Dependence Logic

Exercise 3

Exercise 1. Show that for any downwards closed formulas ϕ and ψ,

$$\mathcal{M} \models_X \phi \vee \psi \iff \mathcal{M} \models^s_X \phi \vee \psi$$

for any model $\mathcal{M}$ and team X. In other words, the lax and strict semantics for disjunction coincide in case both disjuncts are downwards closed.

Exercise 2. Verify the following clauses for pure independence atoms (known as the *Geiger-Paz-Pearl Axioms*):

(1) $\vec{x} \perp \vec{y} \models \pi(\vec{x}) \perp \rho(\vec{y})$, where π is a permutation of $\vec{x}$, and ρ is a permutation of $\vec{y}$ (permutation)

(2) $\vec{x}z \perp \vec{y} \models \vec{x}zz \perp \vec{y}$ (redundancy)

(3) $\models \vec{x} \perp \langle\rangle$ (empty tuple rule).

(4) $\vec{x} \perp \vec{y} \models \vec{y} \perp \vec{x}$ (symmetry rule).

(5) $\vec{x} \perp \vec{y}\vec{z} \models \vec{x} \perp \vec{y}$ (weakening rule)

(6) $\vec{x} \perp \vec{y}, \vec{x}\vec{y} \perp \vec{z} \models \vec{x} \perp \vec{y}\vec{z}$ (exchange rule).

(7) $\vec{x} \perp \vec{x}, \vec{y} \perp \vec{z} \models \vec{x}\vec{y} \perp \vec{z}$ (constancy rule).

Give detailed proofs for items (6) and (7).

Exercise 3. (1) Prove that $\phi \wedge (\psi \vee \chi) \models (\phi \wedge \psi) \vee (\phi \wedge \chi)$ whenever ϕ is downwards closed.

(2) Prove that $(\phi \wedge \psi) \vee (\phi \wedge \chi) \models \phi \wedge (\psi \vee \chi)$ whenever ϕ is closed under unions.

Exercise 4. (1) Give an example of three formulas ϕ, ψ, χ in inclusion logic $\mathrm{FO}(\subseteq)$ such that $\phi \wedge (\psi \vee \chi) \not\models (\phi \wedge \psi) \vee (\phi \wedge \chi)$.

(2) Give an example of three formulas ϕ, ψ, χ in dependence logic $\mathrm{FO}(=(\dots))$ such that $(\phi \wedge \psi) \vee (\phi \wedge \chi) \not\models \phi \wedge (\psi \vee \chi)$.

Exercise 5. Suppose $x \notin Fv(\psi)$. Prove that $\exists x(\phi \wedge \psi) \equiv \exists x\phi \wedge \psi$ and $\exists x(\phi \vee \psi) \equiv \exists x\phi \vee \psi$.