Dependence Logic

September 24 (Tuesday), 2019

Recap: Prenex normal form

- $\exists x \phi \wedge \psi \equiv \exists x(\phi \wedge \psi)$
- $\forall x \phi \wedge \psi \equiv \forall x(\phi \wedge \psi)$
- $\exists x \phi \vee \psi \equiv \exists x(\phi \vee \psi)$
- $\forall x(\phi \vee \psi) \equiv \forall x \phi \vee \psi$ if ϕ is downwards closed
- $\forall x \phi \vee \psi \equiv \exists y \exists z \forall x\big((\phi \wedge y = z) \vee (\psi \wedge y \neq z)\big)$, where y, z are fresh.

Proposition (prenex normal form)

Every formula ϕ is logically equivalent to a formula of the form

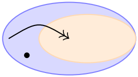
$$Q^1 x_1 \ldots Q^n x_n \theta,$$

where each $Q^i \in \{\forall, \exists\}$, and θ is a quantifier-free formula.

- For every ϕ in FO$(=(\cdots))$, $\phi \equiv \forall \vec{x} \exists \vec{y} \theta$ for some quantifier-free θ.
  Pf. Use $\exists x \forall y \phi(x, y, \vec{v}) \equiv \forall y \exists x \wedge =(\vec{v}, x))$. □
- For every ϕ in FO$(\subseteq)$, $\phi \equiv \exists \vec{x} \forall y \theta$ for some quantifier-free θ.
  Pf. Use $\forall x \phi(x, \vec{v}) \equiv \exists x \forall y(\phi \wedge \vec{v} y \subseteq \vec{v} x)$. □

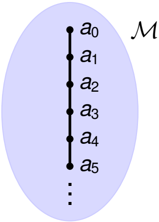
Recap: Infinity and non-well-foundness

$$\phi_\infty := \exists v \forall x \exists y (=(x, y) \wedge =(y, x) \wedge (v \neq y))$$

$\mathcal{M} \models \phi_\infty$ iff $|\mathrm{dom}(\mathcal{M})| = \infty$ iff $\exists f$: 

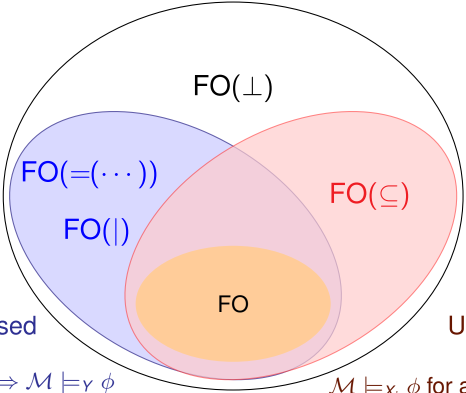
$$\phi_{\mathrm{nwf}} := \exists x \exists y (x \subseteq y \wedge x \prec y)$$

$\mathcal{M} \models \phi_{\mathrm{nwf}} \iff$



x		y
a_1	$\prec$	a_0
a_2	$\prec$	a_1
a_3	$\prec$	a_2
a_4	$\prec$	a_3
⋮		⋮

Expressive power



Downwards closed

$\mathcal{M} \models_X \phi \;\&\; Y \subseteq X \Longrightarrow \mathcal{M} \models_Y \phi$

Union closed

$\mathcal{M} \models_{X_i} \phi$ for all $i \in I \Longrightarrow \mathcal{M} \models_{\bigcup_{i \in I} X_i} \phi$

Flat

$\mathcal{M} \models_X \phi \iff \mathcal{M} \models_{\{s\}} \phi$ for all $s \in X$