Dependence Logic
Exercise 7

Exercise 1. Let $\vec{t}_1, \dots, \vec{t}_n$ be k-ary sequences of terms, and c_l, c_r two constant symbols. Let

$$\eta := (\varphi \owedge \psi) \owedge \exists y \big(((y = c_l \land \varphi) \lor (y = c_r \land \psi)) \land \bigwedge_{i=1}^{n} (\theta_i \land \theta_i') \big),$$

where

$$\begin{aligned} \theta_i := \exists \vec{z}_1 \exists \vec{z}_2 \big(((y = c_l \land \vec{z}_1 = \vec{t}_i \land \vec{z}_2 = \vec{c}_l\ ) \\ \lor (y = c_r \land \vec{z}_1 = \vec{c}_l \land \vec{z}_2 = \vec{t}_i)) \land \vec{t}_i \subseteq \vec{z}_1 \land \vec{t}_i \subseteq \vec{z}_2 \big), \\ \theta_i' := \exists \vec{z}_1 \exists \vec{z}_2 \big(((y = c_l \land \vec{z}_1 = \vec{t}_i \land \vec{z}_2 = \vec{c}_r\ ) \\ \lor (y = c_r \land \vec{z}_1 = \vec{c}_r \land \vec{z}_2 = \vec{t}_i)) \land \vec{t}_i \subseteq \vec{z}_1 \land \vec{t}_i \subseteq \vec{z}_2 \big), \end{aligned}$$

where $\vec{z}_1$, $\vec{z}_2$, $\vec{c}_l := (c_l, \dots, c_l)$, and $\vec{c}_r := (c_r, \dots, c_r)$ are k-ary. Show that for all teams X and all models $\mathcal{M}$ such that $c_l^{\mathcal{M}} \neq c_r^{\mathcal{M}}$,

$$\mathcal{M} \models_X \varphi \underset{\vec{t}_1, \dots, \vec{t}_n}{\lor} \psi \implies \mathcal{M} \models_X \eta.$$

Exercise 2. Let $\sigma = \{E, c, d\}$, where E is a binary relation symbol and c, d are constant symbols. Define $\phi \in \mathrm{FO}(\subseteq)[\sigma]$ as

$$\phi := \exists x (c \subseteq x \land d \neq x \land \forall y (\neg E(x, y) \lor y \subseteq x)).$$

Recall the definition of transitive closure from Exercise 5 and show that for all models $\mathcal{M}$ over σ,

$$\mathcal{M} \models \phi \implies (c^{\mathcal{M}}, d^{\mathcal{M}}) \notin TC(E^{\mathcal{M}}).$$

(Hint: inductively show that the values of x subsume $\{a \in \mathrm{Dom}(\mathcal{M}) \mid (c^{\mathcal{M}}, a) \in TC(E^{\mathcal{M}})\}$).

Exercise 3. Show the converse of the previous exercise: for all models $\mathcal{M}$ over σ,

$$(c^{\mathcal{M}}, d^{\mathcal{M}}) \notin TC(E^{\mathcal{M}}) \implies \mathcal{M} \models \phi.$$

Exercise 4. A sentence $\psi \in \Sigma_1^1[\tau \cup \{R\}]$ is said to be downwards monotone with respect to the relation R if for all τ-structures $\mathcal{M}$ and interpretations P, P' for R

$$\text{if } (\mathcal{M}, P) \models \psi \text{ and } P' \subseteq P, \text{ then } (\mathcal{M}, P') \models \psi.$$

Show using induction on $\phi \in CL_{FO^+}(\Sigma_1^1)$ that if R appears only negatively in ϕ (i.e., all occurrences are of the form $\neg R(\vec{t})$), then ϕ is downwards monotone with respect to R.

Exercise 5. Let $\mathcal{M} = (M, E^{\mathcal{M}})$ be a graph. Define a binary operator $\Phi \colon \mathcal{P}(M^2) \to \mathcal{P}(M^2)$ as follows:

$$\Phi(P) := \{(s(x), s(y)) \mid \mathcal{M}^* \models_s \psi\},$$

where $\psi := R(x, y) \vee E(x, y) \vee \exists z(E(x, z) \wedge R(z, y))$, $\mathcal{M}^* \restriction \tau = \mathcal{M}$, and $R^{\mathcal{M}^*} = P$. Show that any transitive relation $S \subseteq M^2$ such that $E^{\mathcal{M}} \subseteq S$ is a fixed point of Φ.