Dependence Logic

Exercise 1

Exercise 1. Let $M = \{0, 1, 2\}$ and $\tau = \{R\}$, where R is a binary relation symbol. Give an example of two non-isomorphic models $\mathcal{M}$ and $\mathcal{M}'$ such that $\mathsf{dom}(\mathcal{M}) = \mathsf{dom}(\mathcal{M}') = M$ and $|R^{\mathcal{M}}| = |R^{\mathcal{M}'}|$.

Exercise 2. Let $\tau = \{E\}$, where E is a binary relation symbol. Construct a τ-sentence ϕ of first-order logic such that

$$\mathcal{M} \models \phi \iff E^{\mathcal{M}} \text{ is an equivalence relation,}$$

that is, $E^{\mathcal{M}}$ is symmetric, reflexive, and transitive.

Exercise 3. (1) Let $\tau = \emptyset$ (i.e., τ is the empty vocabulary). Construct a τ-sentence ϕ of first-order logic such that for all τ-models $\mathcal{M}$,

$$\mathcal{M} \models \phi \iff |\mathsf{dom}(\mathcal{M})| = 3.$$

(2) Let $\tau = \{f\}$, where f is a unary function symbol. Construct a τ-sentence ϕ of first-order logic such that for all τ-models $\mathcal{M}$,

$$\mathcal{M} \models \phi \Longrightarrow |\mathsf{dom}(\mathcal{M})| \text{ is infinite.}$$

Exercise 4. Let ϕ and ψ be arbitrary first-order formulas. Prove the following:

(1) $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$,

(2) $\neg\exists x\phi \equiv \forall x\neg\phi$.

Exercise 5. Let $\mathcal{M}$ be a model with $\mathsf{dom}(\mathcal{M}) = \{0, 1, 2\}$. Consider the following team X of $\mathcal{M}$ with domain $\{x, y, z\}$:

	x	y	z
s_0	1	2	2
s_1	2	1	2
s_2	0	1	2

Do the following hold:

(1) $\mathcal{M} \models_X =\!(y, z)$?

(2) $\mathcal{M} \models_X x \subseteq y$?

(3) $\mathcal{M} \models_X xx|yz$?

(4) $\mathcal{M} \models_X y \perp z$?

Exercise 6. For any tuple $\vec{x} = (x_1, \ldots, x_n)$ and team X with $x_1, \ldots, x_n \in \mathsf{dom}(X)$, define

$$X[\vec{x}] := \{s(\vec{x}) : s \in X\},$$

where $s(\vec{x}) = (s(x_1), \ldots, s(x_n))$. Show that for any model $\mathcal{M}$ and team X,

- $\mathcal{M} \models_X \vec{x} \subseteq \vec{y} \iff X[\vec{x}] \subseteq X[\vec{y}]$;
- $\mathcal{M} \models_X \vec{x} \mid \vec{y} \iff X[\vec{x}] \cap X[\vec{y}] = \emptyset$;
- $\mathcal{M} \models_X \vec{x} \perp \vec{y} \iff X[\vec{x}] \times X[\vec{y}] = X[\vec{x}\vec{y}]$.