Real Analysis I

Fall 2019

Homework 1

Exercise session: Wed 11 September, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. Let $1 < p < \infty$ and p' be the dual exponent defined via $1/p + 1/p' = 1$. Let f be a Lebesgue measurable function in $\mathbb{R}^n$. Prove that
$$\Big(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x\Big)^{1/p} = \sup\Big\{\Big|\int_{\mathbb{R}^n} f(x)g(x)\, \mathrm{d}x\Big| : \Big(\int_{\mathbb{R}^n} |g(x)|^{p'} \, \mathrm{d}x\Big)^{1/p'} \le 1\Big\}.$$
(Remark: the result is also true for $p = 1$ and $p = \infty$, but this is not required here.)

2. Let $p_i \in (0, \infty]$, $i = 1, \ldots, N$, and define p_{N+1} by setting
$$\frac{1}{p_{N+1}} = \sum_{i=1}^{N} \frac{1}{p_i}.$$
Prove that
$$\Big\| \prod_{i=1}^{N} f_i \Big\|_{p_{N+1}} \le \prod_{i=1}^{N} \|f_i\|_{p_i}.$$

3. Let $0 < p_0 < p_1 \le \infty$ and $\theta \in (0, 1)$. Define $p_\theta \in (p_0, p_1)$ by setting
$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$
Prove that
$$\|f\|_{p_\theta} \le \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}.$$
Conclude that $L^{p_0} \cap L^{p_1} \subset L^p$ for all $p \in (p_0, p_1)$. Can you find a more elementary way to prove this inclusion (one that does not need to establish the above estimate)?

4. Assume $f \in L^{p_0}$ for some $p_0 < \infty$. Prove that
$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

5. Let $p \in [1, \infty)$ and $f, f_k \in L^p$. Show that if $f_k(x) \to f(x)$ a.e. and $\|f_k\|_p \to \|f\|_p$, then
$$\|f - f_k\|_p \to 0.$$