MAST31213 Complexity theory
Master's Programme in Mathematics and Statistics
Fall 2019
Exercise set 4

Read chapters 1.4–1.7 of the book.

Exercise 1. (Continuation of last week's second exercise). Show that the set of decidable languages (over a fixed alphabet) is closed under finite unions and intersections.

Exercise 2. Prove the *linear speed up* theorem: If M is a k-tape Turing machine that decides some language $L \subseteq \{0,1\}^*$ in time $T(n)$, then for any $\varepsilon > 0$ there is a k'-tape Turing machine M' that decides L in time $T'(n) = \varepsilon T(n) + n + 5$. Moreover, if $k > 1$ one can choose $k' = k$ (and $k' = 2$ otherwise). (Hint: replace the alphabet of M by m-tuples of symbols and let one step of M' correspond to m steps of M.)

Exercise 3. Give details for representing Turing machines as binary strings, i.e. describe a procedure of transforming a Turing machine M into a binary string $\llcorner M \lrcorner$, as well as 'decoding' the sequence, such that

- it is possible to recover from $\llcorner M \lrcorner$ a Turing machine that is functionally equivalent to M (i.e. computes the same function in the same running time),
- every string in $\{0,1\}^*$ represents some Turing machine,
- every Turing machine is represented by infinitely many strings.

Exercise 4. Using some fixed encoding μ for Turing machines, we can define the *Kolmogorov complexity* of a string $x \in \{0,1\}^*$ with respect to the encoding as

$$K_\mu(x) = \min\{|\llcorner M \lrcorner| : M \text{ computes } x \text{ from the empty input}\}.$$

That is, $K(x)$ is the length of the smallest 'program' in the encoding that outputs x.

(a) Show that there is an encoding μ_0 of Turing machines such that for any other encoding μ there is a constant c such that for all x, $K_{\mu_0}(x) \leq K_\mu(x) + c$.

In view of this result, we are justified to omit reference to μ and speak of the Kolmogorov complexity of x, $K(x)$ (understanding that a constant may be involved).

(b) Show that for all x, $K(x) \leq |x|$.
(c) Show that for all n there are strings x of length n with $K(x) \geq n$. (These are called *incompressible strings*.)