Real Analysis I

Fall 2019

Homework 5

Exercise session: Wed 9 October, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is such that
$$\int f\varphi = 0$$
for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Show that $f = 0$ a.e.

2. Prove the following variant of Egorov's theorem. Let $f_m, f \colon \mathbb{R}^n \to \mathbb{R}$ be such that $f_m(x) \to f(x)$ for a.e. x. Show that for every $\epsilon > 0$ there is an open set $U \subset \mathbb{R}^n$ so that $|U| < \epsilon$, and so that for all compact sets $K \subset \mathbb{R}^n$ we have that $f_m \to f$ uniformly in $K \setminus U$. Show by example that we cannot in general build the set U so that we would have $f_m \to f$ uniformly in $\mathbb{R}^n \setminus U$.

3. Complete the details of the following new proof of a variant of Lusin's theorem. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ and $f \in L^1$. Given ϵ, choose for every $j \in \mathbb{N}$ a compactly supported continuous function $f_j \in C_c$ so that $\|f - f_j\|_1 < \epsilon/4^j$. Prove that now automatically $f_j \to f$ uniformly on $\mathbb{R}^n \setminus E$ for some set E with $|E| < \epsilon$ – in particular, $f|(\mathbb{R}^n \setminus E)$ is continuous. Show by example that we cannot in general build the set E so that the unrestricted function f would be continuous in $\mathbb{R}^n \setminus E$.

4. Let $A \subset \mathbb{R}$ with $|A| = 0$. For each $k \in \mathbb{N}$ choose an open set $G_k \subset \mathbb{R}$ so that $A \subset G_k$ and $|G_k| < 2^{-k}$. Define $f_k \colon \mathbb{R} \to \mathbb{R}$ by setting
$$f_k(x) = \int_{-\infty}^{x} 1_{G_k}(y)\,\mathrm{d}y = |G_k \cap (-\infty, x]|, \qquad x \in \mathbb{R}.$$
Show that
$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$
defines a continuous and increasing function $f \colon \mathbb{R} \to \mathbb{R}$ for which $\underline{D}f(x) = \infty$ for all $x \in A$.

5. Suppose that $f, g \colon [a, b] \to \mathbb{R}$ are of bounded variation. Show that fg is of bounded variation.