Real Analysis I

Fall 2019

Homework 3

Exercise session: Wed 25 September, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. Let $\varphi \in L^1(\mathbb{R}^n)$ and $\varphi_\epsilon(x) := \frac{1}{\epsilon^n}\varphi(x/\epsilon)$, $\epsilon > 0$. Show that for all $\delta > 0$ we have
$$\lim_{\epsilon \to 0} \int_{|x| \geq \delta} |\varphi_\epsilon(x)| \, \mathrm{d}x = 0.$$
Conclude that if also $\int \varphi = 1$, then $(\varphi_\epsilon)_{\epsilon > 0}$ is an approximate identity.

Proof. By change of variable, $z = x/\epsilon$, $|J| = \epsilon^n$,
$$\int_{|x| \geq \delta} |\varphi_\epsilon(x)| \, \mathrm{d}x = \int_{|x| \geq \delta} \frac{1}{\epsilon^n} |\varphi(x/\epsilon)| \, \mathrm{d}x = \int_{|z| \geq \frac{\delta}{\epsilon}} |\varphi(z)| \, \mathrm{d}z.$$
Hence, by the monotone convergence theorem we have
$$\int_{|z| \geq \frac{\delta}{\epsilon}} |\varphi(z)| \, \mathrm{d}z \to 0$$
as $1/\epsilon \to \infty$.
If $\int \varphi = 1$, then
$$\int \varphi_\epsilon(x) \, \mathrm{d}x = \int \frac{1}{\epsilon^n} \varphi(x/\epsilon) \, \mathrm{d}x = \int \varphi(z) \, \mathrm{d}z = 1,$$
and $\|\varphi_\epsilon\|_1 = \|\varphi\|_1 < \infty$. Hence we conclude that $(\varphi_\epsilon)_{\epsilon > 0}$ is an approximate identity. □

2. Let $K \subset \mathbb{R}^n$ be compact, $V \subset \mathbb{R}^n$ be open and $K \subset V$. Show that there exists a smooth 'cut-off' function $\varphi \in C_c^\infty(V)$ so that $0 \leq \varphi \leq 1$, $\varphi = 1$ on K and
$$|\nabla \varphi(x)| \leq \frac{C}{d(K, \partial V)}.$$

Proof. Let $\varepsilon = \frac{d(K,\partial V)}{2}$ hence $U_\varepsilon = \{x\colon d(x,K) < \varepsilon\} \subset V$. Let us check that $\varphi(x) = 1_{U_\varepsilon} * \eta_\varepsilon(x)$ satisfies the desired conditions. For $x \in K$ we have

$$\varphi(x) = \int_{U_\varepsilon} \eta_\varepsilon(x-y)\,\mathrm{d}y = 1$$

since $|x-y| < \varepsilon$, $\eta_\varepsilon(x) \neq 0$ for $|x| < \varepsilon$, and $\int \eta_\varepsilon = 1$. Since other conditions are immediate, we still need to check the gradient condition. As shown in the lecture notes, we have.

$$\begin{aligned}|\nabla\varphi(x)| &= \Big(\sum_{i=1}^n (D^{e_i}(\varphi(x)))^2\Big)^{1/2} = \Big(\sum_{i=1}^n (1_{U_\varepsilon} * D^{e_i}\eta_\varepsilon(x))^2\Big)^{1/2} \\ &\leq \Big(\sum_{i=1}^n \frac{1}{\varepsilon^2}\|\nabla\eta\|_{L^\infty}^2\Big)^{1/2} = \varepsilon^{-1} n^{1/2}\|\nabla\eta\|_{L^\infty} \sim \frac{C}{d(K,\partial V)}.\end{aligned}$$

□

3. Let (X,μ) be a measure space. For a measurable $f\colon X \to \mathbb{R}$ prove that we have the identity

$$\int_X |f|^p\,\mathrm{d}\mu = p\int_0^\infty \lambda^{p-1}\mu(\{x \in X\colon |f(x)| > \lambda\})\,\mathrm{d}\lambda, \qquad 0 < p < \infty. \tag{0.1}$$

Proof. Assume (0.1) for non-negative functions and let $f = f_+ - f_-$, where $f_+ = f1_{\{f\geq 0\}}$ and $f_- = f1_{\{f<0\}}$. Then

$$\begin{aligned}\int_X |f|^p\,\mathrm{d}\mu &= \int_X f_+^p\,\mathrm{d}\mu + \int_X f_-^p\,\mathrm{d}\mu \\ &= p\int_0^\infty \lambda^{p-1}\mu(\{x \in X\colon f_+(x) > \lambda\})\,\mathrm{d}\lambda \\ &\quad + p\int_0^\infty \lambda^{p-1}\mu(\{x \in X\colon f_-(x) > \lambda\})\,\mathrm{d}\lambda \\ &= \int_0^\infty \lambda^{p-1}\mu(\{x \in X\colon |f(x)| > \lambda\})\,\mathrm{d}\lambda\end{aligned}$$

since $\{x \in X\colon f_+(x) > \lambda\}$ and $\{x \in X\colon f_-(x) > \lambda\}$ are disjoint. Thus, it is enough to prove (0.1) for non-negative functions

First assume that f is a simple function, that is,

$$f = \sum_{i\in\mathcal{I}} a_i 1_{A_i},$$

where $\mathcal{I}$ is a finite index set, $a_i \in \mathbb{R}$ and A_i are measurable and disjoint. In addition, we may assume that $0 \leq a_1 < a_2 < \dots$. Notice

$$f^p = \sum_{i\in\mathcal{I}} a_i^p 1_{A_i}$$

since A_i are disjoint.

By definition of integration , FTC (Fundamental theory of calculus) and additivity of disjoint sets, we have

$$\begin{aligned}
\int_X f^p \,\mathrm{d}\mu &= \sum_{i\in\mathcal{I}} a_i^p \mu(A_i) \\
&= \sum_{i\in\mathcal{I}} \int_0^{a_i} p\lambda^{p-1} \,\mathrm{d}\lambda \cdot \mu(A_i) \\
&= \sum_{i\in\mathcal{I}} \int_0^{a_i} p\lambda^{p-1} \mu(\{x \in X \colon a_{i+1} > f(x) \geq a_i\}) \,\mathrm{d}\lambda \\
&= \int_0^\infty p\lambda^{p-1} \sum_{i\in\mathcal{I}} \mu(\{x \in X \colon a_{i+1} > 1_{A_i}(x) a_i > \lambda\}) \,\mathrm{d}\lambda \\
&= \int_0^\infty p\lambda^{p-1} \mu\Big(\bigcup_{i\in\mathcal{I}} \{x \in X \colon a_{i+1} > 1_{A_i}(x) a_i > \lambda\}\Big) \,\mathrm{d}\lambda \\
&= \int_0^\infty p\lambda^{p-1} \mu(\{x \in X \colon \sum_{i\in\mathcal{I}} 1_{A_i}(x) a_i > \lambda\}) \,\mathrm{d}\lambda \\
&= p \int_0^\infty \lambda^{p-1} \mu(\{x \in X \colon f(x) > \lambda\}) \,\mathrm{d}\lambda.
\end{aligned}$$

Then by results of the course Measure and Integral, we know that for measurable function f there exists monotonic sequence (f_i) of simple functions converging to f. Hence, the result follows by standard monotonicity arguments.

Remark. If μ is a σ-finite measure and assume Fubini's theorem for σ-finite measures, then we get

$$\begin{aligned}
\int_X |f|^p \,\mathrm{d}\mu = \int_X \int_0^{|f|} p\lambda^{p-1} \,\mathrm{d}\lambda \,\mathrm{d}\mu = \int_0^\infty p\lambda^{p-1} \int_{\{|f|>\lambda\}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \\
= \int_0^\infty p\lambda^{p-1} \mu(\{|f| > \lambda\}) \,\mathrm{d}\lambda.
\end{aligned}$$

$\square$

4. Give a proof of the case, where $p_1 = \infty$ in the Marcinkiewicz interpolation theorem.

Proof. Let $f \in L^p$ for some $0 < p_0 < p < \infty$. Let $\lambda, \delta > 0$. As in the lecture notes split $f = f_0 + f_1$, where $f_0 = f 1_{\{|f| > \delta\lambda\}}$ and $f_1 = f 1_{\{|f| \le \delta\lambda\}}$. Clearly,

$$\|f_0\|_{p_0}^{p_0} = \int |f_0|^{p_0} = \int |f_0|^{p_0 - p}|f_0|^p \le (\delta\lambda)^{p_0 - p} \int |f_0|^p \le (\delta\lambda)^{p_0 - p}\|f\|_p^p$$

and $f_1 \le \delta\lambda$, that is, $f_1 \in L^\infty$.

By sublinearity for $\alpha, \beta \ge 0, \alpha + \beta = 1$ we have

$$\lambda = \alpha\lambda + \beta\lambda < |Tf| \le |Tf_0| + |Tf_1|$$

and note that if $|Tf_1| \le \beta\lambda$, then $|Tf_0| > \alpha\lambda$. Since

$$\|Tf_1\|_\infty \le A_\infty \|f_1\|_\infty \le A_\infty \delta\lambda,$$

choosing $\beta = A_\infty \delta$ we conclude that

$$\mu(\{|Tf_1| > \beta\lambda\}) = 0.$$

Hence, we get

$$\begin{aligned}
\mu(\{|Tf| > \lambda\}) \le \mu(\{|Tf_0| > \alpha\lambda\}) &\le \frac{1}{\alpha^{p_0}\lambda^{p_0}}\|Tf_0\|_{p_0,\infty}^{p_0} \\
&\le A_{p_0}^{p_0}\frac{1}{\alpha^{p_0}\lambda^{p_0}}\|f_0\|_{p_0}^{p_0} \\
&= A_{p_0}^{p_0}\frac{1}{\alpha^{p_0}\lambda^{p_0}}\int_{|f| > \delta\lambda} |f|^{p_0}.
\end{aligned}$$

Thus, using (0.1) we get

$$\begin{aligned}
\|Tf\|_p^p &\le p\int_0^\infty \lambda^{p-1} A_{p_0}^{p_0}\frac{1}{\alpha^{p_0}\lambda^{p_0}}\int_{|f| > \delta\lambda} |f|^{p_0}\,\mathrm{d}\lambda \\
&= \frac{A_{p_0}^{p_0}}{\alpha^{p_0}}\int |f|^{p_0} p\int_0^{\frac{|f|}{\delta}} \lambda^{p - p_0 - 1}\,\mathrm{d}\lambda \\
&= \frac{p}{p - p_0}\frac{A_{p_0}^{p_0}}{\alpha^{p_0}}\int |f|^{p_0}\delta^{p_0 - p}|f|^{p - p_0}
\end{aligned}$$

$$= \frac{p}{p-p_0}\frac{A_{p_0}^{p_0}\delta^{p_0-p}}{\alpha^{p_0}}\|f\|_p^p$$

$$\Leftrightarrow \qquad \|Tf\|_p \leq \Big(\frac{p}{p-p_0}\Big)^{1/p}\frac{\delta^{-\theta}}{\alpha^{1-\theta}}A_{p_0}^{1-\theta}\|f\|_p$$

Since $\beta = A_\infty\delta$ and $\alpha = 1-\beta$, we have

$$\frac{\delta^{-\theta}}{\alpha^{1-\theta}} = A_\infty^\theta\frac{\beta^{-\theta}}{(1-\beta)^{1-\theta}}.$$

By standard analysis we get that function $h(\beta) = \frac{\beta^{-\theta}}{(1-\beta)^{1-\theta}}$ attains its minimum at $\beta = \theta$. Furthermore, the function $\varphi(\theta) = \frac{\theta^{-\theta}}{(1-\theta)^{1-\theta}}, \theta \in (0,1)$ attains the maximum at $\theta = 1/2$. Thus, we get

$$\|Tf\|_p \leq 2\Big(\frac{p}{p-p_0}\Big)^{1/p}A_\infty^\theta A_{p_0}^{1-\theta}\|f\|_p.$$

□

5. • A function $g\colon \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous if $\{g > \lambda\}$ is open for all $\lambda \in \mathbb{R}$. Prove that Mf, $f \in L^1_{\text{loc}}$, is measurable by showing that it is lower semicontinuous.

Proof. Let $x \in \{Mf > \lambda\}$. Then $Mf(x) > \lambda$, and also there exists $r > 0$ such that

$$Mf(x) \geq \frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,\mathrm{d}y > \lambda.$$

Remark. We may assume that there exists some r for which the supremum is attained since both $\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,\mathrm{d}y > \lambda$ and $\lim_{r\to\infty}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,\mathrm{d}y > \lambda$, requires that there exists some $B(x,r)$ such that the strict inequality holds.

Since

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,\mathrm{d}y > \lambda,$$

there exists $s > r$ such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,\mathrm{d}y > \frac{1}{|B(x,s)|}\int_{B(x,r)}|f(y)|\,\mathrm{d}y > \lambda.$$

Now take $z \in \mathbb{R}^n$ such that $|x-z| < s-r$ hence for $x_0 \in B(x,r)$ we have $|x_0 - z| \leq |x_0 - x| + |x-z| < r + s - r = s$, i.e. $B(x,r) \subset B(z,s)$.

Using these we get

$$Mf(z) \geq \frac{1}{|B(z,s)|} \int_{B(z,s)} |f(y)| \, \mathrm{d}y \geq \frac{1}{|B(z,s)|} \int_{B(x,r)} |f(y)| \, \mathrm{d}y > \lambda,$$

since $|B(z,s)| = |B(x,s)|$. $\square$

- Prove the following refinement of the estimate $M\colon L^1 \to L^{1,\infty}$. Show that for all $f \in L^1$ and $\lambda > 0$ we have

$$|\{x\colon\ Mf(x) > \lambda\}| \lesssim \frac{1}{\lambda} \int_{|f|>\lambda/2} |f|.$$

Proof. Let $f \in L^1$ and assume the standard $M\colon L^1 \to L^{1,\infty}$ estimate

$$|\{Mf > \lambda\}| \lesssim \frac{1}{\lambda} \int |f|.$$

Write $f = f1_{\{|f| \leq \lambda/2\}} + f1_{\{|f|>\lambda/2\}}$. By sublinearity we have

$$Mf(x) \leq M(f1_{\{|f|\leq\lambda/2\}}) + M(f1_{\{|f|>\lambda/2\}}) \leq \frac{\lambda}{2} + M(f1_{\{|f|>\lambda/2\}}).$$

Hence using above estimate we get

$$|\{Mf > \lambda\}| \leq |\{M(f1_{\{|f|>\lambda/2\}}) > \lambda/2\}| \lesssim \frac{1}{\lambda} \int_{\{|f|>\lambda/2\}} |f|.$$

$\square$