Dependence Logic

Exercise 5

Exercise 1. Prove the following facts about atoms of dependencies:

(1) $=\!(x, y), uv \subseteq xy, u'v' \subseteq xy \models u = u' \to v = v'$

(2) $x|y, uv \subseteq xy, u'v' \subseteq xy \models u \neq v'$

Exercise 2. Prove the following facts about atoms of dependencies:

(1) $x \perp y, uv \subseteq xy, u'v' \subseteq xy \models \exists u''v''(u''v'' \subseteq xy \wedge u'' = u \wedge v'' = v')$

(2) $x \perp_z y, uvw \subseteq xyz, u'v'w' \subseteq xyz \models \exists u''v''w''(u''v''w'' \subseteq xyz \wedge (w = w' \to w'' = w \wedge u'' = u \wedge v'' = v'))$

Exercise 3. A cycle in a directed graph $G = (V, E)$ is a sequence of vertices $(v_0, \ldots, v_n)$ from V where n is a positive integer, $v_0 = v_n$, and (v_i, v_{i+1}) is an edge from E for all $i \in \{0, \ldots, n-1\}$. Construct an inclusion logic sentence $\phi \in \mathrm{FO}(\subseteq)[\tau]$ over $\tau = \{E\}$ such that for all finite directed graphs G,

$$G \models \phi \text{ if and only if } G \text{ contains a cycle.}$$

Exercise 4. Let $\phi = x \perp_y z$ and $V = \{x, y, z, u\}$. Define $\psi_{\phi,V}$ and prove the claim of Theorem 5.2.1 in this case.

Exercise 5. A binary relation $R \subseteq A \times A$ is *transitive* if for all $a, b, c \in A$: if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. The *transitive closure* $TC(R)$ of a binary relation $R \subseteq A \times A$ is defined as $\bigcup_{i \in \mathbb{N}} R_i$ where $R_0 = R$ and $R_{i+1} = R_i \circ R$. The *successor function* S on natural numbers is defined as $S(i) = i + 1$. Let $\tau = \{f, E\}$ where f is a unary function symbol and E a binary relation symbol. Construct a $\Sigma_1^1[\tau]$ formula $\phi(x, y)$ such that for all τ-structures $\mathcal{M}$, where $\mathrm{dom}(\mathcal{M}) = \mathbb{N}$ and $f^{\mathcal{M}} = S$,

$$\mathcal{M} \models_s \phi \text{ if and only } (s(x), s(y)) \in TC(E^{\mathcal{M}}).$$