Real Analysis I

Fall 2019

Homework 2

Exercise session: Wed 18 September, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

In what follows the underlying measure space is $\mathbb{R}^n$ with the Lebesgue measure.

1.
- Prove that the triangle inequality does not hold for $\|\cdot\|_p$, $0 < p < 1$, but that we do have $\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$.
- Recall that $\tau_y f(x) = f(x+y)$. Give an example of $f \in L^\infty$ so that $\|\tau_y f - f\|_\infty \not\to 0$ as $y \to 0$.

Proof. (i) Let $0 < p < 1$. Hence $1/p \in (1, \infty)$. Thus, for $a, b \geq 0$ we have

$$\begin{aligned} & a + b = a^{p/p} + b^{p/p} \leq (a^p + b^p)^{1/p} \\ \Leftrightarrow \qquad & (a+b)^p \leq a^p + b^p, \end{aligned}$$

since for $f\colon \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = x^q$, where $q > 1$, holds $f(x) + f(y) \leq f(x+y)$.
Hence, $\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$ is always true for $0 < p < 1$.
On the otherhand, Let $n = 1$, and take $f = 1_{[0,\frac{1}{2}]}$ and $g = 1_{[\frac{1}{2},1]}$. Then $\|f+g\|_p = 1$ for all p and $\|f\|_p + \|g\|_p = 2^{-1/p} + 2^{-1/p} = 2^{1-1/p}$. Now, since $p \in (0,1)$, we have $1/p > 1 \Leftrightarrow 1 - 1/p < 0$. This means $0 < 2^{1-1/p} < 1$, hence

$$\|f+g\|_p = 1 > 2^{1-1/p} = \|f\|_p + \|g\|_p.$$

(ii) Choose

$$f(x) = \delta_{x_0}(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}.$$

Clearly $f \in L^\infty$. In addition, $|\tau_y f(x) - f(x)| = 1$ if $x = x_0$ or $x = x_0 - y$ for all $y \in \mathbb{R}^n$, otherwise $|\tau_y f(x) - f(x)| = 0$. Thus, $\|\tau_y f - f\|_\infty = 1$ for all y, that is $\|\tau_y f - f\|_\infty \to 1$ as $y \to 0$.

□

2. Let $1 \leq p < \infty$ and $f \in L^p$. Prove that

$$\lim_{|y|\to\infty} \|\tau_y f + f\|_p = 2^{1/p}\|f\|_p.$$

Proof. First, let us prove this for $g \in C_c$. Continuity is not needed and you may choose $g = f1_K$, for some compact set K. Then use a convergence theorem to obtain the result for f. However, we present the more standard proof using the approximation. Denote $K = \operatorname{spt} g$ and define for $y \in \mathbb{R}^n$, $K_y = \{x \colon x + y \in K\}$.

Thus, we have

$$\|\tau_y g + g\|_p^p = \int_{K\cap K_y} |\tau_y g + g|^p + \int_{(K\cap K_y)^c} |\tau_y g - g|^p =: I_1 + I_2.$$

For large enough $|y|$ it is quite obvious that $K\cap K_y = \emptyset$, i.e. $\lim_{|y|\to\infty} I_1 = 0$, hence we do not need to worry about the term I_1. For the term I_2 we further write

$$\begin{aligned} I_2 &= \int_{K^c\cap K_y} |\tau_y g + g|^p + \int_{K\cap K_y^c} |\tau_y g + g|^p + \int_{K^c\cap K_y^c} |\tau_y g + g|^p \\ &= \int_{K_y} |\tau_y g|^p + \int_K |g|^p = \|\tau_y g\|_p^p + \|g\|_p^p \end{aligned}$$

by definition of the sets K and K_y. Now, since $\|\tau_y g\|_p^p = \|g\|_p^p$ we have proven the claim

$$\lim_{|y|\to\infty} \|\tau_y g + g\|_p = 2^{1/p}\|g\|_p.$$

Then, we prove above result for $f \in L^p$. Let $\varepsilon > 0$ and choose $g \in C_c$ such that $\|f - g\|_p < \varepsilon$. Hence, using Minkowski's inequality we get

$$\begin{aligned} \|\tau_y f + f\|_p - 2^{1/p}\|f\|_p &\leq \|\tau_y f - \tau_y g\|_p + \|f - g\|_p - 2^{1/p}\|f - g\|_p \\ &\quad + \|\tau_y g + g\|_p - 2^{1/p}\|g\|_p \\ &\leq (2 - 2^{1/p})\varepsilon + \|\tau_y g + g\|_p - 2^{1/p}\|g\|_p. \end{aligned}$$

For large enough $|y|$ we get

$$\|\tau_y f + f\|_p - 2^{1/p}\|f\|_p \leq (2 - 2^{1/p})\epsilon,$$

which implies

$$\lim_{|y|\to\infty} \|\tau_y f + f\|_p = 2^{1/p}\|f\|_p.$$

□

3. Let $f, g, h \in L^1$. Prove that then

   (a) $f * g = g * f$;

   (b) $(f * g) * h = f * (g * h)$.

Proof. (a) By change of variable, $\varphi(y) = x - y$, $\det(J_\varphi) = 1$, and $\varphi(\mathbb{R}^n) = \mathbb{R}^n$, we have

$$\begin{aligned} f * g(x) &= \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y = g * f(x). \end{aligned}$$

(b) Using Fubini and change of variable we get

$$\begin{aligned} (f * g) * h(x) &= \int (f * g)(y)h(x-y)\,\mathrm{d}y \\ &= \int \int f(z)g(y-z)\,\mathrm{d}z h(x-y)\,\mathrm{d}y \\ &= \int \int f(z)g(y-z)h(x-y)\,\mathrm{d}y\,\mathrm{d}z \\ &= \int f(z) \int g(y)h(x-z-y)\,\mathrm{d}y\,\mathrm{d}z \\ &= \int f(z)(g * h)(x-z)\,\mathrm{d}z = f * (g * h)(x). \end{aligned}$$

□

4. Let $1 \leq p, q, r \leq \infty$ satisfy

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}.$$

Prove that if $f \in L^p$ and $g \in L^q$ we have $f * g \in L^r$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof. Let $1 < p, q, r < \infty$. Note

$$\frac{1}{r'} = 2 - \frac{1}{p} - \frac{1}{q},$$

where r' satisfies $1 = \frac{1}{r} + \frac{1}{r'}$.

In below, integrations are taken over sets such that there is no division by zero. Since in these sets some function equals to zero and then the integration over such set is zero, we abbreviate this from the notation. By Hölder's inequality we have the pointwise inequality

$$\begin{aligned}
|(f*g)(x)|^r &= |\int f(y)g(x-y)\,\mathrm{d}y|^r \\
&\le \Big(\int |f(y)||g(x-y)|\,\mathrm{d}y\Big)^r \\
&= \Big(\int (|f(y)|^p|g(x-y)|^q)^{\frac{1}{r}}|f(y)|^{1-\frac{p}{r}}|g(x-y)|^{1-\frac{q}{r}}\,\mathrm{d}y\Big)^r \\
&\le \Big(\int |f(y)|^p|g(x-y)|^q\,\mathrm{d}y\Big) \\
&\qquad \times \Big(\int |f(y)|^{(1-\frac{p}{r})r'}|g(x-y)|^{(1-\frac{q}{r})r'}\,\mathrm{d}y\Big)^{r/r'}.
\end{aligned}$$

Since

$$r'\Big(\frac{1-\frac{p}{r}}{p}+\frac{1-\frac{q}{r}}{q}\Big) = r'\Big(\frac{1}{p}-\frac{1}{r}+\frac{1}{q}-\frac{1}{r}\Big) = r'\Big(2-\frac{1}{p}-\frac{1}{q}\Big) = 1,$$

by Hölder's inequality we get

$$\Big(\int |f(y)|^{(1-\frac{p}{r})r'}|g(x-y)|^{(1-\frac{q}{r})r'}\,\mathrm{d}y\Big)^{r/r'} \le \|f\|_p^{r-p}\|\|g\|_q^{r-q}$$

Hence, the pointwise inequality attains the form

$$|(f*g)(x)|^p \le \Big(\int |f(y)|^p|g(x-y)|^q\,\mathrm{d}y\Big)\|f\|_p^{r-p}\|\|g\|_q^{r-q}.$$

Thus, using Fubini's theorem we estimate

$$\begin{aligned}
\|f*g\|_r^r &\le \int\int |f(y)|^p|g(x-y)|^q\,\mathrm{d}y\,\mathrm{d}x\|f\|_p^{r-p}\|\|g\|_q^{r-q} \\
&= \int |f(y)|^p\int |g(x-y)|^q\,\mathrm{d}x\,\mathrm{d}y\|f\|_p^{r-p}\|\|g\|_q^{r-q} \\
&= \|f\|_p^p\|\|g\|_q^q\|f\|_p^{r-p}\|\|g\|_q^{r-q} \\
&= \|f\|_p^r\|\|g\|_q^r
\end{aligned}$$

$$\Leftrightarrow \qquad \|f*g\|_r \le \|f\|_p\|\|g\|_q.$$

Now, if $r = \infty$, then $|(f * g)(x)| \leq \|f\|_p \|g\|_q$ for all x and all $p, q \in [1, \infty]$ s.t. $1 = 1/p + 1/q$. Hence, the result follows.

If $r = 1$, then $p = 1$ and $q = 1$, and the result follows by Fubini.

If $p = 1$, then $r = q$. Then the result is obtained by

$$\begin{aligned} \|f * g\|_q &= \Big| \sup_{h \in L^{q'} :\, \|h\| \leq 1} \int \int f(y) g(x-y) h(x) \, \mathrm{d}y \, \mathrm{d}x \Big| \\ &\leq \sup \int |f(y)| \|g\|_q \|h\|_{q'} \, \mathrm{d}y \leq \|f\|_1 \|g\|_q. \end{aligned}$$

If $q = 1$ is similar to above one.

Lastly, we consider $p = \infty$ or $q = \infty$. Hence, assume $q = \infty, p, r \in (1, \infty)$. Then,

$$\frac{1}{p} = 1 + \frac{1}{r} = \frac{r+1}{r} \Rightarrow p = \frac{r}{r+1} > 1$$

which is a contradiction since $r \leq r + 1$ for all $r \geq 0$. Similar contradiction if $q = \infty$. $\square$

5. Let $h \colon \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = e^{-1/t}$ for $t > 0$ and let $h(t) = 0$ otherwise. Prove that h is smooth (i.e. $h \in C^\infty(\mathbb{R})$).

Proof. Take the derivative of $e^{-1/t}$ couple of times and you should see some pattern. Hence, we claim

$$\frac{\mathrm{d}^n}{\mathrm{d}t} e^{-1/t} = e^{-1/t} \frac{\sum_{i=0}^{n-1} a_i t^i}{t^{2n}} = e^{-1/t} \sum_{i=0}^{n-1} a_i t^{i-2n}$$

for $t > 0$ and $n = 1, 2, \ldots$, where $a_i \in \mathbb{R}$.

First, the base case $n = 1$.

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-1/t} = e^{-1/t} \frac{\mathrm{d}}{\mathrm{d}t}(-t^{-1}) = e^{-1/t} t^{-2}.$$

Now, suppose that

$$\frac{\mathrm{d}^n}{\mathrm{d}t} e^{-1/t} = e^{-1/t} \frac{\sum_{i=0}^{n-1} a_i t^i}{t^{2n}} = e^{-1/t} \sum_{i=0}^{n-1} a_i t^{i-2n}$$

holds for some $n \in \mathbb{N}$. Hence, by rules of derivation

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}t} e^{-1/t} &= \frac{\mathrm{d}}{\mathrm{d}t} (e^{-1/t} \sum_{i=0}^{n-1} a_i t^{i-2n}) \\
&= e^{-1/t} t^{-2} \sum_{i=0}^{n-1} a_i t^{i-2n} + e^{-1/t} \sum_{i=0}^{n-1} a_i \frac{\mathrm{d}}{\mathrm{d}t} t^{i-2n} \\
&= e^{-1/t} \sum_{i=0}^{n-1} a_i t^{i-2n-2} + e^{-1/t} \sum_{i=0}^{n-1} a_i (i-2n) t^{i-2n-1} \\
&= e^{-1/t} \sum_{i=0}^{n-1} a_i t^{i-2(n+1)} + e^{-1/t} \sum_{i=1}^{n} a_{i-1} (i-1-2n) t^{i-2(n+1)} \\
&= e^{-1/t} \sum_{i=1}^{n} b_i t^{i-2(n+1)},
\end{aligned}$$

where $b_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$.

Finally, we should check that h is differentiable in the origin. Since $\lim_{t \to -\infty} e^t P_m(-t) = 0$ is easy to verify for any polynomial P_m, we conclude that

$$\lim_{t \to 0+} e^{-1/t} P_m(1/t) = 0.$$

□