dots, \infty$, vector subspaces of $L^p(\mathbb{R}^n)$ and, when equipped with the norm $\|\cdot\|_p$, they are also normed subspaces of $L^p(\mathbb{R}^n)$.

Definition 2.35. If W is a subspace of a normed space $(V, \|\cdot\|)$, we say that W is *dense* in V if $\forall v \in V$ $\exists$ a sequence $w_1, w_2, \ldots \in W$ s.t. $\|w_i - v\| \to 0$ as $i \to \infty$. (That is, $\bar{W} = V$.)

Theorem 2.36. *$C_0^\infty(\mathbb{R}^n)$ is a dense subspace of $L^p(\mathbb{R}^n)$ if $1 \le p < \infty$.*

Proof. Let $f \in L^p(\mathbb{R}^n)$. We need to show: $\exists\ \psi_1, \psi_2, \ldots \in C_0^\infty(\mathbb{R}^n)$ s.t.

$$\|f - \psi_k\|_p \to 0 \text{ as } k \to \infty.$$

(a): Suppose that spt f is compact. Then $f \in L^1$. Choose functions φ_k as inn (2.33). Thm. 2.26 $\Rightarrow$

$$f * \varphi_k \in C^\infty(\mathbb{R}^n).$$

If $d(x, \operatorname{spt} f) > 1/k$ and $y \in \operatorname{spt} \varphi_k$ $(\subset \bar{B}(0, 1/k))$, then $x - y \notin \operatorname{spt} f$. and therefore

$$f * \varphi_k(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_k(y)dy = 0 \ \Rightarrow \ \operatorname{spt}(f * \varphi_k) \ \text{ kompakti.}$$

$$\text{Thm. 2.34} \quad \Rightarrow \ \|f - f * \varphi_k\|_p \xrightarrow{k\to\infty} 0.$$

Hence we may choose $\psi_k = f * \varphi_k$. (Note: Above $\operatorname{spt}(f * \varphi_k)$ is compact since it is both closed and bounded.)

(b): General case: spt f not necessarily compact. Let $\varepsilon > 0$. Denote

$$f_j = f\chi_{B_j}, \quad B_j = B(0, j).$$

Then $f_j \in L^p(\mathbb{R}^n)$, spt f_j is compact. Furthermore, there exists j_0 s.t.

$$\|f - f_j\|_p < \varepsilon/2 \quad \forall j \geq j_0.$$

Part (a) $\Rightarrow$ $\exists\ \psi_1^j, \psi_2^j, \ldots \in C_0^\infty(\mathbb{R}^n)$ s.t.

$$\|f_j - \psi_k^j\|_p < \varepsilon/2 \quad \text{as } k \geq k_j.$$

Then

$$\|f - \psi_{k_j}^j\|_p < \varepsilon \quad \forall j \geq j_0.$$

□

3 Derivative

In this section we study differentiability of integrals, in particular, the question when a function $f\colon [a, b] \to \mathbb{R}$ can be recovered by integrating its derivative f'?

Example 3.1. 1. (Let $g\colon [a, b] \to \mathbb{R}$ be continuous and

$$G(x) = \int_a^x g(t)dt, \quad x \in [a, b].$$

Then G is differentiable and $G'(x) = g(x)$, $x \in [a, b]$.

2. There is a more general result (*Lebesgue's theorem*): Let $g\colon [a, b] \to \mathbb{R}$ be integrable. Then the function $G\colon [a, b] \to \mathbb{R}$,

$$G(x) = \int_a^x g(t)dt,$$

is differentiable a.e. and

$$G'(x) = g(x) \quad \text{a.e. } x \in [a, b].$$

3. Converse to the case 1 (starting from the function):

$$f \in C^1([a,b]) \;\Rightarrow\; \int_a^x f'(t)dt = f(x) - f(a).$$

4. Let $f\colon [0,1] \to [0,1]$ be the Cantor 1/3 function. Then f is a continuous increasing surjection and $f'(t) = 0$ for a.e. $t \in [0,1]$ (f' measurable), but

$$\int_0^1 f'(t)dt = \int_0^1 0dt = 0 \neq 1 = f(1) - f(0).$$

We will study these questions by using "covering theorems" as a tool. Here we try to "almost" cover a given subset of $\mathbb{R}^n$ by closed, pairwise disjoint balls. The disjointness of the balls is required in order to apply the countable additivity of the Lebesgue measure.

3.2 Covering theorems

We denote $kB = B(x, kr)$ if $B = B(x,r)$ and $k > 0$ (or, respectively, $kB = \bar{B}(x, kr)$ if $B = \bar{B}(x,r)$).
Note: We assume that the radius r of a closed ball $\bar{B}(x,r)$ is *positive* ($r > 0$).
($\{y \in \mathbb{R}^n\colon |x-y| < 0\} = \emptyset$, $\{y \in \mathbb{R}^n\colon |x-y| \leq 0\} = \{x\}$)

Theorem 3.3 (Basic covering theorem)**.** *Let $\mathcal{F}$ be an arbitrary family of balls in $\mathbb{R}^n$ s.t.*

$$D = \sup\{d(B)\colon B \in \mathcal{F}\} < \infty,$$

where $d(B) = \sup\{|x-y|\colon x,y \in B\}$ is the diameter of B. Then there exists a countable (possibly finite) family $\mathcal{G} \subset \mathcal{F}$ s.t.

$$B_i \cap B_j = \emptyset \quad \forall B_i, B_j \in \mathcal{G},\; B_i \neq B_j, \quad \text{i.e. the balls in } \mathcal{G} \text{ are pairwise disjoint; and}$$

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B\,.$$

Proof. [1.] Denote

$$\mathcal{F}_j = \{B \in \mathcal{F}\colon D/2^j < d(B) \leq D/2^{j-1}\},\; j \in \mathbb{N},$$

hence $\mathcal{F} = \bigcup_{j=1}^{\infty} \mathcal{F}j$.
Define inductively families $\mathcal{G}_j \subset \mathcal{F}_j$:

(a) Let $\mathcal{G}_1$ be a *maximal* family of pairwise disjoint balls in $\mathcal{F}_1$, i.e.

$$B \in \mathcal{F}_1 \;\Rightarrow\; \exists B' \in \mathcal{G}_1 \text{ s.t. } B \cap B' \neq \emptyset.$$

(Thus: We can not add to $\mathcal{G}_1$ any ball of $\mathcal{F}_1$ without destroying the pairwise disjointness.)

(b) Suppose that families $\mathcal{G}_1, \ldots, \mathcal{G}_{k-1}$ are chosen. Let $\mathcal{G}_k$ be any *maximal* collection of pairwise disjoint balls of $\mathcal{F}_k$ s.t.

$$B \cap B' = \emptyset \quad \forall B' \in \bigcup_{j=1}^{k-1} \mathcal{G}_j.$$

Denote

$$\mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{G}_j,$$

then $\mathcal{G}$ ($\subset \mathcal{F}$) is a family of pairwise disjoint balls.

2. Claim: $\mathcal{G}$ is countable.

Proof: It is enough to show: $\mathcal{G}_j$ countable $\forall j$. Write

$$\mathcal{G}_j = \bigcup_{i=1}^{\infty} \mathcal{G}_{j,i}, \quad \text{where } \mathcal{G}_{j,i} = \{B \in \mathcal{G}_j \colon B \subset \bar{B}(0,i)\},$$

and let us prove that $\mathcal{G}_{j,i}$ is finite (possibly empty), hence $\mathcal{G}_j$'s and therefore $\mathcal{G}$ is countable.

$$\left.\begin{array}{c} B \in \mathcal{G}_{j,i} \;\Rightarrow\; d(B) > D/2^j \text{ and } B \subset \bar{B}(0,i) \\ \bar{B}(0,i) \;\; \text{compact} \end{array}\right\} \overset{(*)}{\Rightarrow} \mathcal{G}_{j,i} \;\; \text{finite.}$$

[Reason for ($*$): assume on the contrary: there are infinitely many disjoint balls in $\mathcal{G}_{j,i}$ $\Rightarrow$ $\exists$ a sequence $x_k \in \bar{B}(0,i)$, $k \in \mathbb{N}$, s.t.

$$|x_k - x_l| \geq D/2^j \quad \forall k \neq l. \tag{3.4}$$

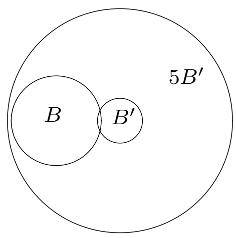
(for example, every x_k is a center of some ball in $\mathcal{G}_{j,i}$.) $\bar{B}(0,i)$ compact $\Rightarrow$ $\exists$ a convergent subsequence of (x_k) that is a contradiction with (3.4).]

3. Claim: For all $B \in \mathcal{F}$ there exists $B' \in \mathcal{G}$ s.t. $B \cap B' \neq \emptyset$ and $B \subset 5B'$. In particular, then

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B\,.$$

Proof: If $B \in \mathcal{F}$, then $B \in \mathcal{F}_k$ for some $k \in \mathbb{N}$. Since $\mathcal{G}_k$ is maximal, there exists $B' \in \bigcup_{j=1}^{k} \mathcal{G}_j$ s.t. $B \cap B' \neq \emptyset$. On the other hand,

$$\left.\begin{array}{c} d(B') > D/2^k \text{ and } d(B) \leq D/2^{k-1} \;\Rightarrow\; d(B) < 2d(B') \\ B \cap B' \neq \emptyset \end{array}\right\} \;\Rightarrow\; B \subset 5B'\,.$$



□

Remark 3.5.

1. The part 2 in the proof follows also from the disjointness of the balls in $\mathcal{G}$ and the separability of $\mathbb{R}^n$ ($\mathbb{Q}^n$ is a countable dense subset of $\mathbb{R}^n$).

2. The basic covering theorem (stated as above) holds for some metric spaces (X,d) provided certain additional assumptions hold. Read the proof again and think of what kind of properties (X,d) should have in order to the proof work.

Definition 3.6. Let $\mathcal{V}$ be a family of balls in $\mathbb{R}^n$. We say that $\mathcal{V}$ is a *Vitali covering* of a set $E \subset \mathbb{R}^n$ if

$$\forall x \in E \text{ and } \forall \varepsilon > 0 \ \exists B \in \mathcal{V} \text{ s.t. } x \in B \text{ and } d(B) < \varepsilon \,.$$

Such a family $\mathcal{V}$ is a closed Vitali covering (respectively, open) if every ball $B \in \mathcal{V}$ is closed (resp. open).

Remark 3.7. If $\mathcal{V}$ is a Vitali covering of E and $R > 0$, then

$$\{B \in \mathcal{V} \colon d(B) < R\}$$

is also a Vitali covering of E.

From the proof of Theorem 3.3 we obtain:

Corollary 3.8. Let $\mathcal{V}$ be a closed Vitali covering of $E \subset \mathbb{R}^n$ s.t. $d(B) < R \ \forall B \in \mathcal{V}$. Then there exists a countable family $\mathcal{G} \subset \mathcal{V}$ of pairwise disjoint balls such that for every finite $\mathcal{G}^* \subset \mathcal{G}$ we have:

$$E \setminus \bigcup_{B \in \mathcal{G}^*} B \subset \bigcup_{B \in \mathcal{G} \setminus \mathcal{G}^*} 5B \,.$$

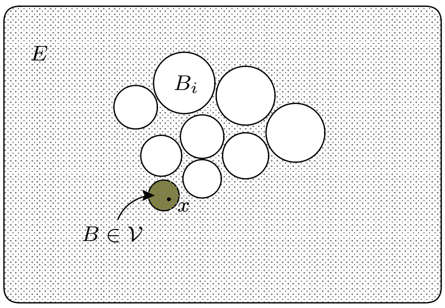
Proof. Let $\mathcal{G} \subset \mathcal{V}$ be as in the proof of Theorem 3.3 and let $\mathcal{G}^* = \{B_1, B_2, \dots, B_m\} \subset \mathcal{G}$ be arbitrary. If

$$E \subset \bigcup_{i=1}^m B_i \,, \quad \text{we are done.}$$

Otherwise, let $x \in E \setminus \bigcup_{i=1}^m B_i$. Then $\bigcup_{i=1}^m B_i$ is compact, and therefore

$$d\big(x, \bigcup_{i=1}^m B_i\big) = \inf\{|x-y| \colon y \in \bigcup_{i=1}^m B_i\} = \min\{|x-y| \colon y \in \bigcup_{i=1}^m B_i\} > 0 \,.$$

Since $\mathcal{V}$ is a Vitali covering of E, $\exists \ B \in \mathcal{V}$ s.t. $x \in B$ and $B \cap \big(\bigcup_{i=1}^m B_i\big) = \emptyset$ ($d(B)$ small enough).



It follows from the part 3 of the proof of Theorem 3.3 that $\exists \ B' \in \mathcal{G}$ s.t. $B \cap B' \neq \emptyset$ and $B \subset 5B'$. In particular $x \in 5B'$. Now $B' \notin \mathcal{G}^*$, because $B \cap \big(\bigcup_{i=1}^m B_i\big) = \emptyset$. Thus $B' \in \mathcal{G} \setminus \mathcal{G}^*$, and so

$$E \setminus \bigcup_{B \in \mathcal{G}^*} B \subset \bigcup_{B \in \mathcal{G} \setminus \mathcal{G}^*} 5B \,.$$

□

Theorem 3.9 (Vitali's covering theorem). *Let $E \subset \mathbb{R}^n$ (not necessarily measurable) and let $\mathcal{V}$ be a closed Vitali covering of E. Then there exists a* countable *subfamily $\mathcal{G} \subset \mathcal{V}$ of* pairwise disjoint *balls s.t.*

$$m\big(E \setminus \bigcup_{B\in\mathcal{G}} B\big) = 0.$$

Proof. $\boxed{1.}$ Suppose fist that E is bounded. Then we may assume that there exists a bounded open $H \subset \mathbb{R}^n$ s.t. $B \subset H$ $\forall B \in \mathcal{V}$. Let $\mathcal{G}$ be as in the proof of Theorem 3.3 (and of Corollary 3.8). Let $\varepsilon > 0$. We will show:

$$m^*\big(E \setminus \bigcup_{B\in\mathcal{G}} B\big) < \varepsilon$$

that implies the claim (since $\varepsilon > 0$ arbitrary). The balls in $\mathcal{G}$ are pairwise disjoint and $\subset H$, and hence

$$\sum_{B\in\mathcal{G}} m(B) = m\big(\bigcup_{B\in\mathcal{G}} B\big) \le m(H) < \infty\,.$$

$\Rightarrow$ $\exists$ a finite $\mathcal{G}^* \subset \mathcal{G}$ s.t.

$$\sum_{B\in\mathcal{G}\setminus\mathcal{G}^*} m(B) < \varepsilon/5^n\,.$$

Corollary 3.8 $\Rightarrow$

$$\begin{aligned} m^*\big(\underbrace{E \setminus \bigcup_{B\in\mathcal{G}} B}_{\subset E\setminus\cup_{B\in\mathcal{G}^*}B}\big) &\le m^*\big(\underbrace{E \setminus \bigcup_{B\in\mathcal{G}^*} B}_{\subset \cup_{B\in\mathcal{G}\setminus\mathcal{G}^*}5B}\big) \le m\big(\bigcup_{B\in\mathcal{G}\setminus\mathcal{G}^*} 5B\big) \\ &\le \sum_{B\in\mathcal{G}\setminus\mathcal{G}^*} m(5B) = 5^n \sum_{B\in\mathcal{G}\setminus\mathcal{G}^*} m(B) < \varepsilon\,. \end{aligned}$$

$\varepsilon > 0$ arbitrary $\Rightarrow$

$$m\big(E \setminus \bigcup_{B\in\mathcal{G}} B\big) = 0\,.$$

$\boxed{2.}$ General case: E not necessarily bounded. Denote

$$A_1 = B(0,1) \text{ and } A_i = B(0,i) \setminus \bar{B}(0,i-1),\ i \ge 2\,.$$

Then the sets A_i are open and disjoint, and

$$m\big(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} A_i\big) = m\big(\bigcup_{i=1}^{\infty} S(0,i)\big) = 0\,, \quad S(0,i) = \{x \in \mathbb{R}^n \colon |x| = i\}\,.^2 \tag{3.10}$$

<u>Idea:</u> Applying the part 1 to the sets $E \cap A_i$ we obtain subfamilies $\mathcal{G}_i \subset \mathcal{V}$. We must take care that the balls in $\mathcal{G}_i$ and $\mathcal{G}_j$ intersect each other. This can be dealt with as follows: A_i open, $x \in A_i \ \Rightarrow\ \exists\ r_x > 0$ s.t. $B(x,r) \subset A_i$ $\forall r \le r_x \Rightarrow$ $\mathcal{V}_i = \{B \in \mathcal{V} \colon B \subset A_i\}$ is a Vitali covering of $E \cap A_i$.
A_i's disjoint $\Rightarrow$

$$\text{if } B \in \mathcal{V}_i \text{ and } B' \in \mathcal{V}_j, i \ne j, \text{ then } B \cap B' = \emptyset. \tag{3.11}$$

[2] Find a simple argument that shows that $m_n\big(S(0,i)\big) = 0$.

1. part $\Rightarrow$ $\exists$ a countable family $\mathcal{G}_i \subset \mathcal{V}_i$ of disjoint balls s.t.

$$m\big((E \cap A_i) \setminus \bigcup_{B \in \mathcal{G}_i} B\big) = 0\,. \tag{3.12}$$

Then $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ satisfies the requirements. Clearly $\mathcal{G}$ is countable and the balls in $\mathcal{G}$ are disjoint (see (3.11)). Moreover,

$$E \setminus \bigcup_{B \in \mathcal{G}} B = \underbrace{\Big((E \setminus \bigcup_{B \in \mathcal{G}} B) \setminus \bigcup_{i=1}^{\infty} A_i\Big)}_{\text{of measure 0 (see (3.10))}} \cup \underbrace{\Big((E \setminus \bigcup_{B \in \mathcal{G}} B) \cap (\bigcup_{i=1}^{\infty} A_i)\Big)}_{=\bigcup_{i=1}^{\infty}\big((E\cap A_i)\setminus \bigcup_{B\in\mathcal{G}_i} B\big)}$$

$$\Rightarrow\ m\big(E \setminus \bigcup_{B \in \mathcal{G}} B\big) \le \sum_{i=1}^{\infty} m\big((E \cap A_i) \setminus \bigcup_{B \in \mathcal{G}_i} B\big) \overset{(3.12)}{=} 0\,.$$

□

Remark 3.13. Also the covering theorem holds for certain "metric measure spaces" (X, d, Γ, μ). Read the proof again and think of what kind of requirements (X, d) and the measure μ should fulfil.

3.14 Maximal function

Let us start with the following useful result:

Let (X, Γ, μ) be a measure space and $g\colon X \to [0, +\infty]$ measurable. The function $[0, +\infty) \to [0, +\infty]$,

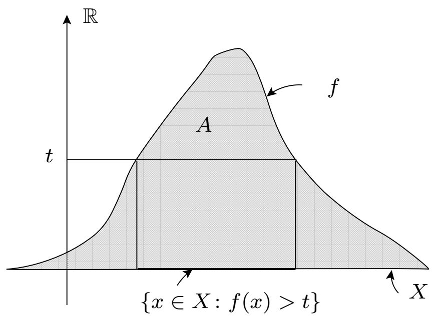
$$t \mapsto \mu\big(\{x\colon g(x) > t\}\big),$$

is called the *distribution function* of g. It is decreasing and hence measurable.

Lemma 3.15. *Let $f\colon X \to [0, +\infty]$ be measurable and $0 < p < \infty$. Then*

$$\int_X f^p d\mu = p \int_0^{\infty} t^{p-1} \mu\big(\{x\colon f(x) > t\}\big) dt\,. \tag{3.16}$$

Proof. Idea: (i) Suppose first that f is simple and prove that (by direct computation) that (3.16) holds. (ii) In the general case, choose a sequence of simple functions $f_k \nearrow f$ and apply the monotone convergence theorem. Details are left as an exercise. □



In the picture above $\int_X f d\mu = \int_0^\infty \mu(\{x \in X : f(x) > t\})dt$ can be interpreted as the "product measure" $(\mu \times m_1)(A)$ of the shaded area A.

Denote $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ if $f\colon \mathbb{R}^n \to \dot{\mathbb{R}}$ is measurable and

$$\int_K |f| < \infty \quad \forall \text{ compact } K \subset \mathbb{R}^n.$$

We say: f is locally integrable (or locally in L^1).

Remark 3.17. 1. $f \in C(\mathbb{R}^n) \;\Rightarrow\; f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

2. $f \in L^1(\mathbb{R}^n) \;\Rightarrow\; f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The converse "$\Leftarrow$" does not hold: for instance $f(x) \equiv 1$.

Our aim is to prove:

$$f \in L^1_{\text{loc}}(\mathbb{R}^n) \;\Rightarrow\; \lim_{r\to 0+} \frac{1}{m\big(B(x,r)\big)} \int_{B(x,r)} f(y)dy = f(x) \text{ a. e. } x \in \mathbb{R}^n.$$

For $A \in \operatorname{Leb} \mathbb{R}^n$ and $m(A) > 0$ we denote

$$\fint_A f(y)dy = \frac{1}{m(A)} \int_A f(y)dy,$$

the "integral mean" of f over A.

Definition 3.18. If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we set

$$Mf(x) = \sup_{B \ni x} \fint_B |f(y)|dy,$$

where B is an (arbitrary) open ball that contains x. Then function $Mf\colon \mathbb{R}^n \to [0,\infty]$ is the (Hardy-Littlewood) *maximal function* of f.

Note: There are different kind of maximal functions in the literature. For instance, the supremum can be taken over balls $B(x,r)$ centered at x. Then we obtain a "centered" maximal function of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, denoted by $\tilde{M}f$,

$$\tilde{M}f(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|dy.$$

It holds: $\tilde{M}f(x) \le Mf(x) \le 2^n \tilde{M}f(x)$ $\forall x \in \mathbb{R}^n$.

Lemma 3.19. *The maximal function* $Mf\colon \mathbb{R}^n \to [0,\infty]$ *is measurable.*

Proof. Write $E_t = \{x : Mf(x) > t\}$. We prove a stronger results that E_t is open[3] $\forall t \in \mathbb{R}$ (and hence, in particular, measurable). Let $x \in E_t$. Then there exists an open ball $B \ni x$ s.t.

$$\fint_B |f(y)|dy > t.$$

[3]A function $u\colon X \to \dot{\mathbb{R}}$ of a topological space X is *lower semicontinuous* if $\{x \in X : u(x) > t\}$ is open $\forall t \in \mathbb{R}$. Upper semicontinuity is defined similarly. Thus u is continuous $\iff$ u is both lower and upper semicontinuous. Proof of Lemma 3.19 $\Rightarrow$ Mf lower semicontinuous.

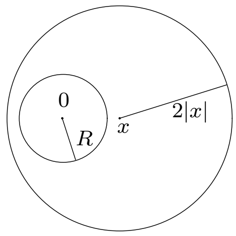
Then $\forall y \in B$ we have:

$$Mf(y) \overset{\sup}{\geq} \fint_B |f(y)|dy > t \;\Rightarrow\; y \in E_t.$$

Hence $B \subset E_t$ and therefore E_t is open. $\square$

Remark 3.20. 1. What can be said about integrability of Mf?
Answer: Mf is "vey seldom" integrable. More precisely: $Mf \in L^1(\mathbb{R}^n) \;\Rightarrow\; f = 0$ a.e.
Reason: Recall that $m\big(B(x,r)\big) = c_n r^n$, where c_n is a constant depending on n. Assume on the contrary that $\int_{\mathbb{R}^n}|f| > 0$, hence $\exists R > 0$ s.t.

$$\underbrace{\int_{B(0,R)} |f(y)|dy}_{\overset{\text{def.}}{=} I} > 0.$$



If $x \in \mathbb{R}^n \setminus B(0,R)$, then $B(0,R) \subset B(x, 2|x|)$ (see the picture) and

$$Mf(x) \geq \frac{1}{m\big(B(x,2|x|)\big)} \int_{B(x,2|x|)} |f(y)|dy \geq \frac{1}{c_n (2|x|)^n} \underbrace{\int_{B(0,R)} |f(y)|dy}_{=I>0}.$$

$$\Rightarrow \int_{\mathbb{R}^n} Mf(x)dx \geq \int_{\mathbb{R}^n\setminus B(0,R)} Mf(x)dx \geq I \int_{\mathbb{R}^n\setminus B(0,R)} c_n^{-1}(2|x|)^{-n}dx = \infty$$

since

$$\int_{\mathbb{R}^n\setminus B(0,R)} c_n^{-1}(2|x|)^{-n}dx = \sum_{i=0}^{\infty} \int_{B(0,2^{i+1}R)\setminus B(0,2^iR)} \underbrace{c_n^{-1}(2|x|)^{-n}}_{\geq c_n^{-1}(2^{i+2}R)^{-n}} \, dx$$

$$\geq \sum_{i=0}^{\infty} \underbrace{m\big(B(0,2^{i+1}R) \setminus B(0,2^iR)\big)}_{=c_n((2^{i+1}R)^n - (2^iR)^n)} c_n^{-1}(2^{i+2}R)^{-n}$$

$$= \sum_{i=0}^{\infty} \underbrace{\frac{2^n - 1}{4^n}}_{=c>0} = \sum_{i=0}^{\infty} c = \infty.$$

2. *Chebyshev's inequality*: $f \in L^1(\mathbb{R}^n) \;\Rightarrow$ an estimate

$$m\big(\{x \in \mathbb{R}^n \colon |f(x)| > t\}\big) \leq \frac{c}{t} \quad \forall t > 0 \tag{3.21}$$

holds with the constant $c = \|f\|_1$. The converse does *not* hold: a measurable function f for which (3.21) holds $\forall t > 0$ with some constant c need not be integrable.

3. We say that a measurable function $f\colon \mathbb{R}^n \to \dot{\mathbb{R}}$ belongs to the *weak L^1-space*, weak-$L^1(\mathbb{R}^n)$, if there is a constant $c = c_f < \infty$ such that (3.21) holds $\forall t > 0$. Thus: $L^1(\mathbb{R}^n) \subset$ weak-$L^1(\mathbb{R}^n)$, but weak-$L^1(\mathbb{R}^n) \not\subset L^1(\mathbb{R}^n)$.

It turns out that the maximal function Mf of an integrable function $f \in L^1(\mathbb{R}^n)$ satisfies (3.21). This is one of the most important properties of Mf. The proof is based on the Basic covering theorem 3.3.

Theorem 3.22 (Hardy-Littlewood)**.** *If $f \in L^1(\mathbb{R}^n)$, then*

$$m\big(\{x \in \mathbb{R}^n \colon Mf(x) > t\}\big) \le \frac{5^n \|f\|_1}{t} \quad \forall t > 0. \tag{3.23}$$

Proof. Fix $t > 0$ and write $M_t = \{x \in \mathbb{R}^n \colon Mf(x) > t\}$. Then $\forall x \in M_t$ $\exists$ an open ball $B_x \ni x$ (not necessarily centered at x) s.t.

$$⨍_{B_x} |f(y)|dy > t.$$

In other words,

$$m(B_x) \le \frac{1}{t}\int_{B_x} |f(y)|dy \quad (\le \frac{\|f\|_1}{t}). \tag{3.24}$$

Let $\mathcal{F} = \{B_x \colon x \in M_t\}$, hence (trivially)

$$M_t \subset \bigcup_{B\in\mathcal{F}} B.$$

Since $m(B_x) = c_n(d(B_x)/2)^n$, then

$$(3.24) \;\Rightarrow\; \sup\{d(B_x)\colon B_x \in \mathcal{F}\} \le 2\left(\frac{\|f\|_1}{c_n t}\right)^{1/n} < \infty.$$

We may apply the Basic covering theorem 3.3 $\Rightarrow$ $\exists$ *countable* subfamily $\mathcal{G} = \{B_1, B_2, \ldots\} \subset \mathcal{F}$ of *disjoint* open balls s.t.

$$M_t \subset \bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B_i\in\mathcal{G}} 5B_i.$$

Hence

$$\begin{aligned} m(M_t) \le m\big(\cup_i 5B_i\big) &\overset{\text{countable union}}{\le} \sum_i m(5B_i) = 5^n \sum_i m(B_i) \\ &\overset{(3.24)}{\le} 5^n \sum_i \frac{1}{t}\int_{B_i} |f(y)|dy \overset{B_i\text{'s disjoint}}{=} \frac{5^n}{t}\int_{\cup_i B_i} |f(y)|dy \\ &\le \frac{5^n}{t}\int_{\mathbb{R}^n} |f(y)|dy. \end{aligned}$$

□

By Remark 3.20, $Mf \in L^1(\mathbb{R}^n) \;\Rightarrow\; f = 0$ a.e. The situation is completely different if $p > 1$.

Theorem 3.25. *Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Then $Mf \in L^p(\mathbb{R}^n)$ and there is a constant $c = c(p, n)$ s.t.*

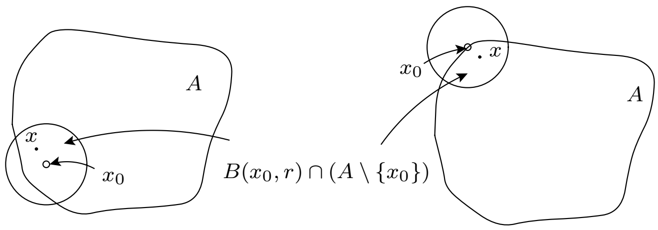
$$\|Mf\|_p \le c\|f\|_p.$$

Proof. The proof uses Lemma 3.15, the Hardy-Littlewood theorem and Fubini's theorem. We omit the details. □

3.26 Lebesgue differentiation theorem

Let $A \subset \mathbb{R}^n$, $h \colon A \to \dot{\mathbb{R}}$ and x_0 an accumulation point of A (i.e. $B(x_0, r) \cap (A \setminus \{x_0\}) \neq \emptyset \ \forall r > 0$). Define

$$\limsup_{x \to x_0} h(x) = \lim_{r \to 0+} \sup\{h(x) \colon x \in B(x_0, r) \cap (A \setminus \{x_0\})\}.$$



Observation: $0 < r_1 < r_2 \Rightarrow$

$$\sup\{h(x) \colon x \in B(x_0, r_1) \cap (A \setminus \{x_0\})\} \le \sup\{h(x) \colon x \in B(x_0, r_2) \cap (A \setminus \{x_0\})\},$$

and therefore the limit exists ($\pm\infty$ allowed). Similarly we can define lim inf.

Motivation: If $f \colon \mathbb{R}^n \to \mathbb{R}$ is continuous, then

$$\lim_{r \to 0+} \fint_{B(x,r)} |f(y) - f(x)| dy = 0 \tag{3.27}$$

$\forall x \in \mathbb{R}^n$ (see the proof of part 3 below). On the other hand, Lusin's theorem says that a measurable function is "almost continuous" (f measurable, $\varepsilon > 0 \Rightarrow \exists$ closed $F \subset \mathbb{R}^n$ s.t. $m(\mathbb{R}^n \setminus F) < \varepsilon$ and $f|F$ continuous), hence a question arises: in what sense (3.27) holds for locally integrable functions.

Theorem 3.28 (Lebesgue differentiation theorem)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$ be a locally integrable function. Then*

$$\lim_{r \to 0+} \fint_{B(x,r)} |f(y) - f(x)| dy = 0 \quad a.e.\ x \in \mathbb{R}^n. \tag{3.29}$$

In particular,

$$\lim_{r \to 0+} \fint_{B(x,r)} f(y) dy = f(x) \quad a.e.\ x \in \mathbb{R}^n. \tag{3.30}$$

Proof. If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let us define

$$\Lambda f(x) = \limsup_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)| dy.$$

(Note: $\Lambda f(x)$ is defined $\forall x \in \mathbb{R}^n$ and the value depends on $f(x)$, thus in particular, on the chosen representative of the equivalence class.)

Then Λf satisfies:

1. $\Lambda f(x) \geq 0 \; \forall x \in \mathbb{R}^n$ (clear).
2. Λ is sub linear, i.e.
$$\Lambda(f+g) \leq \Lambda f + \Lambda g, \quad \forall f, g \in L^1_{\text{loc}}(\mathbb{R}^n).$$
Reason:
$$\begin{aligned}\Lambda(f+g)(x) &= \limsup_{r\to 0} \fint_{B(x,r)} |f(y)+g(y)-f(x)-g(x)|dy \\ &\overset{\triangle\text{-ineq.}}{\leq} \limsup_{r\to 0}\Big(\fint_{B(x,r)} |f(y)-f(x)|dy + \fint_{B(x,r)} |g(y)-g(x)|dy\Big) \\ &\leq \limsup_{r\to 0} \fint_{B(x,r)} |f(y)-f(x)|dy + \limsup_{r\to 0} \fint_{B(x,r)} |g(y)-g(x)|dy \\ &= \Lambda f(x) + \Lambda g(x), \quad \forall x \in \mathbb{R}^n.\end{aligned}$$
3. $g \in C(\mathbb{R}^n) \;\Rightarrow\; \Lambda g(x) = 0 \; \forall x \in \mathbb{R}^n$.
Reason: Fix $x \in \mathbb{R}^n$ and $\varepsilon > 0$.
g continuous at $x \;\Rightarrow\; \exists\, \delta > 0$ s.t. $|g(y)-g(x)| < \varepsilon \quad \forall y \in B(x,\delta)$.
Hence $\forall\; 0 < s \leq \delta$ we have:
$$\fint_{B(x,s)} \underbrace{|g(y)-g(x)|}_{<\varepsilon} dy < \frac{1}{m\big(B(x,s)\big)} \int_{B(x,s)} \varepsilon dy = \frac{\varepsilon m\big(B(x,s)\big)}{m\big(B(x,s)\big)} = \varepsilon$$
$$\Rightarrow \Lambda g(x) = \lim_{r\to 0}\left(\sup_{0<s<r} \underbrace{\fint_{B(x,s)} |g(y)-g(x)|dy}_{<\varepsilon, \text{ if } 0<s\leq\delta}\right) \leq \varepsilon$$
$$\Rightarrow \Lambda g(x) = 0\,, \text{ sinc } \varepsilon > 0 \text{ arbitrary.}$$
4. $\Lambda f \leq Mf + |f|$.
Reason:
$$\fint_{B(x,r)} |f(y)-f(x)|dy \overset{\triangle\text{-ey.}}{\leq} \underbrace{\fint_{B(x,r)} |f(y)|dy}_{\leq Mf(x)} + \underbrace{\fint_{B(x,r)} |f(x)|dy}_{=|f(x)|} \leq Mf(x) + |f(x)|.$$
(Note $|f(x)|$ is *constant* in the last integral.)

Let then $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be given and $t > 0$. It suffices to show that $\forall k \in \mathbb{N}$ (3.29) holds a.e. $x \in B(0,k)$. The validity of (3.29) in $B(0,k)$ is *independend* of the values of f in $\mathbb{R}^n \setminus B(0,2k)$, and therefore we may assume that $f = 0$ in $\mathbb{R}^n \setminus B(0,2k)$ and hence $f \in L^1(\mathbb{R}^n)$.

If $g \in C(\mathbb{R}^n)$, then
$$\Lambda f = \Lambda(f-g+g) \overset{2.}{\leq} \Lambda(f-g) + \Lambda g \overset{3.}{=} \Lambda(f-g) \overset{4.}{\leq} M(f-g) + |f-g|.$$
Hence at least one of the values $M(f-g)(x)$ or $|f(x)-g(x)|$ is at least $\Lambda f(x)/2$, and therefore
$$\{x \in \mathbb{R}^n \colon \Lambda f(x) > t\} \subset \{x \in \mathbb{R}^n \colon M(f-g)(x) > t/2\} \cup \{x \in \mathbb{R}^n \colon |f(x)-g(x)| > t/2\}.$$

Hence

$$m^*\big(\{x\colon \Lambda f(x) > t\}\big) \le \underbrace{m\big(\{x\colon M(f-g)(x) > t/2\}\big)}_{\overset{\text{H.-L.}}{\le} 2\cdot 5^n\|f-g\|_1/t} + \underbrace{m\big(\{x\colon |f(x)-g(x)| > t/2\}\big)}_{\overset{\text{Cheb.}}{\le} 2\|f-g\|_1/t}$$

$$\le \frac{2(5^n+1)\|f-g\|_1}{t}.$$

Theorem 2.36 $\Rightarrow$ continuous functions are dense in L^1 $\Rightarrow$ $\forall \varepsilon > 0\ \exists\ g \in C(\mathbb{R}^n)$ s.t. $\|f-g\|_1 < \varepsilon$

$$\overset{\varepsilon>0 \text{ arbitr.}}{\Longrightarrow} m^*\big(\{x \in \mathbb{R}^n\colon \Lambda f(x) > t\}\big) = 0 \quad \forall t > 0$$

$$\Rightarrow m^*\big(\underbrace{\{x \in \mathbb{R}^n\colon \Lambda f(x) > 0\}}_{\subset \bigcup_k \{x\colon \Lambda f(x) > 1/k\}}\big) \le \sum_{k=1}^{\infty} \underbrace{m^*\big(\{x \in \mathbb{R}^n\colon \Lambda f(x) > 1/k\}\big)}_{=0} = 0$$

$$\Rightarrow \Lambda f(x) = 0 \text{ a.e. } x \in \mathbb{R}^n$$

$$\Rightarrow 0 \le \liminf_{r\to 0} \fint_{B(x,r)} |f(y)-f(x)|dy \le \limsup_{r\to 0} \fint_{B(x,r)} |f(y)-f(x)|dy = 0 \text{ a.e. } x \in \mathbb{R}^n$$

$$\Rightarrow \lim_{r\to 0+} \fint_{B(x,r)} |f(y)-f(x)|dy = 0 \text{ a.e. } x \in \mathbb{R}^n.$$

Finally,

$$\Big|\fint_{B(x,r)} f(y)dy - f(x)\Big| = \Big|\fint_{B(x,r)} f(y)dy - \underbrace{\fint_{B(x,r)} f(x)dy}_{=f(x)}\Big| = \Big|\fint_{B(x,r)} \big(f(y)-f(x)\big)dy\Big|$$

$$\le \fint_{B(x,r)} |f(y)-f(x)|dy \xrightarrow{r\to 0+} 0 \quad \text{a.e. } x \in \mathbb{R}^n.$$

□

Definition 3.31. A point $x \in \mathbb{R}^n$ is *density point* of a set $E \in \operatorname{Leb}\mathbb{R}^n$ if

$$\lim_{r\to 0+} \frac{m\big(E \cap B(x,r)\big)}{m\big(B(x,r)\big)} = 1.$$

Remark 3.32. If $x \in \mathbb{R}^n$ is a density point of E, it need not belong to E.For instance, 0 is a density point of $\mathbb{R}^n \setminus \{0\}$.

Corollary 3.33. Let $E \in \operatorname{Leb}\mathbb{R}^n$. Then almost every $x \in E$ is a density point of E, i.e.

$$\lim_{r\to 0+} \frac{m\big(E \cap B(x,r)\big)}{m\big(B(x,r)\big)} = 1 \quad \text{a.e. } x \in E.$$

Furthermore,

$$\lim_{r\to 0+} \frac{m\big(E \cap B(x,r)\big)}{m\big(B(x,r)\big)} = 0 \quad \text{a.e. } x \in \mathbb{R}^n \setminus E.$$

Proof. $E \in \operatorname{Leb}\mathbb{R}^n \Rightarrow \chi_E \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ since χ_E is measurable and

$$\int_K \chi_E = m(E \cap K) < \infty \quad \forall \text{ compact } K \subset \mathbb{R}^n.$$

Furthermore,

$$\fint_{B(x,r)} \chi_E(y)dy = \frac{m\big(E \cap B(x,r)\big)}{m\big(B(x,r)\big)} \quad \forall x \in \mathbb{R}^n,\ r > 0.$$

Lebesgue diff. theorem 3.28 $\Rightarrow$

$$\lim_{r\to 0+} \frac{m\big(E \cap B(x,r)\big)}{m\big(B(x,r)\big)} = \chi_E(x) \quad \text{a.e. } x \in \mathbb{R}^n.$$

□

A point $x \in \mathbb{R}^n$ is a *Lebesgue point* of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if

$$\lim_{r\to 0+} \fint_{B(x,r)} |f(y) - f(x)|dy = 0.$$

This property *depends* on the value $f(x)$ and hence on the choice of the representative $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. To get rid of this dependence we say that a point $x \in \mathbb{R}^n$ b elongs to the *Lebesgue set*, $\operatorname{Leb}(f)$, of f if $\exists\ A = A(x) \in \mathbb{R}$ s.t.

$$\lim_{r\to 0+} \fint_{B(x,r)} |f(y) - A|dy = 0. \tag{3.34}$$

If such A exists, it is unique since

$$\lim_{r\to 0+} \fint_{B(x,r)} f(y)dy = A.$$

Remark 3.35. 1. $f = g$ a.e. in $\mathbb{R}^n \Rightarrow \operatorname{Leb}(f) = \operatorname{Leb}(g)$. In particular, the Lebesgue set $\operatorname{Leb}(f)$ is well defined in the whole *equivalence class* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, i.e. *does not* depend on the choice of the representative.

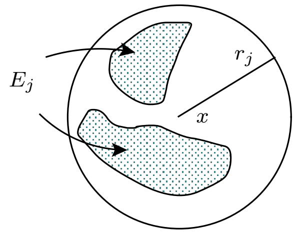
2. Lebesgue diff. theorem 3.28: If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then almost every $x \in \mathbb{R}^n$ belongs to $\operatorname{Leb}(f)$. Furthermore, $A = f(x)$ for a.e. $x \in \mathbb{R}^n$. Thus by modifying f in a set of measure 0 (by setting $f(x) = A(x)$) *we may assume in what follows* that

$$\lim_{r\to 0+} \fint_{B(x,r)} |f(y) - f(x)|dy = 0 \quad \forall x \in \operatorname{Leb}(f).$$

Let $x \in \mathbb{R}^n$. We say the a sequence of measurable sets $E_j \subset \mathbb{R}^n, j \in \mathbb{N}$, *shrinks nice to* x if $\exists$ a constant $c > 0$ and a sequence $r_j > 0$ s.t.

$$E_j \subset B(x, r_j)\ \forall j \text{ and } \lim_{j\to\infty} r_j = 0,$$

$$m\big(B(x, r_j)\big) \le c\, m(E_j)\ \forall j. \tag{3.36}$$



Theorem 3.37. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathrm{Leb}(f)$. If a sequence E_j shrinks nicely to x, then*

$$\lim_{j\to\infty} \fint_{E_j} f(y)dy = f(x).$$

Proof.

$$\Big|\fint_{E_j} f(y)dy - f(x)\Big| = \Big|\fint_{E_j} \big(f(y)dy - f(x)\big)\Big| \le \fint_{E_j} |f(y)dy - f(x)|dy$$

$$\le \underbrace{\frac{m\big(B(x,r_j)\big)}{m(E_j)}}_{\underset{(3.36)}{\le} c} \fint_{B(x,r_j)} |f(y) - f(x)|dy \xrightarrow{r_j\to 0} 0 \text{ since } x \in \mathrm{Leb}(f).$$

□

Now we can study differentiability of the function $F(x) = \int_a^x f(t)dt$ if $f\colon [a,b] \to \mathbb{R}$ is integrable (see Example 3.1.2).

Theorem 3.38. *Let $f\colon [a,b] \to \mathbb{R}$ be integrable and*

$$F(x) = \int_a^x f(t)dt,\ x \in [a,b].$$

YThen F is differentiable a.e. and $F'(x) = f(x)$ for a.e. $x \in [a,b]$.

Proof. Extend f to a function $f\colon \mathbb{R} \to \mathbb{R}$ by setting $f(t) = 0\ \forall t \in \mathbb{R} \setminus [a,b]$, hence $f \in L^1(\mathbb{R})$.
<u>It suffices to show:</u>

$$F'(x) = f(x) \quad \forall x \in \mathrm{Leb}(f) \cap [a,b]$$

since almost every $x \in [a,b]$ belongs to $\mathrm{Leb}(f)$ (Lebesgue diff. theorem 3.28).
Let $x \in \mathrm{Leb}(f)\cap[a,b]$ and let $r_j > 0, j \in \mathbb{N}$, be an arbitrary sequence $r_j \to 0$. Denote $E_j =]x, x+r_j[$,, and so $E_j \subset]x - r_j, x + r_j[= B(x,r_j)$ and $m\big(B(x,r_j)\big) = 2m(E_j)$. Thus the sequence E_j shrinks nicely to x. Theorem 3.37 $\Rightarrow$

$$\frac{F(x+r_j) - F(x)}{r_j} = \frac{1}{r_j}\int_x^{x+r_j} f(t)dt \xrightarrow{j\to\infty} f(x).$$

Similarly, we see that

$$\frac{F(x-r_j) - F(x)}{-r_j} = \frac{1}{r_j}\int_{x-r_j}^{x} f(t)dt \xrightarrow{j\to\infty} f(x)$$

by using sets $E'_j =]x - r_j, x[$. Since (r_j) is an arbitrary sequence, we get

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

□

3.39 Monotonic functions in $\mathbb{R}$

In this section we will study differentiability of monotonic functions by using Vitali's covering theorem as a tool.

Let $\Delta \subset \mathbb{R}$ be an interval. A function $f\colon \Delta \to \mathbb{R}$ is

- *increasing* if $x_1, x_2 \in \Delta, x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$;
- *decreasing* if $x_1, x_2 \in \Delta, x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$;
- *monotonic* if f is increasing or decreasing.

Example 3.40. Write $\mathbb{Q} = \{q_j : j \in \mathbb{N}\}$. Set

$$f(x) = \sum_{\substack{j \in \mathbb{N} \\ q_j \le x}} 2^{-j}, \quad x \in \mathbb{R}.$$

Then $f\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing and $0 < f(x) < \sum_{j \in \mathbb{N}} 2^{-j} = 1 \ \forall x \in \mathbb{R}$. Furthermore, f is continuous in a point $x \in \mathbb{R} \iff x \in \mathbb{R} \setminus \mathbb{Q}$.

Above the set of discontinuity points of f is countable. This holds for all monotonic functions:

Lemma 3.41. *A monotonic function $f\colon [a,b] \to \mathbb{R}$ has at most countably many points of discontinuity.*

Proof. We may assume that f is increasing (g decreasing $\Rightarrow$ $-g$ increasing). For $x \in (a,b)$ we define

$$H(x) = \lim_{y \to x+} f(y) - \lim_{y \to x-} f(y)$$

f increasing and bounded in a neighborhood of x $\Rightarrow$ the limit exists and $H(x) \ge 0$. We denote

$$H_k = \{x \in (a,b)\colon H(x) > 1/k\},\ k \in \mathbb{N},$$

and prove that H_k is finite. Suppose that $x_1, \ldots, x_{2j} \in H_k$ s.t. $x_1 < x_2 < \cdots < x_{2j}$.
For $i = 2, 3, \ldots, j$ choose arbitrary x and y s.t. $x_{2i-2} < y < x_{2i-1} < x < x_{2i}$.
Since f is increasing,

$$\begin{aligned}
f(x_{2i}) - f(x_{2i-2}) &\ge f(x) - f(y) \ge H(x_{2i-1}) > 1/k. \\
\Rightarrow\ f(b) - f(a) &\ge \sum_{i=2}^{j} \big(f(x_{2i}) - f(x_{2i-2})\big) \ge \sum_{i=2}^{j} H(x_{2i-1}) > (j-1)/k \\
&\Rightarrow\ j < k\big(f(b) - f(a)\big) + 1 \\
&\Rightarrow\ \text{the set } H_k \text{ is finite.}
\end{aligned}$$

Furthermore f is discontinuous at $x \in (a,b) \iff H(x) > 0$. Thus

$$\{x \in (a,b)\colon f \text{ discontinuous at } x\} \subset \bigcup_{k=1}^{\infty} H_k$$

which is countable. □

Lebesgue theorem for monotonic functions.

Theorem 3.42. *Let $f\colon [a,b] \to \mathbb{R}$ be a monotonic function. Then the derivative $f'(x)$ exists for a.e. $x \in [a,b]$.*

Proof. We may assume that f is increasing. If $x \in [a,b]$, we define

$$\overline{D}f(x) = \limsup_{\varepsilon \to 0+} \left\{ \frac{f(z)-f(y)}{z-y} \colon a \le y \le x \le z \le b,\ 0 < z-y < \varepsilon \right\} \quad \text{and}$$

$$\underline{D}f(x) = \liminf_{\varepsilon \to 0+} \left\{ \frac{f(z)-f(y)}{z-y} \colon a \le y \le x \le z \le b,\ 0 < z-y < \varepsilon \right\}.$$

f increasing $\Rightarrow$ $0 \le \underline{D}f(x) \le \overline{D}f(x) \le \infty$ $\forall x \in [a,b]$.
<u>We will show:</u>

$$\underline{D}f(x) = \overline{D}f(x) < \infty \quad \text{a.e. } x \in [a,b]. \tag{3.43}$$

Indeed, if $\underline{D}f(x) = \overline{D}f(x) < \infty$, then the derivative $f'(x) = \underline{D}f(x) = \overline{D}f(x)$ exists.
We split the proof of (3.43) in parts:
[1.] Write

$$E_k = \{x \in [a,b] \colon \overline{D}f(x) > k\}, \quad k \in \mathbb{N},$$
$$F_{s,t} = \{x \in [a,b] \colon \underline{D}f(x) < s < t < \overline{D}f(x)\}, \quad 0 < s < t,\ s,t \in \mathbb{Q}.$$

Clearly

$$\{x \in [a,b] \colon \overline{D}f(x) = \infty\} = \bigcap_{k \in \mathbb{N}} E_k$$
$$\{x \in [a,b] \colon \underline{D}f(x) < \overline{D}f(x)\} = \bigcup_{\substack{0<s<t \\ s,t \in \mathbb{Q}}} F_{s,t}. \quad \text{(Note: countable union.)}$$

<u>It's enough to show:</u>

$$m^*(E_k) \le \frac{c}{k}, \quad \forall k \in \mathbb{N},\ \text{for some constant } c > 0,$$
$$m^*(F_{s,t}) = 0 \quad \text{for all } 0 < s < t,\ s,t \in \mathbb{Q},$$

because then

$$\left.\begin{aligned} & m^*\big(\underbrace{\{x \colon \overline{D}f(x) = \infty\}}_{\subset E_k\ \forall k}\big) \le \frac{c}{k}\ \forall k \ \Rightarrow\ m^*\big(\{x \colon \overline{D}f(x) = \infty\}\big) = 0 \\ & m^*\big(\{x \colon \underline{D}f(x) < \overline{D}f(x)\}\big) \le \sum_{\substack{0<s<t \\ s,t \in \mathbb{Q}}} \underbrace{m^*(F_{s,t})}_{=0} = 0 \end{aligned}\right\} \Rightarrow (3.43).$$

[2.] <u>Claim:</u> $m^*(E_k) \le \frac{f(b)-f(a)}{k}$, $k \in \mathbb{N}$.[4]

[4]Compare: $g \in C^1$, $g'(t) > k > 0$ $\forall t \in [c,d] \Rightarrow g(d) - g(c) = \int_c^d g'(t)dt > k(d-c) \Rightarrow m([c,d]) = d - c < (g(d)-g(c))/k$.

Proof: Let $x \in E_k$ be arbitrary. Then

$$\overline{D}f(x) = \lim_{\varepsilon \to 0+} \sup \left\{ \frac{f(z)-f(y)}{z-y} : a \le y \le x \le z \le b,\ 0 < z-y < \varepsilon \right\} > k$$

$$\Rightarrow \forall \varepsilon > 0 \quad \exists \text{ closed interval } I_{x,\varepsilon} = [y,z] \subset [a,b] \text{ s.t. }$$
$$x \in [y,z],\ 0 < z-y = d([y,z]) < \varepsilon \quad \text{and}$$

$$\frac{f(z)-f(y)}{z-y} > k, \quad \text{thus} \quad \underbrace{m\left(]f(y),f(z)[\right)}_{=f(z)-f(y)} > k\, \underbrace{m\big([y,z]\big)}_{=z-y}. \tag{3.44}$$

Hence such intervals $I_{x,\varepsilon} = [y,z]$ form a closed Vitali cover of E_k. By Vitali's covering theorem 3.9 there exists a countable subfamily of *disjoint* closed intervals $I_j = [y_j, z_j] \subset [a,b]$ s.t. (3.44) holds and

$$m\big(E_k \setminus \bigcup_{j\in\mathbb{N}} I_j\big) = 0.$$

Now

$$m^*(E_k) \le m^*\big(E_k \cap \bigcup_{j\in\mathbb{N}} I_j\big) + \overbrace{m^*\big(E_k \setminus \bigcup_{j\in\mathbb{N}} I_j\big)}^{=0} \le m\big(\bigcup_{j\in\mathbb{N}} I_j\big) \le \sum_{j\in\mathbb{N}} m(I_j)$$
$$\overset{(3.44)}{\le} \frac{1}{k} \sum_{j\in\mathbb{N}} m\left(]f(y_j), f(z_j)[\right).$$

f increasing, intevals $I_j = [y_j, z_j]$ disjoint $\Rightarrow$ *open* intervals $]f(y_j), f(z_j)[$ are disjoint. Hence

$$m^*(E_k) \le \frac{1}{k} \sum_{j\in\mathbb{N}} m\left(]f(y_j), f(z_j)[\right) \overset{\text{disjoint}}{=} \frac{1}{k} m\Big(\underbrace{\bigcup_{j\in\mathbb{N}}]f(y_j), f(z_j)[}_{\subset [f(a),f(b)]}\Big)$$
$$\le \frac{f(b)-f(a)}{k}.$$

3. Claim: $m^*(F_{s,t}) = 0 \quad \forall s,t \in \mathbb{Q},\ 0 < s < t.$

Proof: We use Vitali's covering theorem twice. It follows from the definition of outer measure that $\forall \varepsilon > 0\ \exists$ an open $G \supset F_{s,t}$ s.t.

$$m(G) < m^*(F_{s,t}) + \varepsilon.$$

3a. We apply Vitali's covering theorem to the set $F_{s,t}$ (arguing as in part 2).
It follows from definitions of $\underline{D}f(x)$ and $F_{s,t}$ that $\forall x \in F_{s,t}\ \exists$ arbitrary small closed intervals $[y,z],\ y < z$, s.t. $x \in [y,z] \subset G \cap [a,b]$ and

$$\frac{m\big(]f(y),f(z)[\big)}{m\big([y,z]\big)} = \frac{f(z)-f(y)}{z-y} < s. \tag{3.45}$$

(Note: $G \ni x$ open.)
By Vitali's covering theorem there exists disjoint closed intervals $I_j = [y_j, z_j] \subset G \cap [a,b],\ j \in \mathbb{N}$, s.t. (3.45) holds and

$$m\big(F_{s,t} \setminus \bigcup_{j\in\mathbb{N}} I_j\big) = 0.$$

Then

$$m\Big(\bigcup_{j\in\mathbb{N}}]f(y_j),f(z_j)[\Big) \overset{\text{disjoint}}{=} \sum_{j\in\mathbb{N}} m\big(]f(y_j),f(z_j)[\big) \overset{(3.45)}{<} s\sum_{j\in\mathbb{N}} m\big([y_j,z_j]\big)$$

$$\overset{\text{disjoint}}{=} s\, m\Big(\underbrace{\bigcup_{j\in\mathbb{N}}[y_j,z_j]}_{\subset G}\Big) \le s\, m(G) < s(m^*(F_{s,t})+\varepsilon). \tag{3.46}$$

Furthermore,

$$m\big(F_{s,t}\setminus \bigcup_{j\in\mathbb{N}}]y_j,z_j[\big)=0, \tag{3.47}$$

because $\{y_j,z_j\colon j\in\mathbb{N}\}$ is of measure 0. Denote $A=\bigcup_{j\in\mathbb{N}}]y_j,z_j[$.

3b. We apply Vitali's covering theorem to the set $F_{s,t}\cap A$.
It follows from the definitions of $\overline{D}f(x)$ and $F_{s,t}$ that $\forall x\in F_{s,t}\cap A$ there exists arbitrary small closed intervals $[u,v],\ u<v$, s.t. $x\in[u,v]$,

$$\frac{m\big(]f(u),f(v)[\big)}{m\big([u,v]\big)}=\frac{f(v)-f(u)}{v-u}>t, \tag{3.48}$$

and $[u,v]\subset\,]y_j,z_j[$ for some $j\in\mathbb{N}$ (this is possible because $A=\bigcup_j]y_j,z_j[$ is a disjoint union of open intervals).
By Vitali's covering theorem there exists disjoint closed intervals $J_k=[u_k,v_k],\ k\in\mathbb{N}$, as above s.t. every $J_k\subset\,]y_j,z_j[$ for a suitable $j=j_k$ and

$$m\big(F_{s,t}\cap A\setminus\bigcup_{k\in\mathbb{N}}J_k\big)=0. \tag{3.49}$$

Denote $B=\bigcup_k J_k$, hence $B\subset A$ and

$$\left.\begin{array}{l}(3.47)\Rightarrow\ F_{s,t}=(F_{s,t}\cap A)\cup A_0,\quad \text{where } A_0=F_{s,t}\setminus A \text{ of measure } 0\\[2ex] (3.49)\Rightarrow\ F_{s,t}\cap A=(F_{s,t}\cap A\cap B)\cup B_0,\quad \text{where } B_0=F_{s,t}\cap A\setminus B \text{ of measure } 0\end{array}\right\}\Rightarrow$$

$$F_{s,t}=(F_{s,t}\cap A)\cup A_0=\big[(F_{s,t}\cap A\cap B)\cup B_0\big]\cup A_0 \overset{B\subseteq A}{=} (F_{s,t}\cap B)\cup A_0\cup B_0,$$

where $A_0\cup B_0$ is of measure 0. Thus $m^*(F_{s,t})\le m(B),\ B=\bigcup_k J_k$. Then

$$m^*(F_{s,t})\le m\Big(\bigcup_k J_k\Big) \overset{\text{disjoint}}{=} \sum_{k\in\mathbb{N}} m(J_k) \overset{(3.48)}{<} \frac{1}{t}\sum_{k\in\mathbb{N}} m\big(]f(u_k),f(v_k)[\big)$$

$$\overset{\text{disjoint}}{=} \frac{1}{t}m\Big(\underbrace{\bigcup_{k\in\mathbb{N}}]f(u_k),f(v_k)[}_{\subset\bigcup_j]f(y_j),f(z_j)[}\Big) \le \frac{1}{t}m\Big(\bigcup_{j\in\mathbb{N}}]f(y_j),f(z_j)[\Big) \overset{(3.46)}{<} \frac{s}{t}\big(m^*(F_{s,t})+\varepsilon\big).$$

$\varepsilon>0$ arbitrary $\Rightarrow$

$$\left.\begin{array}{c} m^*(F_{s,t})\le \dfrac{s}{t}m^*(F_{s,t})\\ 0<s<t\ \Rightarrow\ \dfrac{s}{t}<1\\ m^*(F_{s,t})\le m\big([a,b]\big)=b-a<\infty\end{array}\right\}\ \Rightarrow\ m^*(F_{s,t})=0\quad \forall s,t\in\mathbb{Q},\ 0<s<t.$$

□

Remark 3.50. The statement of Theorem 3.42 is the best possible. Indeed:
Let $A \subset \mathbb{R}$ be an arbitrary set of measure 0. Then there exists a continuous and increasing function $f\colon \mathbb{R} \to \mathbb{R}$ for which

$$\underline{D}f(x) = \infty \quad \forall x \in A.$$

Theorem 3.51. *If $f\colon [a,b] \to \mathbb{R}$ is increasing, the derivative f' is integrable and*

$$\int_a^b f'(x)dx \le f(b) - f(a). \tag{3.52}$$

Proof. Extend f y setting $f(x) = f(b)$ $\forall x > b$. Theorem 3.42 $\Rightarrow$ $\exists$ derivative $f'(x)$ a.e. $x \in [a,b]$, i.e.

$$f'(x) = \lim_{k\to\infty} \frac{f\big(x+\frac{1}{k}\big) - f(x)}{1/k} \quad \text{a.e. } x \in [a,b]. \tag{3.53}$$

Furthermore, f is measurable as an increasing function, hence the function

$$x \mapsto \frac{f\big(x+\frac{1}{k}\big) - f(x)}{1/k}$$

are measurable $\forall k \in \mathbb{N}$, and consequently

$$x \mapsto \limsup_{k\to\infty} \frac{f\big(x+\frac{1}{k}\big) - f(x)}{1/k}$$

is measurable. Since

$$\limsup_{k\to\infty} \frac{f\big(x+\frac{1}{k}\big) - f(x)}{1/k} = \lim_{k\to\infty} \frac{f\big(x+\frac{1}{k}\big) - f(x)}{1/k} = f'(x) \quad \text{a.e. } x \in [a,b],$$

f' is measurable. We notice (by performing a change of variables $x + 1/k \mapsto x$) that

$$\begin{aligned}\int_a^b k\Big(f\big(x+\frac{1}{k}\big) - f(x)\Big)dx &= k\int_{a+1/k}^{b+1/k} f(x)dx - k\int_a^b f(x)dx \\ &= \underbrace{k\int_b^{b+1/k} \underbrace{f(x)}_{=f(b)}\,dx}_{=f(b)} - k\int_a^{a+1/k} \underbrace{f(x)}_{\ge f(a)}\,dx \\ &\le f(b) - f(a) \quad \forall k \in \mathbb{N}.\end{aligned}$$

f increasing $\Rightarrow$

$$k\Big(f\big(x+\frac{1}{k}\big) - f(x)\Big) \ge 0 \quad \forall x \in [a,b],\ k \in \mathbb{N},$$

and so by Fatou's lemma

$$\begin{aligned}\int_a^b f'(x)dx &\overset{(3.53)}{=} \int_a^b \lim_{k\to\infty} \underbrace{k\Big(f\big(x+\frac{1}{k}\big) - f(x)\Big)}_{\ge 0 \text{ and measurable}}\,dx \overset{\text{Fatou}}{\le} \liminf_{k\to\infty} \underbrace{\int_a^b k\Big(f\big(x+\frac{1}{k}\big) - f(x)\Big)dx}_{\le f(b)-f(a)} \\ &\le f(b) - f(a).\end{aligned}$$

□

Next we will study when an equality holds in (3.52), cf. Example 3.1.4.

3.54 Functions of bounded variation in $\mathbb{R}$

Definition 3.55. The *total variation* of a function $f\colon [a,b] \to \mathbb{R}$ on an interval $[a,x]$, $a \le x \le b$, is

$$V_f(a,x) = \sup \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over all divisions $a = x_0 < x_1 < x_2 < \dots < x_k = x$ of $[a,x]$. We say that f is of *bounded variation* (or has bounded variation) on the interval $[a,b]$ (abbr. $f \in BV$) if $V_f(a,b) < \infty$. A function $g\colon \mathbb{R} \to \mathbb{R}$ is of bounded variation if $V_g(\mathbb{R}) = \sup_{a<b} V_g(a,b) < \infty$.

A trivial observation: $|f(b) - f(a)| \le V_f(a,b)$, since the points a, b form a division of $[a,b]$.

Example 3.56. 1. $f \in C^1\big([a,b]\big) \;\Rightarrow\; f \in BV$.
Proof $f \in C^1\big([a,b]\big) \;\Rightarrow\; f'\colon [a,b] \to \mathbb{R}$ continuous $\Rightarrow\; \exists M = \max\{|f'(x)|\colon x \in [a,b]\} < \infty$. Let $a = x_0 < x_1 < x_2 < \dots < x_k = b$ be an arbitrary division of $[a,b]$. Then

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \overset{\text{VAL}}{\le} M \sum_{i=1}^{k} (x_i - x_{i-1}) = M(b-a)$$
$$\overset{\sup}{\Longrightarrow} V_f(a,b) \le M(b-a). \quad \square$$

2. $f \in C\big([a,b]\big) \;\not\Rightarrow\; f \in BV$.
Let $f\colon [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{jos } 0 < x \le 1, \\ 0, & \text{jos } x = 0. \end{cases}$$

Then f is contiuous but not of bounded variation (exerc.).

We want to prove that every function of bounded variation can be expressed as a difference of two increasing functions.

Lemma 3.57. *Let $f\colon [a,b] \to \mathbb{R}$ be monotonic. Then f is of bounded variation and*

$$V_f(a,b) = |f(b) - f(a)|.$$

Proof. Suppose that f is increasing. Let $a = x_0 < x_1 < x_2 < \dots < x_k = b$ be a division of $[a,b]$. Then

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \overset{f \text{ incr.}}{=} \sum_{i=1}^{k} \big(f(x_i) - f(x_{i-1})\big) = f(x_k) - f(x_0) = f(b) - f(a)$$
$$\overset{\sup}{\Longrightarrow} V_f(a,b) = f(b) - f(a) < \infty.$$

If f is decresing, then $-f$ is increasing, and therefore $V_f(a,b) = V_{-f}(a,b) = f(a) - f(b)$. $\square$

Lemma 3.58. *Let $f\colon [a,b] \to \mathbb{R}$ be of bounded variation. Then*

$$V_f(a,b) = V_f(a,c) + V_f(c,b) \quad \forall c \in \,]a,b[.$$

Proof. Let $c \in]a, b[$. Let

$$a = x_0 < x_1 < \cdots < x_k = c$$
$$c = y_0 < y_1 < \cdots < y_n = b$$

be arbitrary divisions of $[a, c]$ and $[c, b]$, respectively. Then $a = x_0 < x_1 < \cdots < x_k = y_0 < y_1 < \cdots < y_n = b$ is a division of $[a, b]$, and therefore

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |f(y_i) - f(y_{i-1})| \leq V_f(a, b).$$

Taking sup over all divisions of $[a, c]$ and $[c, b]$, we obtain

$$V_f(a, c) + V_f(c, b) \leq V_f(a, b).$$

Conversely: Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a division of $[a, b]$.
Write $k = \min\{i \colon c \leq x_i\}$. Then $\{x_0, x_1, \ldots, x_{k-1}, c\}$ is a division of $[a, c]$ and $\{c, x_k, \ldots, x_n\}$ is a division of $[c, b]$, and therefore

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{k-1} |f(x_i) - f(x_{i-1})| + \underbrace{|f(x_k) - f(x_{k-1})|}_{\leq |f(x_k) - f(c)| + |f(c) - f(x_{k-1})|} + \sum_{i=k+1}^{n} |f(x_i) - f(x_{i-1})|$$

$$\leq \underbrace{\sum_{i=1}^{k-1} |f(x_i) - f(x_{i-1})| + |f(c) - f(x_{k-1})|}_{\leq V_f(a,c)} + \underbrace{|f(x_k) - f(c)| + \sum_{i=k+1}^{n} |f(x_i) - f(x_{i-1})|}_{\leq V_f(c,b)}$$

$$\leq V_f(a, c) + V_f(c, b)$$

$$\overset{\text{sup}}{\Longrightarrow} V_f(a, b) \leq V_f(a, c) + V_f(c, b).$$

□

Now we can easily prove an important characterization of functions of bounded variation.

Theorem 3.59. *A function $f \colon [a, b] \to \mathbb{R}$ is of bounded variation $\iff$ $f = g - h$, where g and h are increasing on the interval $[a, b]$.*

Proof. $\boxed{\Leftarrow}$ g, h increasing $\overset{3.57}{\Longrightarrow}$ g, h of bounded variation $\Rightarrow$ $f = g - h$ of bounded variation (follows easily from the $\triangle$-ineq.).

$\boxed{\Rightarrow}$ Suppose that $f \colon [a, b] \to \mathbb{R}$ is of bounded variation. Then

$$f(x) = V_f(a, x) - \big(V_f(a, x) - f(x)\big), \quad x \in [a, b] \quad (\text{convention: } V_f(a, a) = 0).$$

Claim: $x \mapsto V_f(a, x)$ and $x \mapsto V_f(a, x) - f(x)$ are increasing on the interval $[a, b]$.
Proof: Let $a \leq x_1 \leq x_2 \leq b$. Lemma 3.58 $\Rightarrow$

$$V_f(a, x_2) = V_f(a, x_1) + \underbrace{V_f(x_1, x_2)}_{\geq 0} \geq V_f(a, x_1)$$

and

$$\begin{aligned} V_f(a,x_2) - f(x_2) - \big(V_f(a,x_1) - f(x_1)\big) &= V_f(a,x_1) + V_f(x_1,x_2) - f(x_2) - V_f(a,x_1) + f(x_1) \\ &= \underbrace{V_f(x_1,x_2)}_{\geq |f(x_2)-f(x_1)|} - \big(f(x_2) - f(x_1)\big) \geq 0. \end{aligned}$$

Thus we may choose $g = V_f(a,\cdot)$ and $h = V_f(a,\cdot) - f$. □

Consequences:

Theorem 3.60. *Let $f\colon [a,b] \to \mathbb{R}$ be of bounded variation. Then*

1. *f has at mst countably many points of discontinuity,*
2. *$\exists f'(x)$ for a.e. $x \in [a,b]$.*
3. *f' on integrable.*

Proof. Theorem 3.59 $\Rightarrow$ $f = g - h$, where g and h are increasing.
Lemma 3.41 $\Rightarrow$ g and h have at most countably many points of discontinuity, and therefore the same holds for f.
Theorem 3.42 $\Rightarrow$ $\exists\ g'(x),\ h'(x)$ for a.e. $x \in [a,b]$ $\Rightarrow$ $\exists f'(x) = g'(x) - h'(x)$ for a.e. $x \in [a,b]$.
Furthermore, $|f'(x)| \leq |g'(x)| + |h'(x)|$ for a.e. $x \in [a,b]$, and $|g'|$ and $|h'|$ are integrable (L. 3.51), hence f' is integrable. □

The next result concerning "integral functions" will be very useful in Section 3.67. We also get more examples of functions of bounded variation.

Theorem 3.61. *Let $f\colon [a,b] \to \mathbb{R}$ be integrable, $f \in L^1\big([a,b]\big)$, and*

$$F(x) = \int_a^x f(t)dt, \quad x \in [a,b].$$

Then F is of bounded variation on the interval $[a,b]$, and the variation of F is

$$V_F(a,b) = \int_a^b |f(t)|dt = \|f\|_1. \tag{3.62}$$

Proof. Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be an arbitrary division of $[a,b]$. Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n \Big| \int_a^{x_i} f(t)dt - \int_a^{x_{i-1}} f(t)dt \Big| = \sum_{i=1}^n \Big| \int_{x_{i-1}}^{x_i} f(t)dt \Big| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(t)|dt = \int_a^b |f(t)|dt. \end{aligned}$$

Taking the sup over all divisions of $[a,b]$ we obtain

$$V_F(a,b) \leq \int_a^b |f(t)|dt = \|f\|_1 < \infty. \tag{3.63}$$

Hence F is of bounded variation.
Converse inequality: $\boxed{\text{A.}}$ Let $g\colon [a,b] \to \mathbb{R}$ be continuous (and thus integrable) and

$$G(x) = \int_a^x g(t)dt.$$

Let $\varepsilon > 0$ be arbitrary. Since g is uniformly continuous ($[a,b]$ closed interval), $\exists \delta > 0$ s.t.

$$\left.\begin{array}{l} |x-y| < \delta \\ x, y \in [a,b] \end{array}\right\} \Rightarrow |g(x) - g(y)| < \varepsilon.$$

Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a division of $[a,b]$ s.t. $|x_i - x_{i-1}| < \delta$ $\forall i = 1, 2, \ldots, n$.
Applying $\triangle$-inequality (several times) we get $\forall i = 1, 2, \ldots, n$

$$\begin{aligned}
|G(x_i) - G(x_{i-1})| &= \Big|\int_{x_{i-1}}^{x_i} g(t)dt\Big| = \Big|\int_{x_{i-1}}^{x_i} \big(g(t) - g(x_{i-1})\big)dt + (x_i - x_{i-1})g(x_{i-1})\Big| \\
&\geq (x_i - x_{i-1})|g(x_{i-1})| - \Big|\int_{x_{i-1}}^{x_i} \big(g(t) - g(x_{i-1})\big)dt\Big| \\
&\geq (x_i - x_{i-1})|g(x_{i-1})| - \int_{x_{i-1}}^{x_i} \underbrace{|g(t) - g(x_{i-1})|}_{<\varepsilon} dt \\
&\geq \int_{x_{i-1}}^{x_i} |g(x_{i-1})|dt - \varepsilon(x_i - x_{i-1}) \\
&\geq \int_{x_{i-1}}^{x_i} \big(|g(t)| - |g(t) - g(x_{i-1})|\big)dt - \varepsilon(x_i - x_{i-1}) \\
&= \int_{x_{i-1}}^{x_i} |g(t)|dt - \int_{x_{i-1}}^{x_i} \underbrace{|g(t) - g(x_{i-1})|}_{<\varepsilon} dt - \varepsilon(x_i - x_{i-1}) \\
&\geq \int_{x_{i-1}}^{x_i} |g(t)|dt - 2\varepsilon(x_i - x_{i-1}).
\end{aligned}$$

Taking the sum over $i = 1, 2, \ldots, n$ $\Rightarrow$

$$\begin{aligned}
V_G(a,b) \overset{\sup}{\geq} \sum_{i=1}^n |G(x_i) - G(x_{i-1})| &\geq \underbrace{\sum_{i=1}^n \int_{x_{i-1}}^{x_i} |g(t)|dt}_{=\int_a^b |g(t)|dt} - 2\varepsilon \underbrace{\sum_{i=1}^n (x_i - x_{i-1})}_{=b-a} \\
&= \|g\|_1 - 2\varepsilon(b-a).
\end{aligned}$$

$$\left.\begin{array}{r} \varepsilon > 0 \text{ mv.} \quad \Rightarrow V_G(a,b) \geq \|g\|_1 \\ (3.63) \end{array}\right\} \Rightarrow V_G(a,b) = \|g\|_1 \tag{3.64}$$

if $g\colon [a,b] \to \mathbb{R}$ is continuous.

$\boxed{\text{B.}}$ General case $f \in L^1\big([a,b]\big)$: We set $f(x) = 0$ $\forall x \in \mathbb{R} \setminus [a,b]$, and so $f \in L^1(\mathbb{R})$.
Theorem 2.36 implies that for all $\varepsilon > 0$ there exists $g \in C(\mathbb{R})$ s.t. $\|f - g\|_1 < \varepsilon$.
Denote $g_0 = g|[a,b]$, hence in particular

$$\|f - g_0\|_1 = \int_a^b |f(t) - g(t)|dt \leq \|f - g\|_1 < \varepsilon.$$

Define

$$G_0(x) = \int_a^x g_0(t)dt, \quad x \in [a,b].$$

Then

$$V_{G_0}(a,b) = V_{G_0-F+F}(a,b) \overset{\triangle\text{-ineq.}}{\leq} \underbrace{V_{G_0-F}(a,b)}_{\underset{(3.63)}{\leq} \|g_0-f\|_1<\varepsilon} + V_F(a,b) < \varepsilon + V_F(a,b)$$

$$\Rightarrow \; V_F(a,b) \geq V_{G_0}(a,b) - \varepsilon \overset{(3.64)}{=} \|g_0\|_1 - \varepsilon = \|g_0 - f + f\|_1 - \varepsilon$$
$$\overset{\text{Minkowski}}{\geq} \|f\|_1 - \underbrace{\|g_0 - f\|_1}_{<\varepsilon} - \varepsilon > \|f\|_1 - 2\varepsilon.$$

$$\left.\begin{array}{l} \varepsilon > 0 \text{ arbitr.} \;\; \Rightarrow \; V_F(a,b) \geq \|f\|_1 \\ (3.63) \end{array}\right\} \;\; \Rightarrow \; V_F(a,b) = \|f\|_1.$$

□

Example 3.65. Let $F\colon [0,1] \to \mathbb{R}$,

$$F(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } 0 < x \leq 1, \\ 0, & \text{if } x = 0. \end{cases}$$

Then *there exists no* Lebesgue integrable function $f\colon [0,1] \to \mathbb{R}$ s.t.

$$F(x) = \int_a^x f(t)dt, \quad x \in [a,b], \tag{3.66}$$

i.e. $x \mapsto x \sin \frac{1}{x}$ is not an "integral function".
Reason: For instance, 3.56.2 $\Rightarrow$ F is *not* of bounded variation. On the other hand, if there were a function $f \in L^1\big([0,1]\big)$ for which (3.66) holds, then by Theorem 3.61 F would be of bounded variation.

3.67 Absolutely continuous functions

If $f\colon [a,b] \to \mathbb{R}$ is a differentiable function whose derivative f' is Riemann integrable over an interval $[a,b]$ (ex. if f' continuous), then

$$f(x) = f(a) + \int_a^x f'(t)dt \quad \forall x \in [a,b]. \tag{3.68}$$

Question: more general version by using Lebesgue integral?

Let $f\colon [a,b] \to \mathbb{R}$ be a function such that the derivative $f'(x)$ exists for a.e. $x \in [a,b]$ and f' integrable. In this section we study the question for which functions (3.68) holds. Example 3.1.4: (3.68) does not hold for the Cantor 1/3-function, and so we need an extra condition.

Definition 3.69. A function $f\colon [a,b] \to \mathbb{R}$ is *absolutely continuous* (on $[a,b]$) if for all $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$\sum_{j=1}^{k} |f(b_j) - f(a_j)| < \varepsilon$$

whenever $]a_1, b_1[, \dots,]a_k, b_k[\subset [a,b]$ are disjoint and

$$\sum_{j=1}^{k} \ell\big([a_j, b_j]\big) = \sum_{j=1}^{k} (b_j - a_j) < \delta.$$

(Note: the number of intervals (k) is arbitrary but always finite.)

Integral functions are absolutely continuous:

Lemma 3.70. *Let $f\colon [a,b] \to \mathbb{R}$ be integrable and*

$$F(x) = \int_a^x f(t)dt, \quad x \in [a,b].$$

Then F is absolutely continuous.

Proof. Corollary 2.7 ("absolute continuity of integrals") implies that for all $ve > 0$ there exists $\delta > 0$ s.t.

$$E \in \mathrm{Leb}\big([a,b]\big),\ m(E) < \delta \ \Rightarrow\ \int_E |f(t)|dt < \varepsilon. \tag{3.71}$$

If $]a_1, b_1[, \dots,]a_k, b_k[\subset [a,b]$ are disjoint intervals s.t.

$$\sum_{j=1}^{k} (b_j - a_j) \overset{\text{disjoint}}{=} m\Big(\underbrace{\bigcup_{i=1}^{k}]a_i, b_i[}_{=E}\Big) < \delta,$$

then

$$\begin{aligned}\sum_{i=1}^{k} |F(b_i) - F(a_i)| &= \sum_{i=1}^{k} \Big|\underbrace{\int_a^{b_i} f(t)dt - \int_a^{a_i} f(t)dt}_{= \int_{a_i}^{b_i} f(t)dt}\Big| \\ &\le \sum_{i=1}^{k} \int_{a_i}^{b_i} |f(t)|dt \overset{\text{disjoint}}{=} \int_E |f(t)|dt \overset{(3.71)}{<} \varepsilon.\end{aligned}$$

□

Theorem 3.72. *Let $f\colon [a,b] \to \mathbb{R}$ be absolutely continuous. Then:*

1. *f is (uniformly) continuous,*
2. *f is of bounded variation,*
3. *the derivative $f'(x)$ exists for a.e. $x \in [a,b]$.*
4. *f' is integrable.*

Proof. 1. Clear (take $k = 1$ in the definition).

2. Exerc.

3. f absolutely continuous $\overset{2.}{\Rightarrow}$ f of bounded variation $\overset{3.60}{\Rightarrow}$ $\exists f'(x)$ for a.e. $x \in [a, b]$.
4. f absolutely continuous $\overset{2.}{\Rightarrow}$ f of bounded variation $\overset{3.60}{\Rightarrow}$ f' integrable.

□

Example 3.73. 1. $f\colon [0, 1] \to \mathbb{R}$,

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } 0 < x \leq 1, \\ 0, & \text{if } x = 0, \end{cases}$$

is continuous but *not* of bounded variation (Exerc.) $\overset{3.72}{\Rightarrow}$ f *is not* absolutely continuous.

2. Let $f\colon [a, b] \to \mathbb{R}$ be Lipschitz-function, i.e. $\exists L < \infty$ s.t.

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in [a, b].$$

Claim: f Lipschitz $\Rightarrow$ f absolutely continuous.
Proof: Let $\varepsilon > 0$ and $]a_1, b_1[, \ldots,]a_k, b_k[\subset [a, b]$ be disjoint intervals s.t.

$$\sum_{i=1}^{k} (b_i - a_i) < \varepsilon / L.$$

Then

$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| \leq L \sum_{i=1}^{k} (b_i - a_i) < \varepsilon.$$

"Uniqueness theorem"

Theorem 3.74. *If $f\colon [a, b] \to \mathbb{R}$ is absolutely continuous and $f'(x) = 0$ for a.e. $x \in [a, b]$, then f is constant.*

Proof. Let us show that $f(c) = f(a)$ $\forall c \in]a, b[$. Fix $c \in]a, b[$ and denote

$$E = \{x \in]a, c[\colon f'(x) = 0\},$$

hence $m(E) = c - a$. Let $\varepsilon > 0$ be arbitrary and choose $\delta > 0$ as in the definition of absolutely continuity of f. For every $x \in E$ $(\subset]a, c[)$ there exist arbitrary short intervals $[x, x+h] \subset]a, c[$, $h > 0$, s.t.

$$\frac{|f(x + h) - f(x)|}{h} < \varepsilon,$$

and therefore such intervals form a closed Vitali covering of E. The Vitali covering theorem $\Rightarrow$ $\exists$ disjoint intervals $I_j = [x_j, y_j] \subset]a, c[, j \in \mathbb{N}$, s.t.

$$|f(y_j) - f(x_j)| < \varepsilon(y_j - x_j) \tag{3.75}$$

and

$$m(E \setminus \bigcup_{j \in \mathbb{N}} I_j) = 0.$$

The convergence of measures $\Rightarrow$ $\exists k \in \mathbb{N}$ s.t.

$$m(]a, c[\setminus \bigcup_{j=1}^{k} I_j) < \delta,$$

because

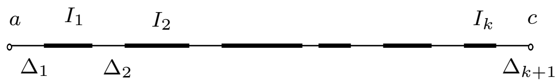
$$\lim_{k\to\infty} m\big(]a,c[\setminus \bigcup_{j=1}^{k} I_j\big) = m\big(]a,c[\setminus \bigcup_{j\in\mathbb{N}} I_j\big) = m\big(E\setminus \bigcup_{j\in\mathbb{N}} I_j\big) = 0.$$

We may assume

$$a < x_1 < y_1 < x_2 < y_2 < \cdots < x_k < y_k < c,$$

and hence $]a,c[\setminus \bigcup_{j=1}^{k} I_j$ is a disjoint unionof open intervals $\Delta_j =]p_j,q_j[,\ j = 1,\dots,k+1$, (see the picture) and

$$\sum_{j=1}^{k+1}(q_j - p_j) = m\big(E\setminus \bigcup_{j=1}^{k} I_j\big) < \delta.$$



We obtain

$$\begin{aligned}
|f(c)-f(a)| &= \Big|\sum_{j=1}^{k}\big(f(y_j)-f(x_j)\big) + \sum_{j=1}^{k+1}\big(f(q_j)-f(p_j)\big)\Big| \\
&\le \sum_{j=1}^{k}\underbrace{|f(y_j)-f(x_j)|}_{\underset{(3.75)}{<}\varepsilon(y_j-x_j)} + \underbrace{\sum_{j=1}^{k+1}|f(q_j)-f(p_j)|}_{\underset{\text{abs. cont.}}{<}\varepsilon} \\
&< \varepsilon\underbrace{\sum_{j=1}^{k}(y_j-x_j)}_{\le c-a} + \varepsilon \\
&\le \varepsilon(c-a+1).
\end{aligned}$$

Letting $\varepsilon\to 0$ we get $f(c) = f(a)$. □

Example 3.76. Let $f\colon [0,1]\to[0,1]$ be the Cantor 1/3-function. L. 3.59 $\Rightarrow$ f of bounded variation (f increasing). On the other hand, $f'(x) = 0$ a.e., but f is not a constant, hence f is not absolutely continuous.

Theorem 3.77. *Let $f\colon [a,b]\to\mathbb{R}$. Then TFAE* [5]

1. *f is absolutely continuous,*
2. *$\exists f'(x)$ for a.e. $x\in[a,b]$, f' is integrable and*

$$f(x) = f(a) + \int_a^x f'(t)dt \quad \forall x\in[a,b],$$

3. *$\exists$ integrable $g\colon [a,b]\to\mathbb{R}$ s.t.*

$$f(x) = f(a) + \int_a^x g(t)dt \quad \forall x\in[a,b].$$

[5]TFAE = "the following are equivalent"

Proof. $\boxed{1. \Rightarrow 2.}$ Suppose that f is absolutely continuous.
3.72 $\Rightarrow$ $\exists f'(x)$ a.e. $x \in [a,b]$, and f' is integrable.
Let
$$F(x) = \int_a^x f'(t)dt, \quad x \in [a,b].$$
Theorem 3.38 $\overset{f' \text{ intva}}{\Longrightarrow}$ $F'(x) = f'(x)$ for a.e. $x \in [a,b]$ $\Rightarrow$ $(F-f)'(x) = 0$ for a.e. $x \in [a,b]$.
$$\left.\begin{array}{c}\text{Lemma 3.70} \Rightarrow F \text{ abs. continuous} \\ \text{assumption: } f \text{ abs. continuous}\end{array}\right\} \overset{\text{easily}}{\Longrightarrow} F - f \text{ abs. continuous.}$$
"Uniqueness theorem" 3.74 $\Rightarrow$ $F - f$ constant, and so
$$F(x) - f(x) = \underbrace{F(a)}_{=0} - f(a) \quad \forall x \in [a,b]$$
$$\Rightarrow f(x) = f(a) + F(x) = f(a) + \int_a^x f'(t)dt \quad \forall x \in [a,b].$$
$\boxed{2. \Rightarrow 3.}$ Choose $g = f'$.
$\boxed{3. \Rightarrow 1.}$ Suppose that $\exists$ integrable $g\colon [a,b] \to \mathbb{R}$ s.t.
$$f(x) = f(a) + \int_a^x g(t)dt \quad \forall x \in [a,b].$$
$$\left.\begin{array}{c}\text{Denote } G(x) = \displaystyle\int_a^x g(t)dt,\ x \in [a,b] \\ \text{assumption: } g \text{ integrable}\end{array}\right\} \overset{3.70}{\Longrightarrow} G \text{ abs. continuous}$$
$$\Rightarrow f = \underbrace{f(a)}_{\text{vakio}} + G \quad \text{abs. continuous.}$$
□

Example 3.78. Let $\alpha > 0$ and
$$f(x) = \begin{cases} x^\alpha \sin \frac{1}{x}, & \text{if } 0 < x \le 1, \\ 0, & \text{if } x = 0. \end{cases}$$
<u>Claim:</u> f is absolutely continuous $\iff$ $\alpha > 1$.
<u>Reason:</u> Example 3.73 $\Rightarrow$ f is not absolutely continuous for $\alpha = 1$. Similarly we may verify that f is not absolutely continuous for $0 < \alpha < 1$.
Suppose $\alpha > 1$: Then:
$$f'(x) = \alpha x^{\alpha-1} \sin \frac{1}{x} - x^{\alpha-2} \cos \frac{1}{x}, \quad 0 < x < 1.$$
Now
$$\lim_{x \to 0+} x^{\alpha-1} \sin \frac{1}{x} = 0 \quad (\text{since } \alpha > 1),$$
and therefore $x \mapsto x^{\alpha-1} \sin \frac{1}{x}$ integrable on the interval $[0,1]$. Furthermore, $h(x) = x^{\alpha-2} \cos \frac{1}{x}$ is integrable on the interval $[0,1]$ since
$$\int_0^1 x^{\alpha-2} \underbrace{|\cos \frac{1}{x}|}_{\le 1} dx \le \int_0^1 x^{\alpha-2} dx = \frac{1}{\alpha - 1}.$$

Hence f' is integrable on the interval $[0,1]$ and

$$f(x) = \int_0^x f'(t)dt, \quad 0 \le x \le 1.$$

Theorem 3.70 $\Rightarrow$ f absolutely continuous for $\alpha > 1$. $\square$

Theorem 3.79. *Let $f\colon [a,b] \to \mathbb{R}$ be increasing. Then*

$$\int_a^b f'(t)dt = f(b) - f(a)$$

if and only if f is absolutely continuous on the interval $[a,b]$.

Proof. Suppose that f is absolutely continuous: Theorem 3.77 $\Rightarrow$

$$\int_a^x f'(t)dt = f(x) - f(a) \quad \forall x \in [a,b].$$

Choosing $x = b$ $\Rightarrow$ claim.
Conversely: Suppose that

$$\int_a^b f'(t)dt = f(b) - f(a). \tag{3.80}$$

Claim: f is absolutely continuous.
Theorem 3.77 $\Rightarrow$ it suffices to prove

$$\int_a^x f'(t)dt = f(x) - f(a) \quad \forall x \in [a,b].$$

Suppose on the contrary: $\exists c \in]a,b[$ s.t.

$$\int_a^c f'(t)dt < f(c) - f(a).$$

Since f is increasing on intervals $[a,c]$ and $[c,b]$, we have

$$\overset{3.51}{\Longrightarrow} \int_a^b f'(t)dt = \underbrace{\int_a^c f'(t)dt}_{<f(c)-f(a)} + \underbrace{\int_c^b f'(t)dt}_{\le f(b)-f(c)} < f(c) - f(a) + f(b) - f(c) = f(b) - f(a).$$

This is a contradiction with the assumption (3.80).
Hence:

$$\int_a^x f'(t)dt = f(x) - f(a) \quad \forall x \in [a,b] \overset{3.77}{\Longrightarrow} f \text{ abs. continuous.}$$

$\square$

Definition 3.81. A function $f\colon [a,b] \to \mathbb{R}$ of bounded variation is *singular* if $f'(x) = 0$ for a.e. $x \in [a,b]$.
(Note: f of bounded variation $\Rightarrow$ $\exists f'(x)$ for a.e. $x \in [a,b]$.)

Example 3.82. 1. Constant functions are singular (and abs. continuous).

2. Cantor 1/3-function is singular.
3. Let $f\colon [a,b] \to \mathbb{R}$ be absolutely continuous. Then

$$f \text{ singular} \quad \Longleftrightarrow \quad f \text{ constant function.}$$

Reason:

$$\left.\begin{array}{r} f'(x) = 0 \text{ a.e.} \\ f \text{ abs. continuous} \end{array}\right\} \overset{3.74}{\Longrightarrow} f(x) \equiv c \text{ constant.}$$

It turns out that every function of bounded variation can be decomposed into an absolutely continuous and a singular part. (Recall: f abs. continuous $\Rightarrow$ f of bounded variation but not conversely.)

Theorem 3.83 (Lebesgue decomposition)**.** *If $f\colon [a,b] \to \mathbb{R}$ is of bounded variation, then*

$$f = g + h,$$

where g is absolutely continuous and h is singular. Furthermore, the decomposition is unique up to an additive constant.

Proof. By Theorem 3.60, $f'(x)$ exists for a.e. $x \in [a,b]$ and, moreover, f' is integrable. Define

$$g(x) = \int_a^x f'(t)dt.$$

Theorem 3.70 $\Rightarrow$ g abs. continuous. Let $h = f - g$. Then h is of bounded variation and

$$\begin{aligned} h'(x) &= (f-g)'(x) = f'(x) - g'(x) \overset{3.38}{=} f'(x) - f'(x) = 0 \text{ a.e. } x \in [a,b] \\ &\Rightarrow h \text{ singular.} \end{aligned}$$

Hence $f = g + h$ is a desired decomposition.
Uniqueness: Suppose that $f = g_1 + h_1 = g_2 + h_2$, where g_1, g_2 are absolutely continuous and h_1, h_2 are singular. Now

$$\left.\begin{array}{ll} w = h_2 - h_1 = g_1 - g_2 & \text{abs. continuous,} \\ w'(x) = h_2'(x) - h_1'(x) = 0 & \text{a.e. } x \in [a,b] \end{array}\right\} \overset{3.74}{\Longrightarrow} w(x) \equiv c \text{ constant function}$$

□

Some additional facts:

1. Let $f\colon [a,b] \to \mathbb{R}$ be absolutely continuous. Then f satisfies *Lusin's condition* (N), that is,

$$E \subset [a,b],\ m(E) = 0 \ \Rightarrow\ m(fE) = 0.$$

2. Another characterization for absolutely continuous functions (see, for instance, [GZ, 7.45]). Let $f\colon [a,b] \to \mathbb{R}$. Then f is absolutely continuous $\Longleftrightarrow$
   (a) f continuous,
   (b) f is of bounded variation,
   (c) $E \subset [a,b],\ m(E) = 0 \ \Rightarrow\ m(fE) = 0$ "condition (N)".
3. Let $f\colon [a,b] \to \mathbb{R}$ be a function s.t. $\exists f'(x)\ \forall x \in [a,b]$ and f' is integrable. Then f s absolutely continuous (see, for instance, [GZ, 7.47]).

THE END

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