# Real Analysis I

Fall 2019

Homework 6

Exercise session: Wed 16 October, 10:15 - 12:00, Exactum CK111; Emil Airta, emil.airta@helsinki.fi.

1. Let $\lambda$ be a Borel measure in $\mathbb{R}^n$ with the property that

$$\lambda(A) = \inf\{\lambda(U)\colon A \subset U,\, U \text{ open}\}.$$

Suppose that

$$\liminf_{r\to 0} \frac{|\overline{B}(x,r)|}{\lambda(\overline{B}(x,r))} \le t$$

for some $t > 0$ and for all $x \in A$. Show that $|A| \le t\lambda(A)$.

*Proof.* Let $\varepsilon > 0$ and $U$ open set such that $\lambda(U) \le \lambda(A) + \varepsilon$. Since for all $x \in A$ we find sequence $r_i \to 0$ as $i \to \infty$ such that $|\bar{B}(x, r_i)| \le (t+\varepsilon)\lambda(\bar{B}(x, r_i))$ and $\bar{B}(x, r_i) \subset U$ for all $i \in \mathbb{N}$. By Vitali's covering theorem we choose a collection $\{\bar{B}_i\}_{i\in\mathcal{I}}$ of disjoint closed balls $\bar{B}_i$ such that $|A \setminus \bigcup_{i\in\mathcal{I}} \bar{B}_i| = 0$ and $\bar{B}_i \subset U$. Thus, we have

$$\begin{aligned}
|A| &= |A \setminus \bigcup_{i\in\mathcal{I}} \bar{B}_i| + \Big|\bigcup_{i\in\mathcal{I}} \bar{B}_i\Big| \\
&= \sum_{i\in\mathcal{I}} |\bar{B}_i| \\
&\le \sum_{i\in\mathcal{I}} (t+\varepsilon)\lambda(\bar{B}_i) \overset{\text{(disjointness!)}}{=} (t+\varepsilon)\lambda\Big(\bigcup_{i\in\mathcal{I}} \bar{B}_i\Big) \\
&\le (t+\varepsilon)\lambda(U) \le (t+\varepsilon)\lambda(A) + \varepsilon
\end{aligned}$$

which is enough since letting $\varepsilon \to 0$ we conclude that $|A| \le t\lambda(A)$. □

2. Define $f\colon [0,1] \to \mathbb{R}$ by setting $f(0) = 0$ and $f(x) = x\sin(1/x)$ otherwise. Show that $f$ is continuous but not of bounded variation.

*Proof.* Continuity is fairly obvious.

Let $x_0 = 0$ and $x_k = 1$. For $k = 1, 2, 3, \ldots, k-1$ set

$$x_i = \frac{1}{i\pi + \pi/2}.$$

By periodicity of sine function for $i = 2, \ldots, k-1$ we have

$$|x_i \sin(x_i) - x_{i-1}\sin(x_{i-1})| = \frac{2}{2i+\pi} + \frac{2}{2i+2+\pi} \geq \frac{C}{i},$$

also above inequality holds for $i = 1$. Therefore,

$$\sum_{i=1}^{k} |x_i \sin(x_i) - x_{i-1}\sin(x_{i-1})| \geq C \sum_{i=1}^{k-1} \frac{1}{i}$$

and recall this is a divergent series, a harmonic series, as $k \to \infty$. By definition of supremum this means that $V_f(0,1) = \infty$.

□

3. Let $f \in BV([a,b])$ be a continuous function on an interval $[a,b]$. Show that $x \mapsto V_f(a,x)$ is continuous. Show also that if $f$ is absolutely continuous, then $x \mapsto V_f(a,x)$ is absolutely continuous.

*Proof.* Assume first that $f$ is a bounded variation and a continuous function on an interval $[a,b]$. Let $\varepsilon > 0$. Choose $\delta$ such that $|f(x) - f(y)| < \varepsilon/2$ whenever $x, y \in [a,b]$ so that $|x-y| < \delta$; recall continuous functions are uniformly continuous in compact sets. Let sequence $(x_i)$ define a division of $[a,b]$ such that there exists $j$ so that $x_j - x_{j-1} < \delta$ and

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \frac{\varepsilon}{2} > V_f(a,b).$$

We can choose such division by definition, also by definition we have

$$V_f(a, x_{j-1}) + V_f(x_j, b) \geq \sum_{\substack{i=1 \\ i\neq j}}^{k} |f(x_i) - f(x_{i-1})|.$$

Thus, we get

$$\begin{aligned}
V_f(x_{j-1}, x_j) &= V_f(a,b) - (V_f(a,x_{j-1}) + V_f(x_j,b)) \\
&< \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \frac{\varepsilon}{2} - (V_f(a,x_{j-1}) + V_f(x_j,b)) \\
&\leq \frac{\varepsilon}{2} + \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| - \sum_{\substack{i=1 \\ i\neq j}}^{k} |f(x_i) - f(x_{i-1})| \\
&= \frac{\varepsilon}{2} + |f(x_j) - f(x_{j-1})| < \varepsilon,
\end{aligned}$$

since $x_{j-1} - x_j < \delta$.

Next, assume that $f$ is an absolutely continuous function hence also bounded variation. Let $\varepsilon > 0$ and let $\delta$ be such that $\sum |f(x_j) - f(x_{j-1})| < \varepsilon/2$ whenever $\sum_i |x_i - x_{i-1}| < \delta$. Then let $(a_j, b_j)$ be some disjoint partition of $[a, b]$ such that $\sum_{j=1}^{k} b_j - a_j < \delta$.

For every $[a_j, b_j]$ we have disjoint partition $a_j = x_0^j < x_1^j < \cdots < x_m^j = b_j$ such that

$$V_f(a, a_j) - V_f(a, b_j) = V_f(a_j, b_j) < \sum_{i=1}^{m} |f(x_i^j) - f(x_i^j)| + \frac{\varepsilon}{2k}.$$

Since $\sum_{j=1}^{k} b_j - a_j < \delta$, we have $\sum_{j=1}^{k} \sum_{i=1}^{m} |x_i^j - x_{i-1}^j| < \delta$. Also, this implies that

$$\sum_i \sum_j |f(x_i^j) - f(x_{i-1}^j)| < \varepsilon/2.$$

Hence,

$$\sum_{j=1}^{k} V_f(a_j, b_j) < \frac{\varepsilon}{2} + k \cdot \frac{\varepsilon}{2k} = \varepsilon,$$

as desired. □

4. Suppose $f, g\colon [a, b] \to \mathbb{R}$ are absolutely continuous. Prove the integration by parts formula:

$$\int_a^x fg' = [f(x)g(x) - f(a)g(a)] - \int_a^x f'g, \qquad x \in [a, b].$$

*Proof.* First, we show that $fg\colon [a, b] \to \mathbb{R}$ is absolutely continuous. Recall continuous functions in bounded domain are bounded.

Let $\varepsilon > 0$ and let $\delta$ be small enough such that

$$\sum_{j=1}^{k} |f(b_j) - f(a_j)| < \frac{\varepsilon}{2\|g\|_\infty}$$

and

$$\sum_{j=1}^{k} |g(b_j) - g(a_j)| < \frac{\varepsilon}{2\|f\|_\infty}$$

whenever intervals $(a_j, b_j)$ are disjoint and $\sum_{j=1}^k b_j - a_j < \delta$. Thus, we have

$$\begin{aligned}\sum_{j=1}^k |f(b_j)g(b_j) - f(a_j)g(a_j)| &\leq \sum_{j=1}^k |g(b_j)||f(b_j) - f(a_j)| \\ &\qquad + \sum_{j=1}^k |f(a_j)||g(b_j) - g(a_j)| \\ &< \varepsilon\end{aligned}$$

for all disjoint intervals $(a_j, b_j)$ with the property $\sum_{j=1}^k b_j - a_j < \delta$.

Since $\varphi = fg$ is absolutely continuous, we can apply Theorem 3.77, ($\varphi$ is absolutely continuous $\Leftrightarrow \exists \varphi'$ a.e. and is integrable, and $\varphi(x) = \varphi(a) + \int_a^x \varphi'(t)\,\mathrm{d}t$ for $x \in [a, b]$), that is,

$$\begin{aligned}f(x)g(x) &= f(a)g(a) + \int_a^x (fg)' \\ &= f(a)g(a) + \int_a^x (f'g + fg') \\ &= f(a)g(a) + \int_a^x f'g + \int_a^x fg'\end{aligned}$$

By rearrangement of the terms we obtain the claim

$$\int_a^x fg' = [f(x)g(x) - f(a)g(a)] - \int_a^x f'g, \qquad x \in [a, b].$$

□

5. Let $f\colon [a, b] \to \mathbb{R}$ be absolutely continuous. Show that $|f(E)| = 0$ if $E \subset [a, b]$ satisfies $|E| = 0$.

*Proof.* Let $\varepsilon > 0$ and let $\delta$ be as in the definition of absolute continuity, i.e.

$$\sum_{i=1}^k |f(x_j) - f(x_{j-1})| < \varepsilon$$

whenever intervals $(x_{j-1}, x_j)$ are disjoint and $\sum_{j=1}^k |x_j - x_{j-1}| < \delta$.

For compact set $K$, with $|K| = 0$, we can choose open cover of disjoint intervals $(a_j, b_j)$ so that $\sum_{j=1}^k b_j - a_j < \delta$.

Now, for each interval $(a_j, b_j)$ we choose a subinterval $(x_j, y_j)$, $x_j = a_j$ and $y_j = b_j$ are allowed, such that $f(x_j) \leq f(x) \leq f(y_j)$ for all $x \in (a_j, b_j)$. Hence, $f((a_j, b_j)) \subset f((x_j, y_j)) \cup f(\{x_j\}) \cup f(\{y_j\})$ and by continuity

$$|f((a_j, b_j))| \leq |f((x_j, y_j)) \cup f(\{x_j\}) \cup f(\{y_j\})| = |f(y_j) - f(x_j)|.$$

Thus, we have

$$|f(K)| \leq \left| f\Big( \bigcup_{j=1}^{k} (a_j, b_j) \Big) \right| \leq \sum_{j=1}^{k} |f(y_j) - f(x_j)| < \varepsilon.$$

Now, let $E$ be any measurable set with $|E| = 0$. Recall for every measurable set $f(E)$ and $\varepsilon > 0$, we can find closed set $K' \subset f(E)$ such that $|f(E) \setminus K'| < \varepsilon$. Also, any compact set $K' \subset f(E)$ is image of some compact set $K$ by continuity. Combining with previous result we get

$$|f(E)| = |f(E) \setminus K'| + |K'| = |f(E) \setminus K'| + |f(K)| < 2\varepsilon$$

which gives the claim as $\varepsilon \to 0$. $\square$