Dependence Logic

Exercise 6

Exercise 1. Let ϕ be an FO($\perp_c$)-sentence with an infinite model or arbitrarily large finite models. Show that ϕ has models in all infinite cardinalities. (Hint: use a similar argument as in Theorem 5.2.4 (Compactness of independence logic) of the lecture notes. In particular, you may assume that first-order sentences satisfy the claim.)

Exercise 2. Let χ be an Σ^1_1-sentence of the form

$$\chi = \exists f_1 \dots \exists f_n Q^1 x_1 \dots Q^m x_m \theta, \tag{1}$$

where $x_1, \dots, x_n$ are pairwise distinct variables, $Q^i \in \{\exists, \forall\}$, θ is a quantifier-free first-order formula, and for each function variable f_i there are pairwise distinct $i_1, \dots, i_p$ such that all terms and subterms in θ with f_i as the outermost symbol are of the form $f_i(x_{i_1}, \dots, x_{i_p})$. Show that χ is equivalent to some FO(=(...))-sentence. (Hint: the converse direction was covered in the proof of Proposition 5.3.3 in the lecture notes.)

Exercise 3. Let $k \geq 1$ and R a k-ary relation symbol. Construct an ESO(k-ary)-sentence ϕ of vocabulary $\tau = \{\leq, R\}$ such for all finite ordered τ-structures $\mathcal{M}$:

$$\mathcal{M} \models \phi \Leftrightarrow |R^{\mathcal{M}}| \text{ is even.}$$

($\mathcal{M}$ being a finite ordered structure means that $|\mathrm{Dom}(\mathcal{M})| = n$ for some $n \in \mathbb{N}$ and $\leq^{\mathcal{M}}$ is a linear order of $\mathrm{Dom}(\mathcal{M})$.)

Exercise 4. Show that $=(x)$ is equivalent to $\forall y(y = x \vee y \mid x)$.

Exercise 5. Let $\phi = g(g(x_1, x_2), x_3) = x_4$. Construct a $\Sigma^1_1[\{g\}]$-sentence ψ such that for all $\{g\}$-structures $\mathcal{M}$ and variable assignments s,

$$\mathcal{M} \models_s \phi \iff \mathcal{M} \models_s \psi,$$

and ψ is of the form

$$\exists f_1 \dots \exists f_n \forall z_1 \dots \forall z_m \theta,$$

where θ is a quantifier-free first-order formula, $f_1, \dots, f_n$ are pairwise distinct function variables, $z_1, \dots, z_m$ are pairwise distinct variables, and for each $f \in \{f_1, \dots, f_n, g\}$ there are pairwise distinct $i_1, \dots, i_p$ such that all terms in θ with f as the outermost symbol are of the form $f(z_{i_1}, \dots, z_{i_p})$. (Hint: see the proof of Theorem 5.3.5 in the lecture notes.)