Dependence Logic
Exercise 9

Exercise 1. Let Φ be a monotone k-ary operator on M, and let

$$S = \{X \subseteq M^k \mid X \subseteq \Phi(X)\}.$$

Show that S is closed under unions, i.e., if $X \in S$ and $Y \in S$, then $X \cup Y \in S$.

Exercise 2. Let ψ be of the form $[\mathrm{LFP}_{\vec{x},R}\,\phi(\vec{x},R)]\vec{t}$, where $\phi(\vec{x},R)$ is a first-order formula. Show that

$$\psi \equiv \theta,$$

where $\theta \in \Pi^1_1$. Here Π^1_1 is a fragment of second-order logic consisting of formulas of the form form

$$\forall Q_1 \cdots \forall Q_n \chi,$$

where $Q_1, ..., Q_n$ are function and relation variables and χ is a first-order formula.

Exercise 3. Let ψ be of the form $[\mathrm{LFP}_{x,R}\,\phi(x,y)]x$, where ϕ is a first-order formula with free variables x and y. Show that ψ is equivalent to the formula

$$[\mathrm{LFP}_{xy,R'}\,\phi'(x,y)]xy,$$

where $\phi' := \phi(R'(t,y)/R(t))$. (Hint: for a model $\mathcal{M}$ with domain M show by induction on α that $\{(a,b) \in M^2 \mid a \in \Gamma^{\alpha}_{\mathcal{M},\phi(x,b)}\} = \Gamma^{\alpha}_{\mathcal{M},\phi'(x,y)}$. If you have not studied transfinite induction you may assume that M is finite.)

Exercise 4. Complete the induction proof of Theorem 6.3.6 for universal quantification by showing that for all $\mathcal{M}$ and X,

$$\mathcal{M} \models_X \psi^+(\vec{x},\vec{y}) \iff (\mathcal{M}, X[\vec{x}]) \models_s \psi(R,\vec{x},\vec{y}) \text{ for all } s \in X,$$

where $\psi(R,\vec{x},\vec{y}) = \forall v \alpha(R,\vec{x},\vec{y}v)$, $\psi^+(\vec{x},\vec{y}) = \forall v \alpha^+(\vec{x},\vec{y}v)$, and the induction assumption is that for all $\mathcal{M}$ and Y,

$$\mathcal{M} \models_Y \alpha^+(\vec{x},\vec{y}v) \iff (\mathcal{M}, Y[\vec{x}]) \models_s \alpha(R,\vec{x},\vec{y}v) \text{ for all } s \in Y.$$