Dependence Logic
Exercise 4

Exercise 1. Let α be a first-order formula. Prove that for any model $\mathcal{M}$ and assignment s,

$$\mathcal{M} \models_{\{s\}} \alpha \iff \mathcal{M} \models_s \alpha.$$

Exercise 2. For any first-order formulas α, β, define $\alpha \to \beta := \neg\alpha \vee \beta$. Prove that

$$\mathcal{M} \models_X \alpha \to \beta \iff \text{for all } Y \subseteq X, \mathcal{M} \models_Y \alpha \text{ implies } \mathcal{M} \models_Y \beta.$$

Exercise 3. Prove that for any model $\mathcal{M}$ and assignment s,

- $\mathcal{M} \models_{\{s\}} =\!(\vec{x}, y)$ and $\mathcal{M} \models_{\{s\}} \vec{x} \perp_{\vec{z}} \vec{y}$ always hold;
- $\mathcal{M} \models_{\{s\}} \vec{x} \subseteq \vec{y} \iff \mathcal{M} \models_s \vec{x} = \vec{y}$;
- $\mathcal{M} \models_{\{s\}} \vec{x} \mid \vec{y} \iff \mathcal{M} \models_s \vec{x} \neq \vec{y}$.

Exercise 4. Let $\phi = \exists x_1 \ldots \exists x_n \theta$, where θ is quantifier-free, be a *sentence* in dependence logic FO(=(...)). Let ϕ^* be the first-order sentence obtained from ϕ by replacing every dependence atom by $\top$. Show that $\phi \equiv \phi^*$.

Exercise 5. Define a new disjunction $\mathbin{\vee\!\!\!\vee}$ as:

- $\mathcal{M} \models_X \phi \mathbin{\vee\!\!\!\vee} \psi$ iff $\mathcal{M} \models_X \phi$ or $\mathcal{M} \models_X \psi$.

Assume that a model always has at least two elements. Prove that

$$\phi \mathbin{\vee\!\!\!\vee} \psi \equiv \exists uv\big(=\!(u) \wedge =\!(v) \wedge \big((u = v \wedge \phi) \vee (u \neq v \wedge \psi)\big)\big),$$

where u, v are fresh variables.

Exercise 6. Let

$$\phi_\infty^* := \exists v \forall x \exists y (=\!(y, x) \wedge v \neq y).$$

Show that for any model $\mathcal{M}$,

$$\mathcal{M} \models \phi_\infty^* \iff |\mathsf{dom}(\mathcal{M})| \text{ is infinite.}$$

That is, the formula ϕ_∞ in Example 4.2.1 can be simplified.