Real Analysis I

Fall 2019

Homework 2

Exercise session: Wed 18 September, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

In what follows the underlying measure space is $\mathbb{R}^n$ with the Lebesgue measure.

1.
   - Prove that the triangle inequality does not hold for $\|\cdot\|_p$, $0 < p < 1$, but that we do have $\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$.
   - Recall that $\tau_y f(x) = f(x+y)$. Give an example of $f \in L^\infty$ so that $\|\tau_y f - f\|_\infty \not\to 0$ as $y \to 0$.

2. Let $1 \leq p < \infty$ and $f \in L^p$. Prove that

$$\lim_{|y|\to\infty} \|\tau_y f + f\|_p = 2^{1/p}\|f\|_p.$$

3. Let $f, g, h \in L^1$. Prove that then
   (a) $f * g = g * f$;
   (b) $(f * g) * h = f * (g * h)$.

4. Let $1 \leq p, q, r \leq \infty$ satisfy

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}.$$

   Prove that if $f \in L^p$ and $g \in L^q$ we have $f * g \in L^r$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

5. Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = e^{-1/t}$ for $t > 0$ and let $h(t) = 0$ otherwise. Prove that h is smooth (i.e. $h \in C^\infty(\mathbb{R})$).