Real Analysis I

Fall 2019

Homework 6

Exercise session: Wed 16 October, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. Let λ be a Borel measure in $\mathbb{R}^n$ with the property that

$$\lambda(A) = \inf\{\lambda(U) \colon A \subset U,\, U \text{ open}\}.$$

   Suppose that

$$\liminf_{r\to 0} \frac{|\overline{B}(x,r)|}{\lambda(\overline{B}(x,r))} \leq t$$

   for some $t > 0$ and for all $x \in A$. Show that $|A| \leq t\lambda(A)$.

2. Define $f \colon [0,1] \to \mathbb{R}$ by setting $f(0) = 0$ and $f(x) = x\sin(1/x)$ otherwise. Show that f is continuous but not of bounded variation.

3. Let $f \in BV([a,b])$ be continuous on the interval $[a,b]$. Show that $x \mapsto V_f(a,x)$ is continuous. Show also that if f is absolutely continuous, then $x \mapsto V_f(a,x)$ is absolutely continuous.

4. Suppose $f, g \colon [a,b] \to \mathbb{R}$ are absolutely continuous. Prove the integration by parts formula:

$$\int_a^x fg' = [f(x)g(x) - f(a)g(a)] - \int_a^x f'g, \qquad x \in [a,b].$$

5. Let $f \colon [a,b] \to \mathbb{R}$ be absolutely continuous. Show that $|f(E)| = 0$ if $E \subset [a,b]$ satisfies $|E| = 0$.