Real Analysis I

Fall 2019

Homework 4

Exercise session: Wed 2 October, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. For a locally integrable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ define the 'rectangular maximal function'

$$M_R f(x_1,x_2) := \sup_{r_1,r_2>0} \frac{1}{2r_1}\int_{x_1-r_1}^{x_1+r_1} \frac{1}{2r_2}\int_{x_2-r_2}^{x_2+r_2} |f(y_1,y_2)|\,\mathrm{d}y_2\,\mathrm{d}y_1.$$

   Show that for all $1<p<\infty$ we have

$$\|M_R f\|_p \lesssim \|f\|_p, \qquad f \in L^p(\mathbb{R}^2).$$

2. Let $\delta \in (0,1)$ and let $E \subset \mathbb{R}^n$ be a measurable set with $|E| < \infty$. Prove that for all $f \in L^1$ we have

$$\int_E [Mf(x)]^\delta \,\mathrm{d}x \le C|E|^{1-\delta}\|f\|_1^\delta,$$

   where $C = C(\delta)$ is a constant.

3. Fix $f \in L^1$ and $\delta \in (0,1)$, and define $w(x) = [Mf(x)]^\delta$. Prove that for almost every x we have

$$Mw(x) \le Cw(x),$$

   where $C = C(\delta)$ is a constant

4. Show that if $E \subset \mathbb{R}^n$ is a measurable set with $|E| < \infty$, then we have for some constant C that

$$\int_E Mf(x)\,\mathrm{d}x \le 2|E| + C\int_{\mathbb{R}^n} |f(x)|\max(\log|f(x)|,0)\,\mathrm{d}x.$$

5. Let (X,μ) be a measure space, $p \in (0,\infty)$, and let T_ϵ, $\epsilon > 0$, be a family of linear operators defined in $L^p(X)$ and taking values in the measurable

functions of X. Assume that there is a dense subset $\mathcal{G} \subset L^p(X)$ so that for all $g \in \mathcal{G}$ we have

$$\lim_{\epsilon \to 0} T_\epsilon g(x) = g(x)$$

for μ-a.e. $x \in X$. For $f \in L^p(X)$ define the related maximal function $T_* f(x) := \sup_{\epsilon > 0} |T_\epsilon f(x)|$. Assume that for all $f \in L^p$ and $\lambda > 0$ we have

$$\mu(\{x \in X : T_* f(x) > \lambda\}) \leq A(\lambda) \|f\|^p_{L^p(X)},$$

where $A \colon (0, \infty) \to (0, \infty)$ is some function.

Prove that for all $f \in L^p(X)$ we have

$$\lim_{\epsilon \to 0} T_\epsilon f(x) = f(x)$$

for μ-a.e. $x \in X$.