Real Analysis I

Fall 2019

Homework 4

Exercise session: Wed 2 October, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. For a locally integrable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ define the 'rectangular maximal function'

$$M_R f(x_1, x_2) := \sup_{r_1, r_2 > 0} \frac{1}{2r_1} \int_{x_1 - r_1}^{x_1 + r_1} \frac{1}{2r_2} \int_{x_2 - r_2}^{x_2 + r_2} |f(y_1, y_2)| \, \mathrm{d}y_2 \, \mathrm{d}y_1.$$

Show that for all $1 < p < \infty$ we have

$$\|M_R f\|_p \lesssim \|f\|_p, \qquad f \in L^p(\mathbb{R}^2).$$

Proof. Notice that we have $M_R f(x_1, x_2) \le M^1 \circ M^2 f(x_1, x_2)$ where $M^1 f(x_1, x_2) = Mf(\cdot, x_2)(x_1)$ and $M^2 f(x_1, x_2) = Mf(x_1, \cdot)(x_2)$ since

$$\begin{aligned} \frac{1}{2r_1} \int_{x_1 - r_1}^{x_1 + r_1} \frac{1}{2r_2} \int_{x_2 - r_2}^{x_2 + r_2} |f(y_1, y_2)| \, \mathrm{d}y_2 \, \mathrm{d}y_1 &\le \frac{1}{2r_1} \int_{x_1 - r_1}^{x_1 + r_1} M^2 f(x_2, y_1) \, \mathrm{d}y_1 \\ &\le M^1 \circ M^2 f(x_1, x_2), \end{aligned}$$

for all (x_1, x_2). Thus, using one-parameter results we get

$$\begin{aligned} \int\int M^1 \circ M^2 f(x_1, x_2)^p \, \mathrm{d}x_1 \, \mathrm{d}x_2 &= \int \|M^1(M^2 f)(\cdot, x_2)\|_p^p \, \mathrm{d}x_2 \\ &\lesssim \int\int (M^2(f)(x_1, x_2))^p \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &= \int\int (M^2(f)(x_1, x_2))^p \, \mathrm{d}x_2 \, \mathrm{d}x_1 \\ &\lesssim \int\int (f(x_1, x_2))^p \, \mathrm{d}x_2 \, \mathrm{d}x_1 = \|f\|_p^p. \end{aligned}$$

□

2. Let $\delta \in (0, 1)$ and let $E \subset \mathbb{R}^n$ be a measurable set with $|E| < \infty$. Prove that for all $f \in L^1$ we have

$$\int_E [Mf(x)]^\delta \, \mathrm{d}x \le C|E|^{1-\delta} \|f\|_1^\delta,$$

where $C = C(\delta)$ is a constant.

Proof. Using the distributional formula of Exercise 3.3. we have

$$\int_E [Mf(x)]^\delta \, \mathrm{d}x = \delta \int_0^\infty \lambda^{\delta-1} |\{x \in E \colon Mf(x) > \lambda\}| \, \mathrm{d}\lambda.$$

Since $\{x \in E \colon Mf(x) > \lambda\} = E \cap \{Mf > \lambda\}$, we have

$$\begin{aligned} \{x \in E \colon Mf(x) > \lambda\}| &\leq \min\{|E|, |\{Mf > \lambda\}|\} \\ &\leq \min\{|E|, \|M\|_{L^1 \to L^{1,\infty}} \lambda^{-1} \|f\|_1\}, \end{aligned}$$

where we applied the weak (1,1) boundedness of the maximal function. Let $a = \|M\|_{L^1 \to L^{1,\infty}} \|f\|_1 |E|^{-1}$. Thus, we get

$$\begin{aligned} &\delta \int_0^\infty \lambda^{\delta-1} |\{x \in E \colon Mf(x) > \lambda\}| \, \mathrm{d}\lambda \\ &= \delta \int_0^a \lambda^{\delta-1} |\{x \in E \colon Mf(x) > \lambda\}| \, \mathrm{d}\lambda \\ &\qquad + \delta \int_a^\infty \lambda^{\delta-1} |\{x \in E \colon Mf(x) > \lambda\}| \, \mathrm{d}\lambda \\ &\leq |E| \delta \int_0^a \lambda^{\delta-1} \, \mathrm{d}\lambda \\ &\qquad + \|M\|_{L^1 \to L^{1,\infty}} \|f\|_1 \delta \int_a^\infty \lambda^{\delta-2} \, \mathrm{d}\lambda \\ &= |E|^{1-\delta} \|M\|_{L^1 \to L^{1,\infty}}^\delta \|f\|_1^\delta + \|M\|_{L^1 \to L^{1,\infty}}^\delta \|f\|_1^\delta \frac{\delta}{1-\delta} |E|^{1-\delta} \\ &= \frac{\|M\|_{L^1 \to L^{1,\infty}}^\delta}{1-\delta} |E|^{1-\delta} \|f\|_1^\delta =: C|E|^{1-\delta} \|f\|_1^\delta. \end{aligned}$$

□

3. Fix $f \in L^1$ and $\delta \in (0,1)$, and define $w(x) = [Mf(x)]^\delta$. Prove that for almost every x we have

$$Mw(x) \leq Cw(x),$$

where $C = C(\delta)$ is a constant.

Proof. First, split

$$\begin{aligned} \int_{B(x,r)} [Mf(y)]^\delta \, \mathrm{d}y &\leq \int_{B(x,r)} [M(f 1_{B(x,2r)})(y)]^\delta \, \mathrm{d}y + \int_{B(x,r)} [M(f 1_{B(x,2r)^c})(y)]^\delta \, \mathrm{d}y \\ &=: I_1 + I_2. \end{aligned}$$

For I_1 we can use result of above exercise, namely

$$\begin{aligned} I_1 &\lesssim |B(x,r)|^{1-\delta}\Big(\int_{B(x,2r)} |f|\Big)^\delta \\ &\lesssim |B(x,r)|\Big(\frac{1}{|B(x,2r)|}\int_{B(x,2r)} |f|\Big)^\delta \\ &\leq |B(x,r)| Mf(x). \end{aligned}$$

Let $y \in B(x,r)$ and let $r' \geq r$ since

$$\int_{B(y,r')} f 1_{B(x,2r)^c} = 0$$

for $r' < r$. Then $B(y,r') \subset B(x,3r')$ and we have

$$\begin{aligned} \frac{1}{|B(y,r')|}\int_{B(y,r')} f 1_{B(x,2r)^c} &\leq \frac{|B(x,3r')|}{|B(y,r')|}\frac{1}{|B(x,3r')|}\int_{B(x,3r')} f \\ &\leq 3^n Mf(x). \end{aligned}$$

Moreover, we get

$$I_2 \leq 3^{n\delta}|B(x,r)|[Mf(x)]^\delta.$$

Combining these estimates we get

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} [Mf(y)]^\delta \,\mathrm{d}y \leq C(\delta) Mf(x)$$

and taking the supremum over $r > 0$ we conclude the claim.

□

4. Show that if $E \subset \mathbb{R}^n$ is a measurable set with $|E| < \infty$, then we have for some constant C that

$$\int_E Mf(x)\,\mathrm{d}x \leq 2|E| + C\int_{\mathbb{R}^n} |f(x)| \max(\log|f(x)|, 0)\,\mathrm{d}x.$$

Proof. Using exercises 3.3 and 3.5 we get

$$\begin{aligned} \int_E Mf(x)\,\mathrm{d}x &= \int_0^\infty |\{x \in E \colon Mf(x) > \lambda\}|\,\mathrm{d}\lambda \\ &= \int_0^2 |\{x \in E \colon Mf(x) > \lambda\}|\,\mathrm{d}\lambda \end{aligned}$$

$$
\begin{aligned}
&+\int_2^\infty |\{x\in E\colon Mf(x)>\lambda\}|\,\mathrm{d}\lambda \\
&\le 2|E| + 2\int_1^\infty |\{x\colon Mf(x)>2\lambda\}|\,\mathrm{d}\lambda \\
&\le 2|E| + C\int_1^\infty \lambda^{-1}\int_{|f|>\lambda} |f(x)|\,\mathrm{d}x\,\mathrm{d}\lambda \\
&= 2|E| + C\int_{\mathbb{R}^n} |f(x)| \int_1^{\max(|f(x)|,1)} \lambda^{-1}\,\mathrm{d}x\,\mathrm{d}\lambda \\
&= 2|E| + C\int_{\mathbb{R}^n} |f(x)|\max(\log|f(x)|,0)\,\mathrm{d}x.
\end{aligned}
$$

□

5. Let (X,μ) be a measure space, $p\in(0,\infty)$, and let T_ε, $\varepsilon>0$, be a family of linear operators defined in $L^p(X)$ and taking values in the measurable functions of X. Assume that there is a dense subset $\mathcal{G}\subset L^p(X)$ so that for all $g\in\mathcal{G}$ we have
$$\lim_{\varepsilon\to 0} T_\varepsilon g(x) = g(x)$$
for μ-a.e. $x\in X$. For $f\in L^p(X)$ define the related maximal function $T_*f(x) := \sup_{\varepsilon>0}|T_\varepsilon f(x)|$. Assume that for all $f\in L^p$ and $\lambda>0$ we have
$$\mu(\{x\in X\colon T_*f(x)>\lambda\}) \le A(\lambda)\|f\|^p_{L^p(X)},$$
where $A\colon(0,\infty)\to(0,\infty)$ is some function.

   Prove that for all $f\in L^p(X)$ we have
$$\lim_{\varepsilon\to 0} T_\varepsilon f(x) = f(x)$$
for μ-a.e. $x\in X$.

Proof. Let $\delta>0$ and $g\in\mathcal{G}$ such that $\|f-g\|^p_{L^p(X)}<\delta$.

Define $\sigma_\varepsilon f(x) = |T_\varepsilon f(x) - f(x)|$. It is enough to show that
$$\mu(\{x\in X\colon \limsup_{\varepsilon\to 0}\sigma_\varepsilon f(x)>\lambda\}) = 0$$
for all $\lambda>0$, since
$$\mu(\{x\in X\colon \limsup_{\varepsilon\to 0}\sigma_\varepsilon f(x)>0\}) = \bigcup_{\lambda>0}\mu(\{x\in X\colon \limsup_{\varepsilon\to 0}\sigma_\varepsilon f(x)>\lambda\})$$

In particular,

$$\liminf_{\varepsilon\to 0} \sigma_\varepsilon f(x) \le \limsup_{\varepsilon\to 0} \sigma_\varepsilon f(x) = 0$$

for almost every x, that is, the limit exists $\lim_{\varepsilon\to 0} \sigma_\varepsilon f(x)$ for a.e. $x \in X$ and equals to zero.

Notice that by linearity of T and triangle inequality

$$\begin{aligned}\sigma_\varepsilon f(x) &= |T_\varepsilon(f-g)(x) + T_\varepsilon g(x) - g(x) + g(x) - f(x)| \\ &\le |T_\varepsilon(f-g)(x)| + \sigma_\varepsilon g(x) + |f(x) - g(x)|.\end{aligned}$$

Furthermore, we know that

$$|T_\epsilon(f-g)(x)| \le T_*(f-g)(x), \text{ and } \qquad \lim_{\varepsilon\to 0} \sigma_\varepsilon g(x) = 0.$$

Thus, by assumptions we have

$$\begin{aligned}&\mu(\{x \in X \colon \limsup_{\varepsilon\to 0} \sigma_\varepsilon f(x) > \lambda\}) \\ &\le \mu(\{x \in X \colon T_*(f-g)(x) > \lambda/2\}) + \mu(\{x \in X \colon |f(x) - g(x)| > \lambda/2\}) \\ &\le (A(\lambda/2) + 2^p/\lambda^p)\|f-g\|^p_{L^p(X)} \le C(\lambda)\delta.\end{aligned}$$

Hence passing limit $\delta \to 0$ we get

$$\mu(\{x \in X \colon \limsup_{\varepsilon\to 0} \sigma_\varepsilon f(x) > \lambda\}) = 0,$$

as claimed.

□