Real Analysis I

Fall 2019

Homework 3

Exercise session: Wed 25 September, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. Let $\varphi \in L^1(\mathbb{R}^n)$ and $\varphi_\epsilon(x) := \frac{1}{\epsilon^n}\varphi(x/\epsilon)$, $\epsilon > 0$. Show that for all $\delta > 0$ we have
$$\lim_{\epsilon \to 0} \int_{|x| \geq \delta} |\varphi_\epsilon(x)| \, \mathrm{d}x = 0.$$
Conclude that if also $\int \varphi = 1$, then $(\varphi_\epsilon)_{\epsilon > 0}$ is an approximate identity.

2. Let $K \subset \mathbb{R}^n$ be compact, $V \subset \mathbb{R}^n$ be open and $K \subset V$. Show that there exists a smooth 'cut-off' function $\varphi \in C_c^\infty(V)$ so that $0 \leq \varphi \leq 1$, $\varphi = 1$ on K and
$$|\nabla \varphi(x)| \leq \frac{C}{d(K, \partial V)}.$$

3. Let (X, μ) be a measure space. For a measurable $f \colon X \to \mathbb{R}$ prove that we have the identity
$$\int_X |f|^p \, \mathrm{d}\mu = p \int_0^\infty \lambda^{p-1} \mu(\{x \in X \colon |f(x)| > \lambda\}) \, \mathrm{d}\lambda, \qquad 0 < p < \infty.$$

4. Give a proof of the case, where $p_1 = \infty$ in the Marcinkiewicz interpolation theorem.

5.
   - A function $g \colon \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous if $\{g > \lambda\}$ is open for all $\lambda \in \mathbb{R}$. Prove that Mf, $f \in L^1_{\mathrm{loc}}$, is measurable by showing that it is lower semicontinuous.
   - Prove the following refinement of the estimate $M \colon L^1 \to L^{1,\infty}$. Show that for all $f \in L^1$ and $\lambda > 0$ we have
   $$|\{x \colon Mf(x) > \lambda\}| \lesssim \frac{1}{\lambda} \int_{|f| > \lambda/2} |f|.$$