# Dependence Logic

Exercise 2

**Exercise 1.** Let $\mathbb{N} = (\mathbb{N}, 0, 1, +, \times, <)$ be the model of natural numbers. Let $X$ be the team illustrated in the following table:

| $x$ | $y$ | $z$ |
|---|---|---|
| 2 | 0 | 2 |
| 1 | 1 | 3 |
| 4 | 2 | 2 |

Prove the following:

(1) $\mathbb{N} \models_X x \neq 0$,

(2) $\mathbb{N} \models_X (y < x + 1) \wedge (y + 1 \neq z)$,

(3) $\mathbb{N} \models_X (x = y) \vee {=}(z)$,

(4) $\mathbb{N} \models_X \exists v(v \subseteq y \wedge v < x)$.

**Exercise 2.** Let $\mathcal{M}$ be a model with $\mathsf{dom}(\mathcal{M}) = \{0, 1, 2\}$. Let $X$ be the team illustrated in the following table with domain $\{v\}$ :

| $v$ |
|---|
| 0 |
| 1 |
| 2 |

Show that $\mathcal{M} \models_X \exists x \, {=}(v, x)$, $\mathcal{M} \models_X \forall x(x \subseteq v)$ and $\mathcal{M} \models_X \forall x(x \perp v)$.

**Exercise 3.** Complete the proof of Theorem 3.6.5 in the lecture notes that formulas of inclusion logic (FO($\subseteq$)) are closed under unions by giving the detailed proof for the case $\phi = \forall x \psi$.

**Exercise 4.** (1) Show that $x \perp_z y$ is neither closed downwards nor closed under unions.

(2) Can you find a formula $\phi$ in dependence logic or inclusion logic such that

$$\mathcal{M} \models_X x \perp_z y \iff \mathcal{M} \models_X \phi$$

for all models $\mathcal{M}$ and teams $X$?

**Exercise 5.** Show that there is no formula $\phi$ of dependence logic (FO(=(·))) such that for all models $\mathcal{M}$ and teams $X$,

$$\mathcal{M} \models_X \phi \iff \mathcal{M} \not\models_X {=}(x, y).$$