Dependence Logic

Exercise 8

Exercise 1. Let M be a set and $\Phi\colon \mathcal{P}(M) \to \mathcal{P}(M)$.

- The function Φ is *monotone* if for all $P \subseteq P' \subseteq M$, $\Phi(P) \subseteq \Phi(P')$.
- The function Φ is *inflationary* if for all $P \subseteq M$, $P \subseteq \Phi(P)$.
- The function Φ is *inductive* if its stages satisfy $\Phi_\alpha \subseteq \Phi_\beta$ for all ordinals $\alpha \leq \beta$, where the stages of Φ are defined by: $\Phi_0 := \emptyset$, $\Phi_{\alpha+1} := \Phi(\Phi_\alpha)$, and $\Phi_\lambda := \cup_{\alpha<\lambda}\Phi_\alpha$ for a limit ordinal λ.

Show that if Ψ is either monotone or inflationary then it is inductive. You may assume M to be finite if you have not studied transfinite induction before.

Exercise 2. Give examples of inductive operators Φ_1 and Φ_2 such that Φ_1 is monotone but not inflationary and Φ_2 is inflationary but not monotone.

Exercise 3. Let $\mathcal{M} = (\mathbb{N}, +, \times, 0, 1)$, i.e., $\mathcal{M}$ consists of the natural numbers together with their natural addition, multiplication, zero, and one. Define $\phi := [\mathrm{LFP}_{x,R}\, \psi]x$ where ψ is defined as

$$x = 1 \vee \exists y(R(y) \wedge x = y \times (1+1)).$$

Show that $\mathcal{M} \models_s \phi$ where $s(x) = 8$. [1]

Exercise 4. Let

$$\varphi = \forall x \neg[\mathrm{GFP}_{x,R}\, \exists y(R(y) \wedge y < x)]x.$$

Let $\mathcal{M} = (\mathbb{N}, <)$ where $<$ is the natural ordering of natural numbers. Show that $\mathcal{M} \models \phi$.[2]

Exercise 5. Denote by posLFP the set of LFP-formulas in negation normal form with no subformulas of the form $[\mathrm{GFP}_{\vec{x},R}\, \phi]\vec{t}$. Construct a sentence $\varphi' \in$ posLFP that is equivalent to the sentence φ from Exercise 4. (Hint: you may use the equivalences stated in the proof of Theorem 2.3.1 and in the beginning of Section 6.3.)

[1] Bonus question (not required): is the set of natural numbers defined by ϕ first-order definable?

[2] Bonus question (not required): what property of (strict) total orders is defined by φ?