CHAPTER 1

SOME BASIC PROPERTIES OF REAL NUMBERS

In the following, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{Z}_{>0}$ denotes the set of all positive integers and $\mathbb{Z}_{\geq 0}$ denotes the set of all non-negative integers. We have the inclusions

$$\mathbb{Z}_{\geq 0} \subset \mathbb{Z}_{>0} \subset \mathbb{Z} \subset \mathbb{R}. \tag{1}$$

1.1. Supremum and infimum of a set of real numbers

Let us consider the set

$$E = \{1, \frac{1}{2}, \frac{1}{3} \dots\} = \{\frac{1}{n}; n \in \mathbb{Z}_{>0}\}.$$

Intuitively, we want to say the following statements:

1. E has a maximal element, or maximum, namely 1;
2. E has no minimal element, or minimum. But the number 0 is the "infimum" of E.

Here, for 1 to be the maximal element of E, it first has to be an element of E, i.e. $1 \in E$. A precise definition of the maximal element of a set of real numbers is the following:

Definition 1.1.1 **(Maximal element)**. — Let A be a non-empty subset of the set of real numbers $\mathbb{R}$. We say that $a \in A$ is the maximal element of A if and only if all elements in A are smaller or equal to a.

$$\forall x \in A, x \leq a. \tag{2}$$

The maximal element, when it exists, is unique and is denoted by $\max(A)$.

In order to make the notion of "supremum" of a set of real numbers rigorous, we need the language of mathematics. The supremum of a set does not necessarily belong to the set.

Definition 1.1.2 **(Supremum)**. — Let $A \subset \mathbb{R}$ be a non-empty set of real numbers. We say that $M \in \mathbb{R}$ is the supremum of A if (and only if)

- M is an upper bound of A: for all elements a of A, a is smaller or equal to M.

$$\forall a \in A, a \leq M; \tag{3}$$

- M is the smallest of all upper bounds of A: "if we move a little bit (ϵ-bit) to the left of M, we can find an element of A".

$$\forall \epsilon > 0, \exists a \in A, a \in (M - \epsilon, M]. \tag{4}$$

  The above condition reads "for all $\epsilon > 0$, there exists an element a of A, such that a is in the interval $(M - \epsilon, M]$".

The second item can also be rephrased as "all number strictly smaller than M is not an upper bound of A". One can similarly define the notion of "infimum" (exercise!).

Let us see an example of how to use this definition to prove things.

Example 1.1.3. — In the example above, 0 is the infimum of the set E.

Proof. — We have to show that 0 verifies the two items in the definition of an infimum.

• First, 0 is a lower bound for E. Indeed, let a be an element of E (we can also write $a \in E$). Then a must be of the form $\frac{1}{n}$ with $n \in \mathbb{Z}_{>0}$ by definition of E. Since for all $n > 0$, $\frac{1}{n} > 0$, 0 is a lower bound for the set E.

• Then we verify that 0 satisfies the second item in the definition of an infimum above. To do that, let $\epsilon > 0$ be an arbitrary positive quantity. We must show that there exists an element a of E such that $a \in [0, 0+\epsilon)$, i.e. $0 \leq a < \epsilon$.

Notice that $0 \leq a$ is given by the fact that 0 is a lower bound of E. Hence we only focus on the other side $a < \epsilon$. Since any element a of E write in the form $\frac{1}{n}$ with $n \in \mathbb{Z}_{>0}$, we see that we should choose a (large!) positive interger n in such a way that $\frac{1}{n} < \epsilon$. This is equivalent to the relation $n > \frac{1}{\epsilon}$.

Let k be the integer(1) such that $k \leq \frac{1}{\epsilon} < k+1$. Since $\frac{1}{\epsilon} > 0$, $k \geq 0$ and $k+1 \in \mathbb{Z}_{>0}$. We take $n = k+1 \in \mathbb{Z}_{>0}$. Then $n > \frac{1}{\epsilon}$ and the element $a = \frac{1}{n}$ satisfies the second item of the definition.

• In conclusion, 0 is the largest lower bound for E, i.e. 0 is the infimum of the set E. □

Notice that in the formulation we implicitely say that the supremum, if it exists(2), is unique. Let us give a proof of this result:

***Proposition 1.1.4* (Uniqueness of the supremum).** — *Let $A \subset \mathbb{R}$ be a non-empty subset of real numbers. If the supremum of A exists and is in $\mathbb{R}$, then it is unique.*

Proof. — Suppose that M, M' and both suprema of A. We are going to argue that this is not possible (proof by contradiction).

Without loss of generality, we can suppose that $M < M'$. In short, since M' is the smallest upper bound of A, $M < M'$ cannot be an upper bound of A and it contradicts the definition of a supremum. With the ϵ-language we can detail this in the following way.

Let $\epsilon = \frac{M'-M}{2} > 0$. Since M' is supremum of A, by definition, there exists an element $a \in A$ such that $a \in (M'-\epsilon, M']$. Since $M'-\epsilon > M$, we have $a > M'-\epsilon > M$. But since M is also supremum of A, the assumption $a \in A$ implies $a < M$. Thus there exists an element a of the set A such that $a > M$ and $a < M$: contradiction. □

It is also important to know how to prove that something does not verify a definition. It is useful to write down for example

***Proposition 1.1.5* (When an upper bound is not the supremum)**

Let $M \in \mathbb{R}$ be an upper bound of a non-empty subset $A \subset \mathbb{R}$. Then M is not the supremum of A if (and only if)

$$\exists \epsilon > 0, \forall a \in A, a \notin (M-\epsilon, M]. \tag{5}$$

i.e. there exists a positive ϵ, such that for all element a of the set A, a is not in the interval $(M-\epsilon, M]$.

Let us see how this works in a example.

Example 1.1.6. — In the example above, 2 is not the supremum of the set E.

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(1) Here $k = \lfloor \frac{1}{\epsilon} \rfloor$ and the function $\lfloor \cdot \rfloor$ is called the floor function of a real number: it extracts in some sense the "interger part" of a real number. However, the existence of such a function is admitted in this course since it is actually a special property of the real numbers called Archimedean property.

(2) The existence of the supremum is a special property of $\mathbb{R}$ and we will not discuss about it in the course.

Proof. — We will show that we can find a ϵ such that if a is an element of the set E, then a is not in the interval $(M-\epsilon, M]$. On a picture we get the idea that for example, $\epsilon = \frac{1}{2}$ should work: this is because there is a "gap" of distance 1 between the number 2 and the set E.

Take $\epsilon = \frac{1}{2}$. Then the interval $(2-\epsilon, 2]$ is equal to $(\frac{3}{2}, 2]$. Since any element a of the set E satisfies $a < 1 < \frac{3}{2}$, we know that a cannot belong to the interval $(\frac{3}{2}, 2]$. This shows that 2 is not the supremum of the set E. □

In the exercise sheet we will encounter more examples.

***Definition 1.1.7* (Boundedness of a set of real numbers).** — Let A be a non-empty subset of $\mathbb{R}$. We say that A is bounded if (and only if) there exist $M, m \in \mathbb{R}$ such that M is an upper bound of A and m is a lower bound of A.

***Example 1.1.8*.** — Consider the set E in the beginning: it is a bounded subset of $\mathbb{R}$.

Proof. — The set E is a non-empty subset of $\mathbb{R}$ and it has an upper bound $1 \in \mathbb{R}$ and a lower bound $0 \in \mathbb{R}$. It is thus bounded. □

***Example 1.1.9*.** — Consider the set $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$: it is not a bounded subset of $\mathbb{R}$.

Proof. — The set $\mathbb{Z}_{>0}$ is a subset of $\mathbb{R}$ and it is non-empty. It is bounded from below by 0 (since all its elements are positive). We now prove that it is not bounded from above, i.e. it does not have an upper bound M in $\mathbb{R}$.

Let us argue by contradiction. Suppose that there exists a real number M in $\mathbb{R}$ such that M is an upper bound of $\mathbb{Z}_{>0}$. By definition of an upper bound, for all positive integers $n > 0$, we have $M \geq n$. This is not possible since for example $N = \lfloor M \rfloor + 1 > M$ and $N \in \mathbb{Z}_{>0}$.

The set $\mathbb{Z}_{>0}$ does not have an upper bound: it is thus unbounded. □

***Remark 1.1.10* (Notations and conventions).** — Any non-empty bounded subset A of $\mathbb{R}$ has an infimum and a supremum: this is a property of the set of real numbers $\mathbb{R}$ called "completeness". If a set $A \subset \mathbb{R}$ is not bounded from below (by any real number), we say that its infimum is $-\infty$. Similarly, if a set $A \subset \mathbb{R}$ is not bounded from above (by any real number), we say that its supremum is $+\infty$. The supremum (resp. infimum) of a set $A \subset \mathbb{R}$ is denoted $\sup(A)$ (resp. $\inf(A)$).

By convention, the supremum of the empty set $\emptyset$ is $-\infty$ and the infimum of the empty set $\emptyset$ is $+\infty$. In some sense this is consistant with the definition: for example for the supremum, one might argue that every real number can be an upper bound for the empty set, so there is no smallest upper bound for $\emptyset$ in $\mathbb{R}$. But it is not worth digging at this stage.

1.2. Developments: quantifiers

The symbol $\forall$ reads "for all" and the symbol $\exists$ reads "there exist(s)". They are the two most commun "quantifiers" in mathematical logic. Sometimes we use the symbol $\exists!$, which means "there exists a unique [...]". We will try to avoid using them too often in this course and write statements in plain natural language.

In the definition of the supremum, the order of the phrase "for all $\epsilon > 0$, there exists an element a of A, such that a is in the interval $(M-\epsilon, M]$" cannot be reversed. Otherwise said, inverting the order of quantifiers can completely change the meaning of a phrase.

***Example 1.2.1* (Another weird definition of the maximal element)**

Let A be a non-empty subset of $\mathbb{R}$. Suppose that there exists a real number $M \in \mathbb{R}$ which satisfies the following properties:

1. M is an upper bound of A: for each element a of the set A, a is smaller or equal to M, i.e.

$$\forall a \in A, a \leq M; \tag{6}$$

2. M satisfies the following property: there exists an element a of A, such that for all $\epsilon > 0$, a is in the interval $(M - \epsilon, M]$:

$$\exists a \in A, \forall \epsilon > 0, a \in (M - \epsilon, M]. \tag{7}$$

Then in fact $M \in A$ and M is the maximal element of A.

Proof. — Let us examine the condition "there exists an element a of A, such that for all $\epsilon > 0$, a is in the interval $(M - \epsilon, M]$". We will show that the only possible a that satisfies this condition must be equal to M. Therefore, M is an element of A (since $a = M$ and $a \in A$) and M is the maximal element of A.

Suppose that $a \neq M$ (proof by contradiction) and a satisfies the second item in the example above. Since M is an upper bound of A and $a \in A$, we know that $a \leq M$. But $a \neq M$, so $a < M$ (the inequality is strict).

To proceed, we have to choose an $\epsilon > 0$ such that a does not belong to the interval $(M - \epsilon, M]$. Take $\epsilon = \frac{M-a}{2}$. Then $a < M - \epsilon$ and $a \notin (M - \epsilon, M]$. This contradicts the second item.

To conclude, we have $a = M$ and by the second item of the example, $a \in A$ so that $M \in A$. But M is also an upper bound of A, so M is the maximal element of A. □

The message here is that one should follow the order of the quantifiers (i.e. respect the order in the logical connections between phrases)!

1.3. Development: operations on sets

We can define some basic algebraic operations on sets. Let A, B be (non-empty) bounded subsets of $\mathbb{R}$: this implies in particular that their suprema and infima exist.

Definition 1.3.1 **(Addition of sets)**. — We define the sum of A and B, denoted by $A + B$, to be the set

$$A + B = \{a + b; a \in A, b \in B\}. \tag{8}$$

Example 1.3.2. — If $A = \{1, 2, 3\}$ and $B = \{-1, 0, 5\}$, then $A + B = \{0, 1, 2, 3, 6, 7, 8\}$.

The supremum of $A + B$ is the sum of supremum of A with the supremum of B.

Proposition 1.3.3 **($\sup$ commutes with $+$)**. — *Let A, B be subsets of $\mathbb{R}$ both bounded from above. We have*

$$\sup(A + B) = \sup(A) + \sup(B). \tag{9}$$

Proof. — The proof is a little bit long but it is a good exercise to read.

• First we prove that the supremum of $A + B$ exists: we prove that $A + B$ is bounded from above by a real number, i.e. $A + B$ has an upper bound.

We show that $\sup(A) + \sup(B)$ is an upper bound of $A + B$. Indeed, all elements in $A + B$ writes as $a + b$ with $a \in A$ and $b \in B$. By definition of the supremum, if $a \in A$ then $a \leq \sup(A)$, and if $b \in B$ then $b \leq \sup(B)$. Summing these two inequalities we have $a + b \leq \sup(A) + \sup(B)$. Thus all elements in $A + B$ are bounded from above by $\sup(A) + \sup(B)$.

• Next, we show that there is no smaller upper bound for $A + B$ than $\sup(A) + \sup(B)$. Take any $\epsilon > 0$. We need to find an element $a + b$ of $A + B$ (and with $a \in A$ and $b \in B$) such that $\sup(A) + \sup(B) - \epsilon < a + b \leq \sup(A) + \sup(B)$. By definition of $\sup(A)$, there exists $a \in A$ such that $a \in (\sup(A) - \frac{\epsilon}{2}, \sup(A)]$. Similarly we can find $b \in B$ such that $b \in (\sup(B) - \frac{\epsilon}{2}, \sup(B)]$. The sum of these two elements yields an element $a + b$ of $A + B$ and it satisfies $a + b \in (\sup(A) + \sup(B) - \epsilon, \sup(A) + \sup(B)]$. Therefore, $\sup(A) + \sup(B)$ is the smallest upper bound for $A + B$.

• In conclusion, $\sup(A + B)$ exists and is equal to $\sup(A) + \sup(B)$. □

We can define other basic operations on sets. See exercise sheet.

1.4. More examples

CHAPTER 2

FUNCTIONS AND SEQUENCES

2.1. Functions and some basic definitions

In this course we are mostly concerned with functions that “eat” a real number and “spit out” a real number, but we first introduce some more general definitions and examples.

To define a function one needs to specify first two sets: one is the set of input X, called “domain of definition”, and the other one is the set of output Y, called “target set” (or “codomaine”). A function f takes an element x of X, called “argument” and gives back an element $y = f(x)$ of Y, called “value”. We can write for example $f : X \to Y$, $x \in X \mapsto y = f(x) \in Y$.

In this course we only consider well-defined functions, that is, functions such that for every element x of X, the value $f(x)$ is uniquely defined and lies in the target set Y. For example, if the domain of definition X and the target set Y are both the set of real numbers $\mathbb{R}$, then the function $x \mapsto x$ is well-defined while $x \mapsto \frac{1}{x}$ is ill-defined (why?). However, $x \mapsto \frac{1}{x}$ is well-defined as a function from $\mathbb{R}\backslash\{0\}$ to $\mathbb{R}\backslash\{0\}$ (why?).

***Example 2.1.1* (A function).** — Let $X = \{1, 2, 3\}$ and $Y = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. Verify that the function $f : x \mapsto \frac{1}{x}$ is a well-defined function from X to Y.

In the above example, the set of all possible values of f is $\{1, \frac{1}{2}, \frac{1}{3}\}$. It is not the full set Y. In general the set of all possible values of $f : X \to Y$ is a subset of Y, called “image” (or “range”) of the function, and is a subset of Y.

***Definition 2.1.2* (Surjection).** — Let $f : X \to Y$ be a function. If the image of f is the full set Y, we say that f is surjective, or that f is a surjection.

Otherwise put, for every element $y \in Y$, we can find (at least one) $x \in X$ such that $f(x) = y$. This can be written as

$$\forall y \in Y, \exists x \in X, f(x) = y.$$

***Example 2.1.3*.** — In the example above, the function f is not a surjection onto Y.

Proof. — To prove that a function $f : X \to Y$ is not surjective we show that

$$\exists y \in Y, \forall x \in X, f(x) \neq y.$$

In plain words, we must find an element y of Y such that for all element x of X, the value $f(x)$ is different from y.

In the example above, consider $y = \frac{1}{4} \in Y$. Then we can verify one by one that no element of X satisfies the condition $f(x) = y = \frac{1}{4}$. □

***Definition 2.1.4* (Injection).** — Let $f : X \to Y$ be a function. If for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$, we say that f is injective, or that f is an injection.

Otherwise put, if $x, x' \in X$ are such that $f(x) = f(x')$ then $x = x'$. This can be written as

$$\forall x, x' \in X, f(x) = f(x') \implies x = x'.$$

Example 2.1.5. — In the example above, the function f is an injection.

***Definition 2.1.6* (Bijection).** — Let $f : X \to Y$ be a function. We call that f is bijective, or that f is a bijection if and only if f is injective and surjective at the same time.

Otherwise put, for every $y \in Y$, there exists one and only one element $x \in X$ such that $f(x) = y$, i.e.

$$\forall y \in Y, \exists! x \in X, f(x) = y.$$

Example 2.1.7. — The function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ such that $f(x) = \frac{1}{x}$ is a bijection.

When a function f is a bijection between X and Y, we establish via f a "correspondence" between elements on these two sets, and we can say that these sets are "equally large". When this is the case, we can also define the inverse function f^{-1} : from Y to X: it is the unique function $g : Y \to X$ which satisfies

$$\forall x \in X, g(f(x)) = x \quad \text{and} \quad \forall y \in Y, f(g(y)) = y.$$

We also abbreviate this by $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

***Remark 2.1.8* (Preimage of a subset of the target set).** — The notation f^{-1} can also be used in another context when we speak of the notion of the preimage of a set in the target set. If $f : X \to Y$ is a function and $B \subset Y$, then f^{-1} applied to B yields a subset X defined as

$$f^{-1}(B) = \{x \in X; f(x) \in B\} \subset X.$$

In short, it is a function between subsets of the target space and the subsets of the domain of definition. Be careful with this notation because it does not work in the same way as the inverse function! For example, for the first example above, one verifies that $f^{-1}(\{\frac{1}{3}, \frac{1}{4}\}) = \{3\}$ but $f(\{3\}) = \{\frac{1}{3}\}$, so that $f\left(f^{-1}(\{\frac{1}{3}, \frac{1}{4}\})\right) \neq \{\frac{1}{3}, \frac{1}{4}\}$.

***Remark 2.1.9* (Graphs of a bijection and its inverse in the case of a real function)**
Let I be an interval of $\mathbb{R}$ and $f : I \to f(I)$ a bijective function. Then the inverse function $g = f^{-1} : f(I) \to I$ exists. One can show that the graph of f and the graph of g are symmetric with respect to the axis $x = y$. Indeed, if (x, y) is in the graph of f, then $y = f(x)$ and consequently $(y, x) = (y, f^{-1}(y))$ is in the graph of g.

One can deduce some other relationships using this observation. For example, if f is derivable at x with $f'(x) \neq 0$, then f^{-1} is derivable at $f(x)$ with derivative $\frac{1}{f'(x)}$. This is an illustration that $f^{-1}(f(x)) = \mathrm{id}_I \implies f'(x) \times (f^{-1})'(f(x)) = 1$.

2.2. Convergence of a sequence

A sequence of real numbers is formally an application $u : \mathbb{Z} \to \mathbb{R}$. Often the domain of definition of u is $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}_{>0}$ and we usually write u_n instead of $u(n)$. The sequence will be written as $(u_n)_{n \in \mathbb{Z}_{\geq 0}}$ or simply (u_n) when the domain of definition is not ambiguous.

Consider the following example of a real sequence:

$$u_n = \frac{1}{n}; \quad \forall n \geq 1.$$

Intuitively one want to say that this sequence "gets closer and closer" to 0 as the value of n increases. A rigorous definition of this phenomenon is the following.

***Definition 2.2.1* (Convergence of a real sequence).** — We say a real sequence $(u_n)_{n\in\mathbb{Z}}$. We say that $(u_n)_{n\in\mathbb{Z}}$ converges as n goes to infinity and the limit of $(u_n)_{n\in\mathbb{Z}}$ is $l \in \mathbb{R}$ if (and only if)

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{Z}, \forall N > n_0, |u_N - l| < \epsilon.$$

In plain language, we sometimes say that sequence $(u_n)_{n\in\mathbb{Z}}$ is ϵ-close to l after index N.
Let us check our intuition that the sequence in the example converges to 0.

Example of a convergent sequence. — Let $\epsilon > 0$. Take $n_0 = \lfloor \frac{1}{\epsilon} \rfloor + 1$. Here, the $\lfloor \cdot \rfloor$ is the floor function. Then for all $N > n_0$,

$$|u_N - 0| = u_N = \frac{1}{N} < \frac{1}{n_0} < \epsilon. \tag{10}$$

By the definition above, we have proven that the sequence $(u_n)_{n\in\mathbb{Z}}$ converges and that its limit is 0. □

We see that the tricky part in the formal proof usually concerns a good choice of the interger n_0. In this course the focus is not the manipulation of this $\epsilon - \delta$ formalism but its different applications.
One should be careful about the precision of our language. When we say that "the" limit of a real sequence $(u_n)_{n\in\mathbb{Z}}$ is $l \in \mathbb{R}$, we imply that l is unique: a real sequence cannot converge to two different limits. We record this fact.

***Proposition 2.2.2* (Uniqueness of the limit).** — *Let $(u_n)_{n>0}$ be a real sequence. If $(u_n)_{n>0}$ converges, then it's limit is unique.*

Let us try to manipulate the notations to prove the following intuitive fact:

Proposition 2.2.3. — *A real converging sequence $(u_n)_{n\in\mathbb{Z}_{>0}}$ is bounded.*

Proof. — Exercise. □

2.3. Subsequence

One can "extract" a subsequence from a given sequence.

***Definition 2.3.1* (Subsequence).** — Let $(u_i)_{i\in\mathbb{Z}_{>0}}$ be a real sequence. We say that $(v_j)_{j\in\mathbb{Z}_{>0}}$ is a subsequence of (u_n) if there exists an increasing sequence of indices $0 < i_1 < i_2 < \dots$ such that for all $j > 0$, $v_j = u_{i_j}$.

In particular, one notices that $i_j \geq j$ for all $j > 0$ (by recurrence).
The most important feature on a subsequence when it comes to the question of convergence is the following:

Proposition 2.3.2. — *Let $(v_j)_{j\in\mathbb{Z}_{>0}}$ be a subsequence of some real sequence $(u_i)_{i\in\mathbb{Z}_{>0}}$. Then if $(u_i)_{i\in\mathbb{Z}_{>0}}$ is converging towards $l \in \mathbb{R}$, the subsequence $(v_j)_{j\in\mathbb{Z}_{>0}}$ is also converging and its limit is $l \in \mathbb{R}$.*

Studying subsequences of a sequence gives good tests for convergence. A real sequence $(u_i)_{i\in\mathbb{Z}_{>0}}$ is not converging if either of the two following cases happens:
- It has a non-bounded subsequence;
- It has two converging subsequences but with different limits.

***Remark 2.3.3* (Bolzano-Weierstrass theorem).** — The above two situations are actually all that can happen for a non-converging sequence: this is a consequence of the Bolzano-Weierstrass theorem, which states that every bounded sequence must have a converging subsequence.

2.4. Divergence of a sequence

Consider the following sequences:

$$v_n = n; \quad \forall n \geq 1;$$

$$w_n = (-1)^n; \quad \forall n \geq 1.$$

Both sequences do not seem to “get closer and closer” to any real number. How can we verify this intuition?

One can for example argue in the following way (without invoking the $\epsilon - \delta$ formalism):

- The sequence v_n is not bounded, thus it cannot be convergent;
- The sequene w_n has two convergent subsequences w_{2n+1} and w_{2n} with different limits.

Here the two types of divergence are different in nature: the first one is not bounded and “escapes” to infinity, while the second one is bounded but “oscillates” with large jumps.

2.5. Operations on sequences

Just like functions (a sequence is a function $\mathbb{Z} \to \mathbb{R}$ anyways), we can perform operations on sequences like addition, multiplication, composition. Let us record some results on convergence.

***Proposition 2.5.1* (Operations).** — *The set of converging real sequence is closed under addition and multiplication. In other words, let $(u_n)_{n>0}$ and $(v_n)_{n>0}$ be two converging real sequences. Then the following limits exist and can be calculated as:*

$$\begin{aligned} \lim_{n\to\infty}(u_n + v_n) &= \lim_{n\to\infty} u_n + \lim_{n\to\infty} v_n; \\ \lim_{n\to\infty}(u_n \cdot v_n) &= \lim_{n\to\infty} u_n \cdot \lim_{n\to\infty} v_n. \end{aligned} \tag{11}$$

***Proposition 2.5.2* (Comparaison).** — *Let $(u_n)_{n>0}$ and $(v_n)_{n>0}$ be two converging real sequences with limits respectively l and l'. If for all $n > 0$, $u_n \leq v_n$, then $l \leq l'$.*

***Example 2.5.3*.** — Let $u_n = \frac{1}{e^n}$ and $v_n = \frac{1}{1+n}$. Since for all $x > 0$, $e^x > 1 + x$, we have $u_n < v_n$ for all $n > 0$. If the sequences converges then $\lim u_n \geq \lim v_n$.

Attention! Even if for all $n > 0$ we have strict inequality $u_n < v_n$ for two converging real sequences, we can only conclude with a loose inequality on their limits.

***Proposition 2.5.4* (Composition with a continuous function)**

Let $(u_n)_{n>0}$ be a converging real sequence towards $l \in \mathbb{R}$. Let I be an open interval containing l and $f : I \to \mathbb{R}$ be continuous at l. Then the sequence $(f(u_n))_{n>0}$ converges towards $f(l)$.

***Example 2.5.5*.** — The sequence $u_n = \cos(\frac{1}{n})$ is converging to 1.

***Example 2.5.6*.** — The sequence $(u_n)_{n>0}$ with $u_n = 2\exp\left(\left(\frac{1}{1+\frac{1}{n}}\right)^{\sqrt{2}}\right)$ converges...

Careful! For example, the function $x \mapsto \frac{1}{x}$ is not continuous at $x = 0$, so we can never apply this to the sequence $\frac{1}{u_n}$ with u_n going to 0.

2.6. Monotone convergence theorem

The following theorem is very useful in practice when we want to prove that a sequence has a limit without knowing what the limit exactly is.

Theorem 2.6.1 **(Monotone convergence theorem)**. — *If a real sequence* $(u_n)_{n\in\mathbb{Z}_{>0}}$ *is decreasing and bounded from below, then its infimum is the limit.*

As an exercise, formulate the version for an increasing real sequence bounded from above.
Let us see how to apply this to some examples.

Example 2.6.2. — The sequence $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is convergent.

Proof. — The sequence is increasing. Let us prove that it is bounded from above. Notice the inequality:

$$\frac{1}{k^2} \leq \frac{1}{(k-1)(k)} = \frac{1}{k-1} - \frac{1}{k}$$

for all $k > 2$ and we have, by summing up from $k = 2$ to n,

$$u_n \leq 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{n} = 2 - \frac{1}{n} \leq 2.$$

Then the sequence is increasing and bounded from above: it is thus converging. □

Example 2.6.3. — For all $s > 2$, the sequence $v_n = \sum_{k=1}^{n} \frac{1}{k^s}$ is convergent.

Proof. — The sequence v_n is increasing and bounded from abvoe by the limit of the sequence u_n in the example above. □

2.7. Squeeze theorem

The squeeze theorem roughly says that if a sequence (u_n) is in between two other convergent sequences **with the same limit**, then the sequence (u_n) is converging towards the same limit.

Theorem 2.7.1. — *Let* $v_n \leq u_n \leq w_n$ *be three sequences. If* $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = l$*, then* $\lim_{n\to\infty} u_n = l$.

Of course, the comparaison does not have to be true for the first terms!

Example 2.7.2. — The sequence $(u_n)_{n>0}$ with $u_n = \frac{\cos(n)}{n}$ is converging to 0.

Example 2.7.3. — Using the inequality $\sin(x) \leq x \leq \tan(x)$ for $x \in (0, \frac{\pi}{2})$, prove that the sequence $u_n = n\sin(\frac{1}{n})$ is converging to 1.

Example 2.7.4. — A real sequence $(u_n)_{n>0}$ is converging to 0 if and only if $(|u_n|)_{n>0}$ is converging to 0.

Example 2.7.5. — The sequence $u_n = \sum_{i=1}^{n} \frac{(-1)^{n-1}}{n}$ is converging.

Proof. — Consider $v_n = u_{2n-1}$ and $w_n = u_{2n}\ldots$ □

2.8. Back to the source

All the theorems above are consequence of the definition of the convergence of a real sequence. Although they are very useful (that is why we have them), sometimes it is easier to go back to the definition of the convergence with the $\epsilon - \delta$ language to prove results. Here are some examples.

Example 2.8.1. — Let $(u_n)_{n>0}$ be a converging real sequence towards $l \in \mathbb{R}$. Show that the sequence $(v_n)_{n>0}$ defined by $v_{2k} = u_k, v_{2k-1} = u_k$ for all $k > 0$ is also converging towards l.