Real Analysis I

Fall 2019

Homework 5

Exercise session: Wed 9 October, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is such that

$$\int f\varphi = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Show that $f = 0$ a.e.

Proof. By Lebesgue differentiation theorem, it is enough to show that for all balls $B(x,r)$

$$\int_{B(x,r)} f = 0.$$

Let $\varepsilon > 0$. As shown in Exercise 3.2 for every $B(x, r+\varepsilon)$ we find φ such that $\varphi \in C_c^\infty(B(x, r+\varepsilon))$ such that $\varphi = 1$ in $B(x,r)$. Hence,

$$0 = \int f\varphi = \int_{B(x,r)} f + \int_{B(x,r+\varepsilon)\setminus B(x,r)} f\varphi.$$

By dominated convergence theorem

$$\lim_{\varepsilon \to 0} \int_{B(x,r+\varepsilon)\setminus B(x,r)} f\varphi = \int_{\partial B(x,r)} f\varphi = 0$$

since $|\partial B(x,r)| = 0$. Thus $\int_{B(x,r)} f = 0$ which concludes the claim. $\square$

2. Prove the following variant of Egorov's theorem. Let $f_m, f \colon \mathbb{R}^n \to \mathbb{R}$ be such that $f_m(x) \to f(x)$ for a.e. x. Show that for every $\varepsilon > 0$ there is an open set $U \subset \mathbb{R}^n$ so that $|U| < \varepsilon$, and so that for all compact sets $K \subset \mathbb{R}^n$ we have that $f_m \to f$ uniformly in $K \setminus U$. Show by example that we cannot in general build the set U so that we would have $f_m \to f$ uniformly in $\mathbb{R}^n \setminus U$.

Proof. • Let $\varepsilon > 0$ and let $f_m, f\colon \mathbb{R}^n \to \mathbb{R}$ be such that $f_m(x) \to f(x)$ for a.e. x. By Egorov's theorem we have that for every $B(0,N)$ there exists open set$^{(*)}$ $U_N \subset B(0,N)$ such that $|U_N| < 2^{-N}\varepsilon$ and $f_m \to f$ uniformly in $B(0,N) \setminus U_N$. Let $U = \bigcup_{N=1}^{\infty} U_N$ hence $|U| \leq \sum_{N=1}^{\infty} |U_N| < \varepsilon$. Now for every compact set K there exists $N > 0$ such that $K \subset B(0,N)$. Hence there exists U_N such that $f_m \to f$ uniformly in $B(0,N) \setminus U_N$, especially $f_m \to f$ uniformly in $K \setminus U$.

$^{(*)}$*Remark.* For every $\varepsilon > 0$ and every measurable set F there exists open set G such that $F \subset G$ and $|G \setminus F| < \varepsilon$.

• For simplicity let $n = 1$ and set $f_m = 1_{[m,m+1]}$ hence $f_m(x) \to 0$ as $m \to \infty$ for all $x \in \mathbb{R}$. Now it is clear that for every compact set K there exists m_0 such that $f_m(x) = 0$ for all $x \in K$ whenever $m > m_0$. However, every measurable set U such that $|U| < \infty$ there exists m and $x \in \mathbb{R} \setminus U$ such that we have $f_m(x) - f(x) = 1$. So we can not build U such that $f_m \to f$ uniformly in $\mathbb{R} \setminus U$.

□

3. Complete the details of the following new proof of a variant of Lusin's theorem. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $f \in L^1$. Given ε, choose for every $j \in \mathbb{N}$ a compactly supported continuous function $f_j \in C_c$ so that $\|f - f_j\|_1 < \varepsilon/4^j$. Prove that now automatically $f_j \to f$ uniformly on $\mathbb{R}^n \setminus E$ for some set E with $|E| < \varepsilon$ – in particular, $f|(\mathbb{R}^n \setminus E)$ is continuous. Show by example that we cannot in general build the set E so that the unrestricted function f would be continuous in $\mathbb{R}^n \setminus E$.

Proof. • As suggested let us consider the set $\{|f - f_j| > 2^{-j}\}$. By Chebyshev's inequality

$$|\{|f - f_j| > 2^{-j}\}| \leq 2^j \|f - f_j\|_1 < 2^{-j}\varepsilon.$$

Hence,

$$|E| := \left| \bigcup_{j=1}^{\infty} \{|f - f_j| > 2^{-j}\} \right| < \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon$$

and $f_j \to f$ uniformly on $\mathbb{R}^n \setminus E$. Recall this implies that $f|(\mathbb{R}^n \setminus E)$ is continuous, namely for every $\varepsilon > 0$ there exists j_ε such that for all $z \in \mathbb{R}^n \setminus E$ $|f_j(z) - f(z)| < \varepsilon/3$ whenever $j \geq j_\varepsilon$ and there exists $\delta > 0$ such that $|f_j(x) - f_j(y)| < \varepsilon/3$ whenever $|x - y| < \delta$ hence we have

$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| < \varepsilon.$$

- Let $f = 1_{\mathbb{Q}}$. Then it is obvious that f is not continuous anywhere. However, $f|\mathbb{Q}$ and $f|(\mathbb{R} \setminus \mathbb{Q})$ are both constant functions and obviously also continuous.

$\square$

4. Let $A \subset \mathbb{R}$ with $|A| = 0$. For each $k \in \mathbb{N}$ choose an open set $G_k \subset \mathbb{R}$ so that $A \subset G_k$ and $|G_k| < 2^{-k}$. Define $f_k \colon \mathbb{R} \to \mathbb{R}$ by setting

$$f_k(x) = \int_{-\infty}^{x} 1_{G_k}(y)\,\mathrm{d}y = |G_k \cap (-\infty, x]|, \qquad x \in \mathbb{R}.$$

Show that

$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$

defines a continuous and increasing function $f \colon \mathbb{R} \to \mathbb{R}$ for which $\underline{D}f(x) = \infty$ for all $x \in A$.

Proof. First, observe that $f(x)$ is well-defined for all $x \in \mathbb{R}$ since

$$f(x) = \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} |G_k \cap (-\infty, x]| \leq \sum_{k=1}^{\infty} |G_k| \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Let $\varepsilon > 0$. Fix N such that $\sum_{k=N}^{\infty} |G_k| \leq \varepsilon/2$ and let $x, y \in \mathbb{R}$ such that $0 < x - y < \varepsilon/2N$ hence

$$\begin{aligned}
|f(x) - f(y)| &= \Big|\sum_{k=1}^{\infty} \int_{-\infty}^{x} 1_{G_k}(z)\,\mathrm{d}z - \int_{-\infty}^{y} 1_{G_k}(z)\,\mathrm{d}z\Big| = \Big|\sum_{k=1}^{\infty} \int_{y}^{x} 1_{G_k}(z)\,\mathrm{d}z\Big| \\
&= \sum_{k=1}^{\infty} |G_k \cap (y, x]| = \sum_{k=1}^{N} |G_k \cap (y, x]| + \sum_{k=N}^{\infty} |G_k \cap (y, x]| \\
&\leq N|x - y| + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Thus, function f is continuous. Also clearly f is increasing since $(-\infty, y] \subset (-\infty, x]$ for all $y < x$.

We still need to check

$$\underline{D}f(x) = \liminf_{\varepsilon \to 0+} \left\{ \frac{f(z) - f(y)}{z - y} : y \leq x \leq z, 0 < z - y < \varepsilon \right\} = \infty$$

for all $x \in A$.

Let $x \in A$. We know that $x \in \bigcap_{k=1}^{N} G_k$ for any positive integer N since $A \subset G_k$ for all k. In addition, we know that $\bigcap_{k=1}^{N} G_k$ is open since G_ks are open. Hence, small enough ε we have $[y, z] \subset \bigcap_{k=1}^{N} G_k$ whenever $y \leq x \leq z$ and $0 < z - y < \varepsilon$.

Therefore, we get

$$\frac{f(z) - f(y)}{z - y} \geq \sum_{k=1}^{N} \frac{f_k(z) - f_k(y)}{z - y} = \sum_{k=1}^{N} \frac{|G_k \cap (y, z]|}{z - y} = N$$

since $f_k(z) - f_k(y)$ is non-negative for all k. and $(y, z] \subset G_k$ for $k \leq N$. As $N \to \infty$, we conclude that $\underline{D}f(x) = \infty$ for all $x \in A$. □

5. Suppose that $f, g \colon [a, b] \to \mathbb{R}$ are of bounded variation. Show that fg is of bounded variation.

Proof. Recall $f \colon [a, b] \to \mathbb{R}$ is bounded variation if

$$V_f(a, b) = \sup \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| < \infty$$

where the supremum is taken over all divisions $a = x_0 < x_1 < \cdots < x_k = b$.

First, notice for $x \in [a, b]$ we have

$$|f(x)| \leq |f(a)| + |f(x) - f(a)| \leq |f(a)| + V_f(a, b)$$

hence $\max_{x \in [a,b]} |f(x)| \leq |f(a)| + V_f(a, b) < \infty$.

Let $(x_i)_{i=1}^{k}$ be some division of the interval $[a, b]$. Then

$$\begin{aligned}
\sum_{i=1}^{k} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| &\leq \sum_{i=1}^{k} |f(x_i)g(x_i) - f(x_{i-1})g(x_i)| \\
&\quad + \sum_{i=1}^{k} |f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^{k} |g(x_i)||f(x_i) - f(x_{i-1})| \\
&\quad + \sum_{i=1}^{k} |f(x_{i-1})||g(x_i) - g(x_{i-1})|
\end{aligned}$$

$$\leq \max_{x\in[a,b]} |g(x)| V_f(a,b) + \max_{x\in[a,b]} |f(x)| V_g(a,b).$$

Taking the supremum over all divisions we get the claim

$$V_{fg}(a,b) \leq \max_{x\in[a,b]} |g(x)| V_f(a,b) + \max_{x\in[a,b]} |f(x)| V_g(a,b) < \infty.$$

$\square$