Real Analysis I

Fall 2019

Homework 1

Exercise session: Wed 11 September, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. Let $1 < p < \infty$ and p' be the dual exponent defined via $1/p + 1/p' = 1$. Let f be a Lebesgue measurable function in $\mathbb{R}^n$. Prove that

$$\Big(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \Big)^{1/p} = \sup \Big\{ \Big| \int_{\mathbb{R}^n} f(x)g(x) \, \mathrm{d}x \Big| : \Big(\int_{\mathbb{R}^n} |g(x)|^{p'} \, \mathrm{d}x \Big)^{1/p'} \le 1 \Big\}.$$

(Remark: the result is also true for $p = 1$ and $p = \infty$, but this is not required here.)

Proof. First, we show "$\ge$". Let $g \in L^{p'}(\mathbb{R}^n)$ such that $\|g\|_{p'} \le 1$. By Hölder's inequality we have

$$\begin{aligned} \Big| \int_{\mathbb{R}^n} f(x)g(x) \, \mathrm{d}x \Big| &\le \int_{\mathbb{R}^n} |f(x)g(x)| \, \mathrm{d}x \\ &\le \Big(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \Big)^{1/p} \Big(\int_{\mathbb{R}^n} |g(x)|^{p'} \, \mathrm{d}x \Big)^{1/p'} \\ &\le \Big(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \Big)^{1/p}. \end{aligned}$$

Then the bound "$\le$". Here we consider separately cases $\|f\|_p = 0$, $0 < \|f\|_p < \infty$ and $\|f\|_p = \infty$. The first one is trivial; $fg = 0$ for almost everywhere (for all measurable g s.t. $\|g\|_{p'} < \infty$), since $f = 0$ for almost everywhere.

The case $0 < \|f\|_p < \infty$. Now, write

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x &= \int_{\mathbb{R}^n} f(x) \cdot f(x)|f(x)|^{p-2} \, \mathrm{d}x \\ &=: \int_{\mathbb{R}^n} f(x)\varphi_f(x) \, \mathrm{d}x, \end{aligned}$$

where $\varphi_f(x) = f(x)|f(x)|^{p-2}$. (If $p < 2$, then set $\varphi_f(x) = 0$ for $x \in \{f = 0\}$). Then observe that

$$\begin{aligned}\|\varphi_f\|_{p'}^{p'} &= \int_{\mathbb{R}^n} |f(x)|^{p'} |f(x)|^{p'(p-2)} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} |f(x)|^{p'(p-1)} \,\mathrm{d}x \\ &= \|f\|_p^p < \infty,\end{aligned}$$

since $p' = p/(p-1)$. Furthermore, we have

$$\|\varphi_f\|_{p'} = \|f\|_p^{p/p'} = \|f\|_p^{p-1}.$$

Using these observations we get

$$\begin{aligned}\|f\|_p = \|f\|_p^p \|f\|_p^{-(p-1)} &= \int f(x) \frac{\varphi_f(x)}{\|\varphi_f\|_{p'}} \,\mathrm{d}x \\ &=: \int f(x) g(x) \,\mathrm{d}x,\end{aligned}$$

where

$$g(x) := \frac{f(x)|f(x)|^{p-2}}{\|f\|_p^{p-1}}$$

and $\|g\|_{p'} = 1$. Hence, we may conclude that not only we get "$\leq$" but we found the specific function to attain the supremum.

The case $\|f\|_p = \infty$. Let $B(\bar{0}, k) =: B_k$ be origin centred ball with radius k. Clearly, $B_k \nearrow \mathbb{R}^n$ and also $F_k := \{|f| < k\} \cap B_k \nearrow \mathbb{R}^n$. Notice that for $k \in (0, \infty)$

$$\|f 1_{F_k}\|_p = \Big(\int |f(x)|^p 1_{F_k}(x) \,\mathrm{d}x \Big)^{1/p} \leq k |B_k|^{1/p} < \infty,$$

where 1_{F_k} is the characteristic function of the set F_k.

Thus, by the previous case there exists a function $g_k := g 1_{F_k}$ such that $\|g_k\|_{p'} = 1$ and

$$\|f 1_{F_k}\|_p = \int f 1_{F_k} g_k \leq \sup_{g : \|g\|_{p'} \leq 1} \Big| \int fg \Big|.$$

The above inequality is by definitions since the p' norm of $g_k 1_{F_k}$ equal to 1. By monotone convergence theorem $\|f 1_{F_k}\|_p \to \|f\|_p = \infty$ this implies that

$$\sup_{g : \|g\|_{p'} \leq 1} \Big| \int fg \Big| = \infty.$$

$\square$

2. Let $p_i \in (0, \infty]$, $i = 1, \dots, N$, and define p_{N+1} by setting

$$\frac{1}{p_{N+1}} = \sum_{i=1}^{N} \frac{1}{p_i}.$$

Prove that

$$\Big\| \prod_{i=1}^{N} f_i \Big\|_{p_{N+1}} \le \prod_{i=1}^{N} \|f_i\|_{p_i}.$$

Proof. Let $p_i \in (0, \infty)$. We prove this using induction. Let us begin with the base case $N = 2$. Hence,

$$\begin{aligned} \frac{1}{p_{N+1}} &= \frac{1}{p_1} + \frac{1}{p_2} \\ \Leftrightarrow \qquad 1 &= \frac{p_{N+1}}{p_1} + \frac{p_{N+1}}{p_2} \end{aligned}$$

Using above as the Hölder exponents we have

$$\begin{aligned} &\left(\int |f_1|^{p_{N+1}} |f_2|^{p_{N+1}} \right)^{1/p_{N+1}} \\ &\le \left(\int |f_1|^{p_1} \right)^{1/p_1} \left(\int |f_2|^{p_2} \right)^{1/p_2}, \end{aligned}$$

which proves the base case.

Now we assume that

$$\Big\| \prod_{i=1}^{N} f_i \Big\|_{q_{N+1}} \le \prod_{i=1}^{N} \|f_i\|_{q_i}.$$

holds for some $N \ge 2$ and q_{N+1} defined via

$$\frac{1}{q_{N+1}} = \sum_{i=1}^{N} \frac{1}{q_i}.$$

For exponents

$$1 = \frac{p_{N+2}}{p_{N+1}} + \sum_{i=1}^{N} \frac{p_{N+2}}{p_i} =: \frac{p_{N+2}}{p_{N+1}} + \frac{p_{N+2}}{q_{N+1}}$$

Hölder's inequality gives us

$$\begin{aligned}
&\Big\| \prod_{i=1}^{N+1} f_i \Big\|_{p_{N+2}} \\
&= \Big(\int |f_{N+1}|^{p_{N+2}} \Big| \prod_{i=1}^{N} f_i \Big|^{p_{N+2}} \Big)^{1/p_{N+2}} \\
&\leq \Big(\int |f_{N+1}|^{p_{N+1}} \Big)^{1/p_{N+1}} \Big(\int \Big| \prod_{i=1}^{N+1} f_i \Big|^{q_{N+1}} \Big)^{1/q_{N+1}} \\
&= \|f_{N+1}\|_{p_{N+1}} \Big\| \prod_{i=1}^{N} f_i \Big\|_{q_{N+1}}.
\end{aligned}$$

Since q_{N+1} is defined via

$$\frac{1}{q_{N+1}} = \sum_{i=1}^{N} \frac{1}{p_i},$$

by the induction assumption we get

$$\Big\| \prod_{i=1}^{N+1} f_i \Big\|_{p_{N+2}} \leq \|f_{N+1}\|_{p_{N+1}} \Big\| \prod_{i=1}^{N} f_i \Big\|_{q_{N+1}} \leq \|f_{N+1}\|_{p_{N+1}} \prod_{i=1}^{N} \|f_i\|_{p_i} = \prod_{i=1}^{N+1} \|f_i\|_{p_i}.$$

Then the case where we allow $p_i = \infty$. Now, suppose $p_N = \infty$ and $p_i \in (0, \infty)$. I.e. we have $1/p_N = 0$ hence

$$\frac{1}{p_{N+1}} = \sum_{i=1}^{N-1} \frac{1}{p_i}.$$

By above proof we get

$$\Big\| \prod_{i=1}^{N+1} f_i \Big\|_{p_{N+1}} \leq \|f_N\|_{\infty} \Big\| \prod_{i=1}^{N-1} f_i \Big\|_{p_{N+1}} \leq \prod_{i=1}^{N+1} \|f_i\|_{p_i}.$$

Rest follows similarly. $\square$

3. Let $0 < p_0 < p_1 \leq \infty$ and $\theta \in (0,1)$. Define $p_\theta \in (p_0, p_1)$ by setting

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Prove that

$$\|f\|_{p_\theta} \le \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}.$$

Conclude that $L^{p_0} \cap L^{p_1} \subset L^p$ for all $p \in (p_0, p_1)$. Can you find a more elementary way to prove this inclusion (one that does not need to establish the above estimate)?

Proof. Let $0 < p_0 < p_1 < \infty$ and $\theta \in (0,1)$. Define $p_\theta \in (p_0, p_1)$ by setting

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Observe that above can be written in the form of Hölder conjugates

$$1 = p_\theta \cdot \frac{1-\theta}{p_0} + p_\theta \cdot \frac{\theta}{p_1}.$$

By Hölder's inequality with above exponents, we get

$$\begin{aligned}
\left(\int |f|^{p_\theta} \right)^{1/p_\theta} &= \left(\int |f|^{p_\theta(1-\theta)} |f|^{p_\theta \theta} \right)^{1/p_\theta} \\
&\le \left(\int |f|^{p_0} \right)^{(1-\theta)/p_0} \left(\int |f|^{p_1} \right)^{\theta/p_1} \\
&= \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}.
\end{aligned}$$

If $p_1 = \infty$, then we have

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0}$$

and

$$\begin{aligned}
\left(\int |f|^{p_\theta} \right)^{1/p_\theta} &= \left(\int |f|^{p_\theta(1-\theta)} |f|^{p_\theta \theta} \right)^{1/p_\theta} \\
&= \left(\int |f|^{p_0} |f|^{p_\theta \theta} \right)^{1/p_\theta} \\
&\le \left(\int |f|^{p_0} \right)^{(1-\theta)/p_0} \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \{ |f(x)|^{p_\theta \theta} \}^{1/p_\theta} \\
&= \|f\|_{p_0}^{(1-\theta)} \|f\|_{p_1}^{\theta}.
\end{aligned}$$

The more elementary approach: Let $f \in L^{p_0} \cap L^{p_1}$ for $0 < p_0 < p_1 < \infty$ (for $p_1 = \infty$ adjust the proof slightly, we omit the details) and let $p \in (p_0, p_1)$. Split

$f = f_1 + f_2 := f1_{\{|f|\leq 1\}} + f1_{\{|f|>1\}}$, and notice that

$$|f|^p \leq 2^p(|f_1|^p + |f_2|^p) \leq 2^p(|f_1|^{p_0} + |f_2|^{p_1})$$

$$\Rightarrow \qquad \|f\|_p^p \leq 2^p(\|f_1\|_{p_0}^{p_0} + \|f_2\|_{p_1}^{p_1})$$

Hence, by monotonicity

$$\|f\|_p \leq 2(\|f\|_{p_0}^{p_0} + \|f\|_{p_1}^{p_1})^{\frac{1}{p}} < \infty.$$

□

4. Assume $f \in L^{p_0}$ for some $p_0 < \infty$. Prove that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Proof. "$\leq$" Without loss of generality, we may assume homogeneity i.e. $\|f\|_{p_0} = 1$. Also, we may assume that $\|f\|_\infty < \infty$. Using the result of the previous exercise, for all $p > p_0$ (and $p < \infty$) we have

$$\|f\|_p \leq \|f\|_{p_0}^{p_0/p} \|f\|_\infty^{(p-p_0)/p} = \|f\|_\infty^{(p-p_0)/p} = \|f\|_\infty^{1-1/p}.$$

Taking the limit leads us to the bound

$$\lim_{p\to\infty} \|f\|_p \leq \|f\|_\infty.$$

Lastly, we need to prove the bound "$\geq$". We may assume that that the limit $\lim_{p\to\infty} \|f\|_p$ exists and $\lim_{p\to\infty} \|f\|_p < \infty$. First, suppose $\|f\|_\infty < \infty$. Let $\varepsilon > 0$. Set $A_\varepsilon = \{x \in \mathbb{R}^d \colon |f(x)| > \|f\|_\infty - \varepsilon\}$ and notice that $0 < \mu(A_\varepsilon) < \infty$ for all $\varepsilon < \|f\|_\infty$. Hence, we have

$$\|f\|_p \geq \Big(\int_{A_\varepsilon} |f|^p \Big)^{1/p} \geq \mu(A_\varepsilon)^{1/p}(\|f\|_\infty - \varepsilon).$$

Recall that

$$\lim_{p\to\infty} \|f\|_p \mu(A_\varepsilon)^{-1/p} = \lim_{p\to\infty} \mu(A_\varepsilon)^{-1/p} \lim_{p\to\infty} \|f\|_p = \lim_{p\to\infty} \|f\|_p$$

for all ε. This implies

$$\lim_{p\to\infty} \|f\|_p \geq \|f\|_\infty - \varepsilon$$

for all ε. Hence, we get

$$\lim_{p\to\infty} \|f\|_p \geq \|f\|_\infty.$$

Then, let us assume that $\|f\|_\infty = \infty$ and still assuming that the limit exists. As shown above for all $t > 0$ we have

$$\lim_{p\to\infty} \|f\|_p \geq \lim_{p\to\infty} \|f^t\|_p = \|f^t\|_\infty,$$

where $f^t = f 1_{\{|f|<t\}}$. By monotonicity we get that $\lim_{t\to\infty} \|f^t\|_\infty = \|f\|_\infty = \infty$. We conclude that

$$\lim_{p\to\infty} \|f\|_p = \infty.$$

□

5. Let $p \in [1, \infty)$ and $f, f_k \in L^p$. Show that if $f_k(x) \to f(x)$ a.e. and $\|f_k\|_p \to \|f\|_p$, then

$$\|f - f_k\|_p \to 0.$$

Proof. Recall that

$$|f_k - f|^p \leq 2^p(|f_k|^p + |f|^p)$$

is equivalent with

$$0 \leq 2^p(|f_k|^p + |f|^p) - |f_k - f|^p.$$

By Fatou's lemma we have

$$0 \leq \int \lim_{k\to\infty} 2^p(|f_k|^p + |f|^p) - |f_k - f|^p \leq 2^{p+1}\|f\|_p^p + \liminf_{k\to\infty} \int -|f_k - f|^p.$$

Since

$$0 \leq 2^{p+1}\|f\|_p^p \leq 2^{p+1}\|f\|_p^p - \limsup_{k\to\infty} \int |f_k - f|^p,$$

we need to have

$$0 \leq \liminf_{k\to\infty} \int |f_k - f|^p \leq \limsup_{k\to\infty} \int |f_k - f|^p = 0,$$

that is, we have

$$\|f - f_k\|_p \to 0,$$

as desired.

□